Effective parameters of clusters of cylinders embedded in a nonviscous fluid or gas

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This work presents a comprehensive analysis of the homogenization method recently used by Torrent *et al.* [Phys. Rev. Lett. **96**, 204302 (2006)] to get the effective parameters of two-dimensional clusters of rigid cylinders in air. Here, the method is developed by studying the scattering of sound by clusters of fluid cylinders embedded in a nonviscous fluid or gas. This general problem is studied in the long wavelength limit (homogenization) by multiple scattering theory. Asymptotic relations are derived and employed to formulate a method of homogenization based on the scattering properties of the cluster. Exact formulas for the effective parameters (i.e., effective sound speed and effective density) are obtained as a function of the location of each cylinder and the physical parameters (density and speed of sound) of cylinders and the embedded medium. Results for several fluid-fluid composite systems are reported. The case of rigid (infinite density) cylinders in air is deeply analyzed, showing results for ordered and disordered lattices. It is concluded that the method provides a tool of designing acoustic metamaterials with prefixed refractive properties.

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I. INTRODUCTION

Phononic crystals, which are composites made of inhomogeneous distributions of some elastic materials periodically embedded in a matrix with different elastic properties, are widely studied from the late 1990s boosted by some experimental works.[1–3] Their potential applications like, for example, filters or noise barriers, are based on the existence of elastic band gaps, a range of frequencies where wave propagation is forbidden.[1,2] However, a recent work by Cervera *et al.*[4] showed that its properties in the low frequency region (well below the band gap) can be used to fabricate acoustic lenses, a result that opened a new field of applications for PCs, like refractive devices.[5–8] Then, newly acoustic interferometers were characterized[9] and further theoretical works[10,11] supported the experimental findings. Moreover, homogenization theories applied to the determination of the effective parameters of PCs have been recently published,[12–16] though homogenization of heterogeneous medium has been previously studied by using statistical approaches.[17]

Several models have been applied to determine the effective parameters of the resulting homogenized medium (i.e., its effective sound speed c_{eff} and effective density ρ_{eff}) at large wavelengths. The more simple consists of assuming that parameters of a mixture of materials should be their volume average, but it does not work neither to fit c_{eff} (Ref. 4) nor to approach ρ_{eff}.[16] A heuristic model was introduced in Ref. 4 to obtain c_{eff} that works at low filling fractions of sonic scatterers. Later, more exact theories have been developed to obtain reliable results in the full range of filling fractions. For example, a plane wave expansion was used by Krokhin *et al.*[12] to obtain c_{eff} for an infinite periodic system. More recently, Mey and co-workers[15] used the previously developed average T-matrix approach (ATA)[17] to demonstrate that Berryman's expression for ρ_{eff} can be successfully applied to get the experimentally measured c_{eff}. Finally, the last work on this topic has successfully solved the problem of homogenization of finite clusters made of rigid cylinders in air by using exact multiple scattering theory, obtaining simultaneously ρ_{eff} and c_{eff}.[16] Additionally, that work established a lower limit of the cluster size from which the homogenization approach is valid. Finally, it is interesting to note that, by using the coherent-potential-approximation (CPA) method, Hu and Chan[14] studying the refraction of water waves by cylinder arrays derived expressions for ρ_{eff} and c_{eff} analogous to the ones found by Torrent *et al.*[16] In an analogous problem, in the field of electromagnetic wave propagation, Bush and Sokoulis developed the so-called "energy-density CPA,"[18] and they particularly proved that, for example, the frequency-independent long-wavelength dielectric constant reduces to the expression given by the well-known Maxwell-Garnett theory. It is expected that a similar "energy-density CPA" applied to acoustic systems will converge in the long-wavelength limit to the expressions reported here.

