Analysis of equivalent anisotropy arising from dual isotropic layers of acoustic media

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The equivalence between a single mass-anisotropic layer and two isotropic layers is analyzed by studying two systems: one consists of an anisotropic layer sandwiched between two arbitrarily chosen isotropic media; and the other consists of two isotropic layers, of a total thickness equal to that of the anisotropic layer, sandwiched between the same pair of isotropic media. The equivalence is established by matching the transmission and reflection coefficients of the two systems for an arbitrarily chosen incident angle. The first-order equivalence leads to exactly the same set of relations as often quoted in the literature. However, it was concluded that a full second-order equivalence is not possible unless the incident is normal to the surface, or the materials are isotropic. One of the requirements for the second-order equivalence is that the two isotropic layers must have their impedances matched. Together with the first order equivalence requirements, this gives a complete set of conditions for determining all the materials properties of the two isotropic layers. On the other hand, the unattainable full second-order equivalence can be alleviated by a proper placement of layers: by placing the heavier layer adjacent to the medium of greater acoustic impedance. Numerical examples show that this remedy in fact is more important than following the partial requirement for the second order equivalence when the equivalent isotropic layers are used in acoustic cloaking applications. © *2012 Acoustical Society of America*. [http://dx.doi.org/10.1121/1.4744927]

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I. INTRODUCTION

Anisotropic materials are an important ingredient in the quest for acoustic cloaking. The particular anisotropy needed in various acoustic cloaking designs is mass-orthotropy: that is, the mass density must be a diagonal tensor having different values for different entries. The mass-anisotropy is not a commonly observed feature in natural materials. This peculiar anisotropy is derived from the analogy between the governing equations for electromagnetic and acoustic waves, when cloak designs for electromagnetic waves were ported to acoustics, such as the well-known design by Cummer and Schurig (2007).

In the past few years, the idea that two layers of isotropic materials can be combined to effectively behave like a mass-anisotropic layer has created an intense interest in the metamaterial research community. This idea comes from a paper by Schoenberg and Sen (1983) that studies the equivalent acoustic properties of a half space filled with periodic layers of different isotropic media. However, the original paper did not explore this equivalence in depth, except a mere mentioning that this is the effective medium for long wavelength propagation. It does not even contain a displayed equation for the now often-quoted equivalent properties. This idea was introduced to acoustic cloak design by Cheng *et al*. (2008) and Torrent and Sánchez-Dehesa (2008). Since then, many cloaking designs based on this or similar ideas of homogenizing layered structures to achieve necessary mass-anisotropy have been proposed, such as Qiu *et al*. (2009), Farhat *et al*. (2009), Chen and Chan (2010), Ren *et al*. (2010), Norris and Nagy (2010), and Urzhumov *et al*. (2010). Yang *et al*. (2010) used the idea to design acoustic super-scatterers, which is opposite of cloaking. Recently, Smith (2011) applied the asymptotic homogenization method to study periodically mixed layers of two isotropic media, and obtained the same effective material properties in the low frequencies; but when approaching a local resonant frequency, the effective medium changed dramatically to having an anisotropic modulus.

When this idea is approached from the perspective of finding the isotropic materials to mimic the desired anisotropy, it is noted that this equivalence does not provide sufficient conditions to completely determine the properties of the isotropic layers. This opens many questions such as how to best utilize this equivalence, how to place the two layers, whether there are limitations on the incident angle, and what are comparative merits of different choices. Given the great interest in the recent years, in this paper, this equivalence is analyzed in depth in an attempt to answer these questions. Although the equivalence can be established for any number of layers, this paper will focus exclusively on the case of dual layers.

Specifically, the equivalence of the following two systems is considered: the *anisotropic system:* a thin anisotropic

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layer is sandwiched between two arbitrarily chosen isotropic media 1 and 2; and the *isotropic system:* two thin isotropic layers A and B, of a total thickness equal to that of the anisotropic layer, sandwiched between the two isotropic media 1 and 2. For simplicity, this equivalence is studied in a Cartesian coordinate system. The equivalence will be established based on the transmission and reflection of a planar incident wave by the two systems. The analysis is limited to the case when the layer's surface is one of the principal axes of the anisotropic layer.

This paper is organized as the following: In Sec. II, the mass-anisotropic acoustic media is briefly reviewed, and reflection and transmission coefficients for the anisotropic system are derived. Section III presents the transmission and reflection coefficients for the isotropic system. Section IV studies the equivalence of the two systems and the first order approximation. Section V explores second order relations between the two systems. Section VI discusses a strategy for suppressing the difference of the two systems. The paper is concluded in Sec. VII.

II. PLANE WAVE REFLECTION AND TRANSMISSION BY ANISOTROPIC LAYER

A coordinate system is set up such that the layer is bounded by the y-axis and the line $x = h$, where h is the thickness of the layer. In addition, the x- and y-axes coincide with the material's principal directions. The governing equation in such anisotropic media consists of the following ingredients, based on Cummer and Schurig (2007) for a polar coordinate system, for waves of a constant frequency ω:

$$i\omega\rho_x \upsilon_x = -\frac{\partial p}{\partial x}, \tag{1}$$

$$i\omega\rho_y \upsilon_y = -\frac{\partial p}{\partial y}, \tag{2}$$

$$\frac{i\omega}{K} p = -\frac{\partial \upsilon_x}{\partial x} - \frac{\partial \upsilon_y}{\partial y}, \tag{3}$$

where ρ_x and ρ_y are the mass densities in the two principal directions, K is the bulk modulus, p is the acoustic pressure, and υ_x and υ_y are the particle velocity components. Eliminating $\boldsymbol{\upsilon}$ gives the following governing equation:

$$\frac{1}{\rho_x}\frac{\partial^2 p}{\partial x^2} + \frac{1}{\rho_y}\frac{\partial^2 p}{\partial y^2} + \frac{\omega^2}{K} p = 0. \tag{4}$$

Consider a planar incident wave of unit amplitude from Medium 1 impinges onto the anisotropic layer. Assume the incident wave is expressible as

$$p^{\text{inc}} = \mathrm{e}^{i[k_1(x\cos\theta_1 + y\sin\theta_1) - \omega t]}, \tag{5}$$

where θ_1 is the angle the direction of propagation of the incident wave with respect to the x-axis, and k_1 is the wave number in Medium 1. The reflected wave, also in Medium 1, propagates in a direction that forms an angle π–θ_1 with respect to the x-axis. The transmitted wave in Medium 2 propagates in a direction that forms an angle of θ_2 with respect to the x-axis. There are two waves in the anisotropic layer: one propagating forward with an angle θ, and one propagating backward with an angle π–θ. Note that in the anisotropic layer, the angle is the direction normal to the line of constant phases, which is generally different from the direction of propagation. These waves and their associated angles, as well as the geometry of the problem setup, are depicted in Fig. 1.

The notational convention is the following: medium-specific quantities belonging to isotropic media will be signified by subscripts representing the media they belong to; and those without subscript belong to the anisotropic layer. When the same symbol is used in both systems, the one belonging to the anisotropic system will be signified by an overhead tilde.

The waves in the field are expressible as

$$p^{\text{ref}} = \tilde{R}\mathrm{e}^{i[k_1(-x\cos\theta_1 + y\sin\theta_1) - \omega t]}, \tag{6}$$

$$p^{\text{trm}} = \tilde{T}\mathrm{e}^{i[k_2(x\cos\theta_2 + y\sin\theta_2) - \omega t]}, \tag{7}$$

$$p_a^{+} = A_{+}\mathrm{e}^{i[k(x\cos\theta + y\sin\theta) - \omega t]}, \tag{8}$$

$$p_a^{-} = A_{-}\mathrm{e}^{i[k(-x\cos\theta + y\sin\theta) - \omega t]}, \tag{9}$$

where $\tilde{R}$ and $\tilde{T}$ are the reflection and transmission coefficients (since the incident wave has an unit amplitude), respectively, A_{-} and A_{+} are the amplitudes of the waves in the anisotropic layer. The angles are related by Snell's law as

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2} = \frac{\sin\theta}{c}, \tag{10}$$

where c's are the wave speeds of different media. Snell's law for anisotropic media remains the same as for isotropic media.

The interfaces at $x = 0$ and $x = h$ require the continuities of pressure p and the velocity component υ_x, which can be expressed in terms of acoustic pressure using Eq. (1). These continuity conditions can be written as

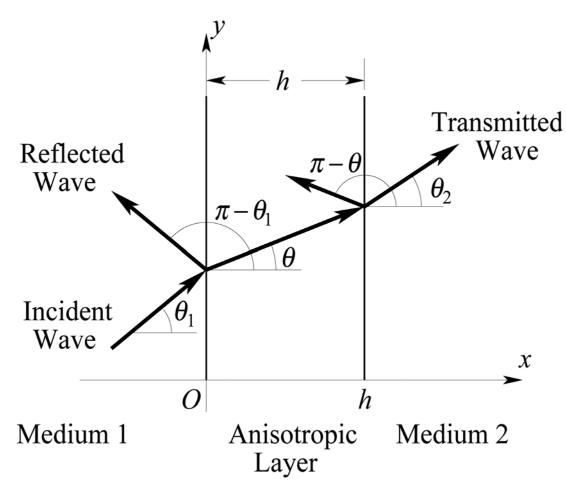

FIG. 1. Geometry of an anisotropic layer sandwiched between two isotrpoic media.

$$1 + \tilde{R} = A_+ + A_-, \tag{11}$$

$$\tilde{\xi}_1 (1 - \tilde{R}) = A_+ - A_-, \tag{12}$$

$$A_+ e^{i\beta} + A_- e^{-i\beta} = \tilde{T} e^{ik_2 h \cos\theta_2}, \tag{13}$$

$$A_+ e^{i\beta} - A_- e^{-i\beta} = \tilde{\xi}_2 \tilde{T} e^{ik_2 h \cos\theta_2}, \tag{14}$$

where

$$\beta = kh \cos\theta, \tag{15}$$

and

$$\tilde{\xi}_1 = \frac{\rho_x c \cos\theta_1}{\rho_1 c_1 \cos\theta} \qquad \tilde{\xi}_2 = \frac{\rho_x c \cos\theta_2}{\rho_2 c_2 \cos\theta}. \tag{16}$$