In this work a comprehensive report of the method employed in Ref. 16 to obtain the effective parameters of two-dimensional (2D) clusters of rigid cylinders in air is presented. Here, the method is developed for the general case of fluid cylinders, with some density ρ_{cyl} and sound speed c_{cyl}, embedded in a background made of a nonviscous fluid or gas with parameters ρ_b and c_b. The method uses the multiple scattering theory (MST)[9,19–21] to characterize the homogenization of arbitrary 2D clusters consisting of periodic or nonperiodic arrays of cylinders. The property of MST, which considers each scatterer individually, has allowed us to obtain analytical expressions in the general case of cylinders with arbitrary sections forming structures with arbitrary external shape. Numerical results have been obtained for several fluid-fluid systems (air columns in water, water columns in air, mercury columns in water, and water columns in mercury), and for a case of practical interest, the one consisting of periodic arrays of circular rigid rods ($\rho_{cyl}=\infty$) in air ($\rho_b=\rho_{air}$). Moreover, this work also reports results showing how the effective parameters of the cluster change when positional or structural disorder exists in the lattice. Though disordering in the lattice has been previously studied for both phononic[9,10,22] and photonic[23–26] crystals, the main emphasis of those works was to analyze the effects of disordering at wavelengths comparable to the lattice parameter.

The paper is organized as follows. In Sec. II MST will be briefly summarized and the concept of effective T matrix is introduced. In Sec. III the homogenization conditions are explained, showing that the method is an improvement of the ATA method.[17] The effective parameters will be obtained in Sec. IV for a general fluid-fluid composite. Numerical results are reported and discussed for several fluid-fluid composite systems and for the case of infinite-density cylinders in air, which is a good approximation to actual structures using solid cylinders made of huge-density materials. It will be shown that the method allows one to calculate the effective parameters of a finite-size sonic crystal and to determine even its effective shape. In Sec. V the method is applied to the case of disordered lattices of cylinders, which in practice is the most common situation. Finally, in Sec. VI a summary of the work is presented.

II. MULTIPLE SCATTERING THEORY

A. Scattering coefficients

Consider a cluster of N parallel cylinders with an arbitrary section located at positions $\vec{R}_\alpha$ $(\alpha=1,2,\ldots,N)$, and embedded in a background (b). Let us also consider that an external sound field (P^{ext}) with temporal dependence $e^{-i\omega t}$ impinges the cluster. At any arbitrary point (r,θ) of the 2D space, the external field can be expanded as a linear combination of Bessel functions J_q:

$$P^{ext}(r,\theta)=\sum_q A_q^{ext} J_q(kr)e^{iq\theta}, \qquad (1)$$

where $k=\omega/c_b$.

The total scattered field will be given by the sum of the scattered fields by each α-cylinder, P_α^{scat}:

$$P^{scat}(r,\theta)=\sum_{\alpha=1}^{N} P_\alpha^{scat}(r,\theta)=\sum_{\alpha=1}^{N}\sum_{q=-\infty}^{\infty}(A_\alpha)_q H_q(kr_\alpha)e^{iq\theta_\alpha}, \qquad (2)$$

where H_q is the qth order Hankel function of first kind, and (r_α,θ_α) are the polar coordinates with the origin translated to the center of the α-cylinder, i.e., $\vec{r}_\alpha=\vec{r}-\vec{R}_\alpha$, as shown in Fig. 1. $(A_\alpha)_q$ are the coefficients to be determined.

The total field that impinges the α cylinder can be expressed by

$$P_\alpha(r_\alpha,\theta_\alpha)=\sum_{s=-\infty}^{\infty}(B_\alpha)_s J_s(kr_\alpha)e^{iq\theta_\alpha}, \qquad (3)$$

these coefficients are related with the $(A_\alpha)_q$ by means of the T matrix formalism[27]

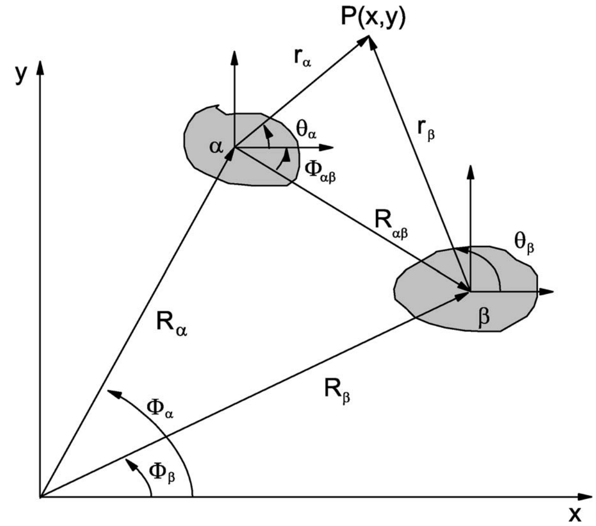

FIG. 1. System of coordinates and definition of variables employed in the expressions of multiple scattering theory.

$$(A_\alpha)_q=\sum_s (T_\alpha)_{qs}(B_\alpha)_s \qquad (4)$$

$(T_\alpha)_{qs}$ being the elements of $\mathcal{T}_\alpha$, the T matrix of the α-cylinder.

The field impinging over the α-cylinder [see Eq. (3)] can be expressed as a sum of the external field [see Eq. (1)] and the fields scattered by all the cylinders, P_β^{scat}, except α. After an easy manipulation it is possible to find the relationship between B_α and A_β coefficients for $\alpha\neq\beta$:

$$(B_\alpha)_q=\sum_s A_s^{ext}J_{s-q}(kR_\alpha)e^{i(s-q)\Phi_\alpha} + \sum_{\beta\neq\alpha}(A_\beta)_s H_{q-s}(kr_{\alpha\beta})e^{i(s-q)\theta_{\alpha\beta}}. \qquad (5)$$

Now, by multiplying this equation by $(T_\alpha)_{rq}$ and summing for all q we get

$$(A_\alpha)_r-\sum_s\sum_\beta (G_{\alpha\beta})_{rs}(A_\beta)_s=(S_\alpha)_r, \qquad (6)$$

where

$$(G_{\alpha\beta})_{rs}=\sum_q(1-\delta_{\alpha\beta})(T_\alpha)_{rq}H_{q-s}(kr_{\alpha\beta})e^{i(s-q)\theta_{\alpha\beta}} \qquad (7)$$

and

$$(S_\alpha)_r=\sum_s\sum_q (T_\alpha)_{rq}J_{s-q}(kR_\alpha)e^{i(s-q)\Phi_\alpha}A_s^{ext}. \qquad (8)$$

By truncating the angular momenta $|s|\leq q_{max}$ and $|r|\leq q_{max}$, Eq. (6) is in matrix form $\mathcal{MA}=\mathcal{S}$, where the $\mathcal{M}$ matrix is a square matrix of dimension $N(2q_{max}+1)\times N(2q_{max}+1)$, and $\mathcal{S}=\mathcal{T}\tilde{\mathcal{J}}\mathcal{A}^{ext}$ is a column matrix of dimension $N(2q_{max}+1)$. The elements of $\mathcal{M}$ are

$$(M_{\alpha\beta})_{rs}=\delta_{rs}\delta_{\alpha\beta}-(G_{\alpha\beta})_{sr}. \qquad (9)$$

The inversion of this matrix gives the solution we are looking for; i.e., $\mathcal{A}=\mathcal{M}^{-1}\mathcal{S}$.

$$(A_\alpha)_r = \sum_{\beta=1}^{N} \sum_s (M^{-1}_{\alpha\beta})_{rs}(S_\beta)_s. \quad (10)$$

Thus the solution for a given cluster is a function of positions and properties of each cylinder, and of the external field. Let us stress that for the cases reported here the maximum value q_{max} employed is 5.