They can be solved to give

$$\tilde{R} = \frac{(\tilde{\xi}_1 - \tilde{\xi}_2)\cos\beta + i(1 - \tilde{\xi}_1\tilde{\xi}_2)\sin\beta}{(\tilde{\xi}_1 + \tilde{\xi}_2)\cos\beta - i(1 + \tilde{\xi}_1\tilde{\xi}_2)\sin\beta}, \tag{17}$$

$$\tilde{T} = \frac{2\tilde{\xi}_1 e^{-ik_2 h \cos\theta_2}}{(\tilde{\xi}_1 + \tilde{\xi}_2)\cos\beta - i(1 + \tilde{\xi}_1\tilde{\xi}_2)\sin\beta}. \tag{18}$$

This set of expressions appears to be identical to the one for an isotropic layer. The difference for an anisotropic layer is that its wave speed c, and in turn any parameters that are related to the wave speed, is a function of angle θ. Substituting the wave expression in either Eqs. (8) or (9) into Eq. (4) gives the following angular dependence of the sound speed in the anisotropic layer

$$c^2 = K\left[\frac{\cos^2\theta}{\rho_x} + \frac{\sin^2\theta}{\rho_y}\right]. \tag{19}$$

It is also noted that, although media 1 and 2 have been assumed to be isotropic, the derivation above remains unchanged if either or both media are anisotropic.

III. PLANE WAVE REFLECTION AND TRANSMISSION BY TWO ISOTROPIC LAYERS

For the two isotropic layers sandwiched between two isotropic media, the process of finding the transmission and reflection coefficients is the same as the case for an anisotropic layer, except that this case has one more interface, giving two more boundary conditions. For brevity, the derivation is omitted. Denote the thicknesses of the layers as h_A and h_B. The waves and the geometry of this problem setup are depicted in Fig. 2. The reflection and transmission coefficients are found to be

$$R = \frac{\Delta_R}{\Delta} \qquad T = \frac{\Delta_T}{\Delta}, \tag{20}$$

where

$$\begin{aligned}\Delta = 2e^{ik_2 h\cos\theta_2}\{&\cos\beta_A[(\xi_3 + \xi_1\xi_2)\cos\beta_B \\ &- i(1 + \xi_1\xi_2\xi_3)\sin\beta_B] + i\sin\beta_A[-(\xi_2 \\ &+ \xi_1\xi_3)\cos\beta_B + i(\xi_1 + \xi_2\xi_3)\sin\beta_B]\},\end{aligned} \tag{21}$$

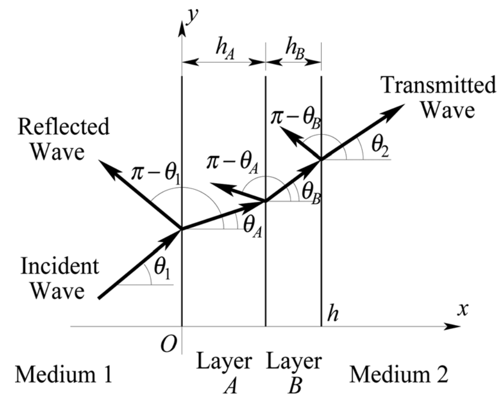

FIG. 2. Geometry of two isotropic layers sandwiched between two isotrpoic media.

$$\begin{aligned}\Delta_R = 2e^{ik_2 h\cos\theta_2}\{&\cos\beta_A[(-\xi_3 + \xi_1\xi_2)\cos\beta_B \\ &+ i(1 - \xi_1\xi_2\xi_3)\sin\beta_B] + i\sin\beta_A[(\xi_2 \\ &- \xi_1\xi_3)\cos\beta_B + (\xi_1 - \xi_2\xi_3)\sin\beta_B]\},\end{aligned} \tag{22}$$

$$\Delta_T = 4\xi_1\xi_2, \tag{23}$$

and

$$\beta_A = k_A h_A \cos\theta_A \quad \beta_B = k_B h_B \cos\theta_B, \tag{24}$$

$$\xi_1 = \frac{\cos\theta_1}{\cos\theta_A}\frac{\rho_A c_A}{\rho_1 c_1} \quad \xi_2 = \frac{\cos\theta_A}{\cos\theta_B}\frac{\rho_B c_B}{\rho_A c_A} \quad \xi_3 = \frac{\cos\theta_2}{\cos\theta_B}\frac{\rho_B c_B}{\rho_2 c_2}, \tag{25}$$

angles θ_A and θ_B represent the directions, as angles between the propagation directions and the surface normal, of the forward-traveling waves in layers A and B, respectively. Furthermore, the propagation angles are also related by Snell's law as

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2} = \frac{\sin\theta_A}{c_A} = \frac{\sin\theta_B}{c_B}. \tag{26}$$

IV. THE ZEROTH- AND THE FIRST-ORDER EQUIVALENCES

It can be readily shown that, as the zeroth order approximation, that is, setting $\sin\alpha = 0$ and $\cos\alpha = 1$, where α represents any of β_A, β_B, and β, the two sets of reflection and transmission coefficients become identical by noting that

$$\frac{\tilde{\xi}_2}{\tilde{\xi}_1} = \frac{\xi_3}{\xi_1\xi_2} = \frac{\cos\theta_2}{\cos\theta_1}\frac{\rho_1 c_1}{\rho_2 c_2} = \zeta. \tag{27}$$

Essentially, ζ characterizes the acoustic properties of the surrounding media, and is independent of the sandwiched layer(s).