B. Effective T matrix

By using Graf's theorem[28] the Hankel functions of Eq. (2) can be translated to the origin of coordinates, then the total scattered field at a given point outside the cluster can be cast in the following expression:

$$P^{scat}(r,\theta) = \sum_p A_p^{SC} H_p(kr) e^{ip\theta}; \quad r > R_\alpha^+, \quad (11)$$

where R_α^+ stands for the greater of R_α, and

$$A_p^{SC} \equiv \sum_\alpha \sum_q (A_\alpha)_q J_{p-q}(kR_\alpha) e^{i(q-p)\Phi_\alpha}. \quad (12)$$

Here, A_p^{SC} defines the coefficients of a single cylinder (SC) that has the same scattering properties of the cluster. Then, the total scattered field can be expressed as the scattered field by only one cylinder (of arbitrary shape) located at the origin. The relation between A_s^{SC} and A^{ext} will define the T matrix of this hypothetical cylinder. To obtain this effective T matrix, Eq. (10) is inserted in equation above, then

$$A_p^{SC} = \sum_{\alpha,\beta} \sum_{q,r} (M^{-1}_{\alpha\beta})_{qr}(S_\beta)_r J_{p-q}(kR_\alpha) e^{i(q-p)\Phi_\alpha}. \quad (13)$$

Let us introduce matrix $\mathcal{J}$ whose matrix elements are defined by

$$(J_\alpha)_{pq} = J_{p-q}(kR_\alpha) e^{i(q-p)\Phi_\alpha}. \quad (14)$$

After an easy manipulation it is possible to write $\mathcal{A}^{SC} \equiv \mathcal{T}^{eff}\mathcal{A}^{ext}$, where $\mathcal{T}^{\mathrm{eff}}(\omega) = \mathcal{J}\mathcal{M}^{-1}\mathcal{T}\tilde{\mathcal{J}}$. Matrix $\tilde{\mathcal{J}}$ is the complex conjugate of the transpose of $\mathcal{J}$. The dependence in ω has been introduced in order to stress the frequency dependence of all these matrices. The elements of the defined effective T matrix are

$$T_{ps}^{\mathrm{eff}} = \sum_{\alpha,\beta} \sum_{r,q,t} (J_\alpha)_{pq}(M^{-1}_{\alpha\beta})_{qr}(T_\beta)_{rt}(J_\beta)^*_{st}. \quad (15)$$

III. HOMOGENIZATION

Here, the cluster of cylinders will be treated as a big single cylinder. In other words, it is expected that for long wavelengths, the cluster will behave like a homogeneous cylinder if cylinders forming the cluster are regularly distributed. The goal is to determine the parameters of this effective homogeneous cylinder; that is, its effective external shape, speed of sound, and density.

To determine these parameters, the pressure scattered by the cluster is compared with the pressure scattered by a single effective cylinder, and it is imposed that, for $k \to 0 (\lambda \to \infty)$, it is not possible to distinguish between the cluster and the homogeneous cylinder. Mathematically it can be expressed as

$$\lim_{k\to 0} \frac{P^{scat}(r,\theta;k) - P^{scat}_{cyl}(r,\theta;k)}{P^{scat}(r,\theta;k)} = 0. \quad (16)$$

Here, P^{scat} stands for the scattered pressure by the cluster and P^{scat}_{cyl} stands for the scattered pressure by the effective cylinder.

Note that Eq. (16) defines the effective cylinder by means of the relative difference of the scattered pressure, and this difference must hold independently of the incident field and must be true for every (r,θ). For $r > R_\alpha^+$ this equation can be expressed as

$$\lim_{k\to 0} \frac{\sum_q [A_q^{SC} - A_q^{cyl}] H_q(kr) e^{iq\theta}}{\sum_q A_q^{sc} H_q(kr) e^{iq\theta}} = 0, \quad (17)$$

where it is assumed that the scattered pressure by the cylinder is defined with the coefficients A^{cyl}. After using the T matrix formalism

$$\lim_{k\to 0} \frac{\sum_q \sum_s [T_{qs}^{eff} - T_{qs}^{cyl}] A_s^{ext} H_q(kr) e^{iq\theta}}{\sum_q \sum_s T_{qs}^{eff} A_s^{ext} H_q(kr) e^{iq\theta}} = 0. \quad (18)$$