As the first order approximation, setting $\sin\alpha = \alpha$, $\cos\alpha = 1$ gives, for the anisotropic layer,

$$\tilde{R} = \frac{(1 - \zeta) + i\tilde{P}_-}{(1 + \zeta) - i\tilde{P}_+} \quad \tilde{T} = \frac{2e^{-ik_2 h\cos\theta_2}}{(1 + \zeta) - i\tilde{P}_+}, \tag{28}$$

and, for the two isotropic layers,

$$R = \frac{(1-\zeta) + iP_-}{(1+\zeta) - iP_+} \quad T = \frac{2e^{-ik_2 h \cos\theta_2}}{(1+\zeta) - iP_+}, \tag{29}$$

where the following parameters have been introduced:

$$\tilde{P}_\pm = \left(\frac{1}{\tilde{\xi}_1} \pm \tilde{\xi}_2\right)\beta, \tag{30}$$

$$P_\pm = \left(\frac{1}{\xi_1} \pm \frac{\xi_3}{\xi_2}\right)\beta_A + \left(\frac{1}{\xi_1 \xi_2} \pm \xi_3\right)\beta_B. \tag{31}$$

For the two systems to behave equivalently, it is necessary that $P_\pm = \tilde{P}_\pm$ for an arbitrary incident angle θ_1 and arbitrary choices of materials for media 1 and 2. Also, for the two systems to be geometrically compatible, the total thickness must be equal; that is, $h = h_A + h_B$. Denote $h_B = rh_A$, then, $h = (1+r)h_A$.

For the anisotropic layer, substituting $\tilde{\xi}$'s and β in Eqs. (15) and (16) gives

$$\tilde{P}_\pm = \omega h \left[\frac{\rho_1 c_1}{\cos\theta_1} \frac{\cos^2\theta}{\rho_x c^2} \pm \frac{\cos\theta_2}{\rho_2 c_2}\rho_x\right]. \tag{32}$$

Combining the Snell's law in Eq. (10) with the expression for c^2 in Eq. (19) gives

$$\left(\frac{\cos\theta}{c}\right)^2 = \frac{\rho_x}{K} - \frac{\rho_x}{\rho_y}\frac{\sin^2\theta_1}{c_1^2}. \tag{33}$$

Then, Eq. (32) becomes

$$\frac{\tilde{P}_\pm}{k_1 h_A} = (1+r)\left[\frac{K_1}{\cos\theta_1}\left(\frac{1}{K} - \frac{1}{\rho_y c_1^2}\right) + \frac{\rho_1}{\rho_y}\cos\theta_1 \pm \frac{\rho_x c_1}{\rho_2 c_2}\cos\theta_2\right], \tag{34}$$

where $K_1 = \rho_1 c_1^2$ has been used.

For the dual isotropic layers, substituting the expressions for ξ's and β's in Eqs. (24) and (25) into Eq. (31) gives

$$P_\pm = \omega\left[\frac{\rho_1 c_1}{\cos\theta_1}\left(\frac{\cos^2\theta_A}{\rho_A c_A^2}h_A + \frac{\cos^2\theta_B}{\rho_B c_B^2}h_B\right) \pm \frac{\cos\theta_2}{\rho_2 c_2}(\rho_A h_A + \rho_B h_B)\right]. \tag{35}$$

Expressing angles θ_A and θ_B in terms of the incident angle θ_1 via Snell's law in Eq. (26) gives

$$\frac{P_\pm}{k_1 h_A} = \frac{K_1}{\cos\theta_1}\left[\frac{1}{K_A} + \frac{r}{K_B} - \frac{1}{c_1^2}\left(\frac{1}{\rho_A} + \frac{r}{\rho_B}\right)\right] + \cos\theta_1 \rho_1\left(\frac{1}{\rho_A} + \frac{r}{\rho_B}\right) \pm \frac{c_1}{\rho_2 c_2}\cos\theta_2(\rho_A + r\rho_B), \tag{36}$$

where $\cos\theta_2$ has not been converted to θ_1 because medium 2 is also arbitrary.

The equivalence would require matching the coefficients for $\cos\theta_1$, $1/\cos\theta_1$, and $\cos\theta_2$ terms in Eqs. (34) and (36), which gives the following well-known set of three conditions:

$$(1+r)\rho_x = \rho_A + r\rho_B, \tag{37}$$

$$\frac{1+r}{\rho_y} = \frac{1}{\rho_A} + \frac{r}{\rho_B}, \tag{38}$$

$$\frac{1+r}{K} = \frac{1}{K_A} + \frac{r}{K_B}. \tag{39}$$

This also shows that the equivalence at the first order is complete; that is, there is no any other restriction besides the obvious low frequency limitation.

The first two conditions completely determine the mass densities of the two isotropic layers. They appear to have two sets of solutions, but one of the sets is the same as the other by replacing r with $1/r$, which is merely swapping the two layers. Furthermore, they yield real and positive mass densities for the isotropic layers only when $\rho_x > \rho_y$. Imposing the condition $\rho_A > \rho_B$ to give the layers the essential identifiable characteristics, one set of the solutions drops. The remaining set can be written as

$$\rho_A = \frac{1}{2}\left\{(1+r)\rho_x + (1-r)\rho_y + \sqrt{(\rho_x - \rho_y)[(1+r)^2\rho_x - (1-r)^2\rho_y]}\right\}, \tag{40}$$

$$\rho_B = \frac{1}{2r}\left\{(1+r)\rho_x - (1-r)\rho_y - \sqrt{(\rho_x - \rho_y)[(1+r)^2\rho_x - (1-r)^2\rho_y]}\right\}. \tag{41}$$

The mass densities of the two isotropic layers as determined by this equivalence are shown in Fig. 3 for three different thickness ratios: $r = 0.5$, 1, and 2. It is always true that

$$\rho_A > \rho_x \quad \text{and} \quad \rho_B < \rho_y. \tag{42}$$

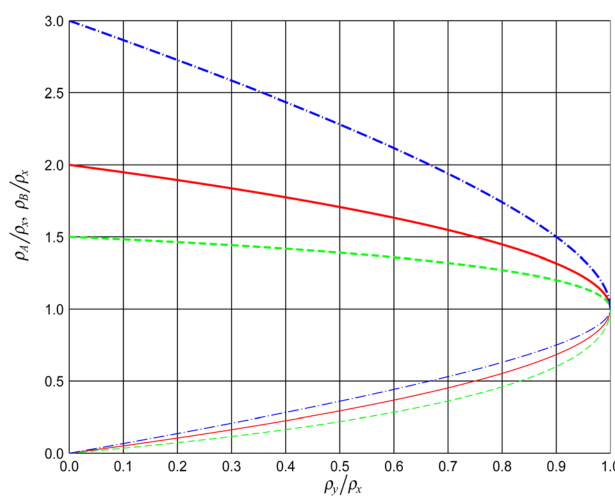

FIG. 3. (Color online) Mass densities of the equivalent two isotropic layers determined by those of the anisotropic layer, at three different thickness ratios. Solid curves: $r = 1$. Dashed curves: $r = 0.5$, Dot-dashed curves: $r = 2$. Thicker curves: ρ_A; thinner curves: ρ_B.

In general, using a thicker lighter layer requires a heavier heavier layer, and vise versa. The extremes of the mass densities are reached when $\rho_y \to 0$, leading to

$$\rho_A^{\max} = (1+r)\rho_x \quad \rho_B^{\min} = 0. \tag{43}$$

For the special case $r=1$,

$$\rho_{A,B} = \rho_x \pm \sqrt{\rho_x}\sqrt{\rho_x - \rho_y}. \tag{44}$$

However, this set of conditions is not sufficient to completely determine the bulk modulus for the two isotropic layers. This means there is an infinite number of choices for the bulk modulus for the two layers. Alternatively, it is possible to trade the freedom for higher order equivalence such that the equivalence would work over a broader frequency range.

V. SECOND ORDER APPROXIMATION

As the second order approximation, setting $\sin\alpha = \alpha$ and $\cos\alpha = 1 - 1/2\alpha^2$ gives, for the anisotropic layer,

$$\tilde{R} = \frac{(1-\zeta) + \hat{\imath}\tilde{P}_- - \tilde{Q}_-}{(1+\zeta) + \hat{\imath}\tilde{P}_+ - \tilde{Q}_+}, \tag{45}$$

$$\tilde{T} = \frac{2e^{-ik_2h\cos\theta_2}}{(1+\zeta) + \hat{\imath}\tilde{P}_+ - \tilde{Q}_+}, \tag{46}$$

and for the dual isotropic layers,

$$R = \frac{(1-\zeta) + \hat{\imath}P_- - Q_-}{(1+\zeta) + \hat{\imath}P_+ - Q_+}, \tag{47}$$

$$T = \frac{2e^{-ik_2h\cos\theta_2}}{(1+\zeta) + \hat{\imath}P_+ - Q_+}, \tag{48}$$

where the following new parameters have been introduced,

$$\tilde{Q}_\pm = \frac{1}{2}(1\pm\zeta)\beta^2, \tag{49}$$

$$Q_\pm = \frac{1}{2}(1\pm\zeta)(\beta_A^2 + \beta_B^2) - \left(\frac{1}{\xi_2} \pm \frac{\xi_3}{\xi_1}\right)\beta_A\beta_B. \tag{50}$$

Through a similar process, $\tilde{Q}_\pm$ and $Q_\pm$ can be expressed in terms of the incident angle θ_1 as

$$\frac{\tilde{Q}\pm}{(k_1h_A)^2} = \frac{1}{2}(1\pm\zeta)(1+r)^2\left[\frac{\rho_x c_1^2}{K} - \frac{\rho_x}{\rho_y}\sin^2\theta_1\right] \tag{51}$$

and

$$\begin{aligned}\frac{Q\pm}{(k_1h_A)^2} =& c_1^2\left\{\left[\frac{1}{2}\left(\frac{1}{c_A^2} + \frac{r^2}{c_B^2}\right) + r\frac{\rho_A}{K_B}\right]\right.\\ &\left.\pm\zeta\left[\frac{1}{2}\left(\frac{1}{c_A^2} + \frac{r^2}{c_B^2}\right) + r\frac{\rho_B}{K_A}\right]\right\}\\ &-\sin^2\theta_1\left\{\left[\frac{1}{2}(1+r^2) + r\frac{\rho_A}{\rho_B}\right]\right.\\ &\left.\pm\zeta\left[\frac{1}{2}(1+r^2) + r\frac{\rho_B}{\rho_A}\right]\right\}.\end{aligned} \tag{52}$$

Note that all the terms added at the second order appear in the real parts of either the denominator or the numerator; while the first order equivalence conditions are derived from the imaginary parts. This means that enforcing the equivalence at the second order does not interfere with the first order equivalence. Thus, the additional requirement is $Q_\pm = \tilde{Q}_\pm$. Since both ζ and θ_1 are arbitrary, the equivalence would require the following 4 conditions

$$\frac{1}{c_A^2} + \frac{r^2}{c_B^2} + 2r\frac{\rho_A}{K_B} = (1+r)^2\frac{\rho_x}{K}, \tag{53}$$

$$\frac{1}{c_A^2} + \frac{r^2}{c_B^2} + 2r\frac{\rho_B}{K_A} = (1+r)^2\frac{\rho_x}{K}, \tag{54}$$

$$1 + r^2 + 2r\frac{\rho_A}{\rho_B} = (1+r)^2\frac{\rho_x}{\rho_y}, \tag{55}$$

$$1 + r^2 + 2r\frac{\rho_B}{\rho_A} = (1+r)^2\frac{\rho_x}{\rho_y}. \tag{56}$$

The first two conditions require

$$\rho_A K_A = \rho_B K_B \quad \text{or} \quad \rho_A c_A = \rho_B c_B. \tag{57}$$

That is, the two layers must be impedance-matched. This gives one more condition on the bulk modulus of the two layers. Combined with the first order equivalence conditions, the bulk modulus of both layers can now be completely determined as

$$K_A = \frac{\rho_x}{\rho_A}K \quad K_B = \frac{\rho_x}{\rho_B}K. \tag{58}$$

Then, Eqs. (53) and (54) become identical, which can be further reduced to Eq. (37). In other words, they do not add more conditions.

Unfortunately, Eqs. (55) and (56) require $\rho_A = \rho_B$, and in turn, $\rho_x = \rho_y$. This means that a full second-order equivalence can be achieved only in one of the following two scenarios: (1) when the anisotropic layer degenerates to an isotropic layer, by which the two equivalent isotropic layers would also be identical to the "anisotropic" layer, or (2) when the incident is normal to the surface: $\theta_1 = 0$, as the unmatched term is modulated by $\sin^2\theta_1$. For any real anisotropic layer, it cannot be fully imitated with two isotropic layers up to the second order accuracy.

VI. PROPER LAYER PLACEMENT FOR ERROR SUPPRESSION

With the unavailability of a full second-order equivalence, an alternative is to suppress the effects of the unmatched terms. One of the mechanisms is limiting the incident angle θ_1 to small angles, while requiring the anisotropy not too strong, that is $\rho_x/\rho_y \approx 1$. These two conditions can be combined to suppress unmatched term to the second order smallness. Such limitations may make this mechanism impractical in many situations.

Another mechanism is through a proper layer placement. Since $P_\pm$ and $Q_\pm$ are the first order and second order, respectively, small terms, the second-order approximations for the reflection and transmission coefficients for the two isotropic layers can be alternatively written as

$$R \approx \frac{1-\zeta}{1+\zeta} + \frac{\hat{\imath}P-}{1+\zeta} - \frac{\hat{\imath}(1-\zeta)P+}{(1+\zeta)^2} + \frac{P-P+}{(1+\zeta)^2} - \frac{Q-}{1+\zeta} + \frac{(1-\zeta)Q+}{(1+\zeta)^2}, \tag{59}$$

$$T \approx 2e^{-\mathrm{i}k_2 h \cos\theta_2}\left[\frac{1}{1+\zeta} - \frac{\hat{\imath}P+}{(1+\zeta)^2} + \frac{Q+}{(1+\zeta)^2}\right]. \tag{60}$$

As ζ is always positive, the factor $(1+\zeta)$ in the denominator serves to suppress the effects due to $Q_\pm$ (as well as $\tilde{Q}_\pm$). The larger the ζ value, the smaller the influences $Q_\pm$ would have. Recall that ζ essentially represents the impedance ratio of Medium 1 to Medium 2. If the two surrounding media are different, one way of layer placement, which shall be referred to as the *proper placement,* always achieves a larger ζ than the other way. The proper placement is to place the heavier layer adjacent to the medium of higher impedance.

As an example to demonstrate the significance of the layer placement, consider an acoustic cloak following the design of Cummer and Schurig (2007). In this design, the material properties vary in the following manner: the radial sound speed remains a constant, but the radial mass density increases and approaches to infinity as the radius decreases, from the outer rim of the cloak to the interface between the cloak and the cloaked object. That is, the radial impedance increases in the direction of approaching to the cloaked object.

In this example, a rigid cylinder of radius a is being cloaked, in a host medium of water, of mass density of $\rho_0 = 1000\,\mathrm{kg/m^3}$, and a sound speed of $c_0 = 1350\,\mathrm{m/s}$. The cloak has an outer radius of $b = 1.5a$, and is comprised of 5 anisotropic layers, which are then converted into 10 isotropic layers of equal thickness. The conversion uses the first-order equivalence conditions in Eqs. (37)–(39), as well as the impedance match of the second-order in Eq. (57). The material properties of both anisotropic and the equivalent isotropic layers are listed in Table I. In Table I, all properties are normalized by those of the host medium, and $K_0 = \rho_0 c_0^2$ is the bulk modulus of water. The layers are numbered inwards, with Layer 10 being the closest to the rigid cylinder.

The *normalized total scattering cross section,* denoted as σ, is a scalar quantity that describes the overall scattering strength of a scatterer. It is the total energy scattered by the scatterer over a closed surface enclosing the scatterer, normalized by the power of the incident wave and the geometric cross section (in the two-dimensional case, the diameter) of the scatterer. The computation procedure has been discussed elsewhere (Cai and Sanchez-Dehesa, 2007, 2008). The computation gives analytically exact values of the T-matrix for the scatterer comprising of multiple uniform layers of either mass-anisotropic or isotropic media. The T-matrix represent the linear transformation from wave expansion coefficients of the incident wave to those of the scattered wave. For an axisymmetric scatterer, the T-matrix is a diagonal matrix, and the normalized total scattering cross-section can then be calculated from

$$\sigma = \frac{2}{\pi k_0 a}\sum_{n=-\infty}^{\infty} |[T]_n|^2, \tag{61}$$

where $[T]_n$ is the T-matrix entry at the nth row and the nth column (n runs from $-\infty$ to ∞), and k_0 is the wave number in the host.

The normalized scattering cross section of the cloaked rigid cylinder by the aforementioned isotropic cloaks with different layer placements is compared with the anisotropic cloak in Fig. 4 over the frequency range from $k_0 a = 0$ to 5. One of the layer placements, shown as the solid curve, is as suggested in this paper, which, for the case of cloaking, is to place the heavier layer of the pair closer to the object. This is the layer placement listed in Table I. The other layer placement, shown as the dashed curve, is the opposite: placing the softer layer of the pair closer to the cloaked object.

First and the foremost, Fig. 4 shows that the equivalence works at low frequencies as expected. The computed scattering cross section for all the three cloaks coincide, up to a frequency $k_0 a \approx 0.5$. Afterwards, the curves for the isotropic cloaks start to deviate from the one for the anisotropic cloak,

TABLE I. Material Properties for Cloaks Comprising of 5 Anisotropic Layers and 5 Pairs of Impedance-Matched Isotropic Layers

	Anisotropic Layers			Isotropic Layers	
Layer	ρ_r/ρ_0	ρ_θ/ρ_0	K/K_0	$\rho_{A,B}/\rho_0$	$K_{A,B}/K_0$
1	3.2222	0.31345	0.35802	0.15910	7.2510
2				6.28534	0.18354
3	3.8571	0.25926	0.42857	0.13188	12.5342
4				7.5824	0.21801
5	5.0000	0.20000	0.55556	0.10102	27.497
6				9.8990	0.28061
7	7.6667	0.13044	0.85185	0.065497	99.712
8				15.268	0.42775
9	21.000	0.047619	2.3333	0.023823	2056.8
10				41.976	1.1673

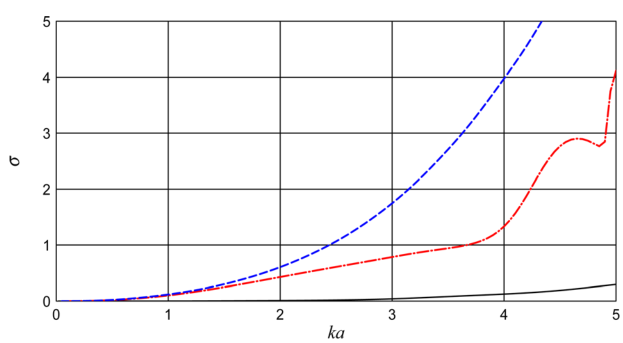

FIG. 4. (Color online) Normalized total scattering cross section σ of a rigid cylinder cloaked by cloaks comprising of 5 isotropic layer pairs. Each pair is impedance-matched. Solid curve: anisotropic cloak. Dot-dashed curve: proper placement—placing heavier layer in each pair closer to the object. Dashed curve: improper placement—placing softer layer in each pair closer to the object.

indicating that the second order effects have become noticeable. The curves for the two isotropic cloaks start to deviate after $k_0 a \approx 1.5$. Second, Fig. 4 shows clearly that the effect of layer placement is rather significant, especially after $k_0 a > 1.5$. For example, at $k_0 a = 2$, the scattering cross section is reduced by one third merely by a proper placement of the layers without any change of other parameters; at $k_0 a = 3$, the reduction is by a half, and at $k_0 a = 4$, the reduction is by two thirds. In other words, with a proper layer placement, it is possible to maintained a relatively small scattering cross section over an extended frequency range.

The significance of the layer placement also suggests that meeting the partial requirement for the second order equivalence in Eq. (57) may not be as important as the layer placement. An observation to support this hypothesis is that, according to Eqs. (59) and (60), a large ζ suppresses $Q_\pm$ entirely, not just those unmatched terms. If this is the case, it might be better to keep the choices for bulk modulus open for more, if any at all for cloak design, material choices. While pondering this prospect, it must be pointed out that, in fact, the requirement in Eq. (57) may bring a problem of its own. As indicated in Table I, the mass densities in each pair of the isotropic layers can differ by one or even three orders of magnitudes. Having both layers impedance-matched means that the wave number in one of the layers is much greater than the other. This may cause the layer with the larger wave number (the heavier layer) prematurely exceeding the low frequency limit (the small β_A, β_B, and β assumption), which would be the opposite of what seeking the second-order equivalence attempts to achieve.

In light of this, two other convenient yet simple alternatives are evaluated and observed. One is to require both layers to have the same bulk modulus, which, according to Eq. (39), would be equal to that of the anisotropic layer. This is the assumption used by Cheng *et al.* (2008). Another is to require the bulk modulus of the two layers proportional to their respective mass densities, which gives both layers the same sound speed and the same wave number. This is the assumption used by Torrent and Sanchez-Dehesa (2008). The resulting normalized total scattering cross section for the two alternatives are shown in Figs. 5 and 6, respectively. In each of these figures, two layer placements are compared with the anisotropic cloak. One of the layer placement is

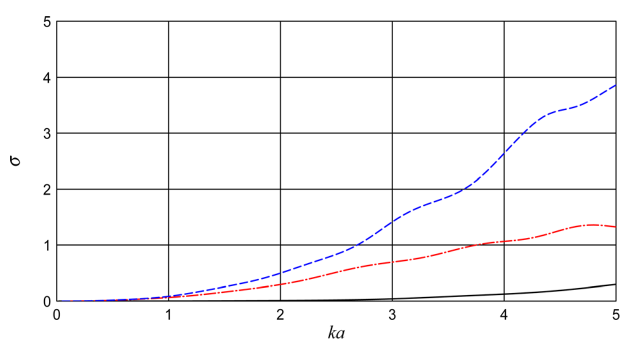

FIG. 5. (Color online) Normalized total scattering cross section σ of a rigid cylinder cloaked by cloaks comprising of 5 isotropic layer pairs. Each pair has the same bulk modulus. Solid curve: anisotropic cloak. Dot-dashed curve: proper placement—placing heavier layer in each pair closer to the object. Dashed curve: improper placement—placing softer layer in each pair closer to the object.

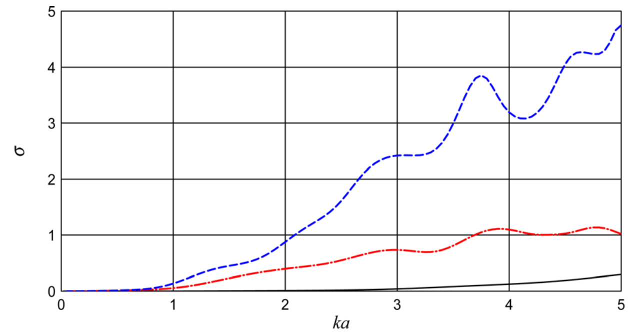

FIG. 6. (Color online) Normalized total scattering cross section σ of a rigid cylinder cloaked by cloaks comprising of 5 isotropic layer pairs. Each pair has the same sound speed and wave number. Solid curve: anisotropic cloak. Dot-dashed curve: proper placement — placing heavier layer in each pair closer to the object. Dashed curve: improper placement— placing softer layer in each pair closer to the object.

called the proper layer placement as suggested in this paper, whose result is shown in dot-dashed curve. The other, the improper placement, is shown in dashed curve. In both cases, the proper layer placement consistently produces better cloaking effects. The case with matched wave speeds gives a slightly better performance in higher frequencies beyond $k_0 a = 3$. It is also important to note that, both alternatives perform better than the one that follows the partial requirement for the second order equivalence.

However, a cautionary note might be in order here. The above observations are limited to cloaking applications, which presents some extreme challenges, such as extremely high material property gradients near the cloaked object, where extreme anisotropy is required. For other applications when the material requirements are not so extreme, partial fulfillment of the second order equivalence may still be a worthy effort.

VII. CONCLUSIONS

In this paper, the equivalence between a single mass-anisotropic layer and two isotropic layers is studied. The first order equivalence is derived from the wave transmission and reflections by the layer(s). Second order equivalence is explored. One more equivalence requirement is obtained from the second order approximations: that the two isotropic layers must have their impedance matched. Together with the first order equivalence requirements, this gives a complete set of conditions for determining all the materials properties of the two layers. However, it is concluded that full second order equivalence is not possible unless the incident is normal to the surface, or the materials are isotropic. On the other hand, the unattainable second order equivalence can be alleviated by a proper placement of layers: by placing the heavier layer adjacent to the medium of greater acoustic impedance. Numerical examples show that this remedy in fact is more important than following the partial requirement for the second order equivalence when the equivalent isotropic layers are used in acoustic cloaking applications.

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