The theory must be independent of the coefficients A_s^{ext}, so it is possible to assume that all them are zero except one, separating then the above equation into $2s_{max}+1$ equations. For these equations to be true independently of the polar coordinates, the following relations must be satisfied:

$$\lim_{k\to 0} \frac{T_{qs}^{eff} - T_{qs}^{cyl}}{T_{qs}^{eff}} = 0. \quad (19)$$

It can be shown (see Appendix) that elements T_{sq} for both the cluster and the single cylinder go to zero as $\approx \hat{T}_{sq} k^n$, n being a positive integer bigger than zero that depends on s and q, and $\hat{T}_{sq}$ are the k-independent coefficients of the lower order term in the k-expansion of the corresponding T-matrix elements.

Therefore the homogenization conditions become

$$\hat{T}_{sq}^{\mathrm{eff}} = \hat{T}_{sq}^{\mathrm{cyl}} \; \forall \, s,q. \quad (20)$$

These equations form a set of $s \times q$ equations for the unknown parameters of the uniform effective cylinder. However, in the present work the shape of the effective cylinder will be proposed and, then, the effective parameters to determine will be only ρ_{eff} and c_{eff}.

IV. EFFECTIVE PARAMETERS

In this section the diagonal terms of $\hat{T}_{sq}$ (see Appendix) are used to determine the effective parameters of the homogenized cluster. Calculations are reported for the case of hexagonal arrays of infinite density cylinders, a problem that fairly approaches, for example, the real case of solid cylinders embedded in air.

A. The isotropic term and the bulk modulus

To get the exact expression of $\hat{T}_{00}^{\mathrm{eff}}$ (isotropic term), we notice that in Eq. (A20) when $p=0$ also $q=0$, and besides, if $s=0$ implies that r runs to 1, −1, and 0, then

$$\hat{T}_{00}^{\mathrm{eff}} = \sum_{r=\pm 1,0} \sum_{\alpha,\beta} (\hat{J}_\alpha)_{00} (\hat{M}_{\alpha\beta}^{-1})_{0r} \hat{T}_{rr} (\hat{J}_\beta)_{0r}^*. \tag{21}$$

From Eq. (A16), it is clear that

$$\sum_s^o \boldsymbol{M}_{0s} \hat{\boldsymbol{M}}_{sr}^{-1} = \hat{\boldsymbol{M}}_{0r}^{-1} = \delta_{0r} \boldsymbol{I}. \tag{22}$$

Also, taking into account that $(\hat{J}_\alpha)_{00}=1$, the isotropic term becomes

$$\hat{T}_{00}^{\mathrm{eff}} = \sum_{\alpha,\beta} \delta_{\alpha\beta} \hat{T}_{00} = N \hat{T}_{00}, \tag{23}$$

where N is the number of cylinder and T_{00} the isotropic term of the cylinder from which the cluster is made of. This result is independent of the shape of the cluster.

In Ref. 27 it is shown that for a cylinder of arbitrary shape, the isotropic element $\hat{T}_{00}^{cyl}$ is given by

$$\hat{T}_{00}^{cyl} = \frac{iA_{cyl}}{4}\left[\frac{1}{\bar{\rho}_{cyl}\bar{c}_{cyl}^2} - 1\right]. \tag{24}$$

Hereafter, an overline over a variable will be used to denote the corresponding quantity normalized to that of the background material b; for example, $\bar{\rho} \equiv \rho/\rho_b$ and $\bar{c} \equiv c/c_b$.

If $\hat{T}_{00}$ is the term of a circular cylinder [see Eq. (A3a)] of radius R_a, speed of sound $c_{cyl}=c_a$, and density $\rho_{cyl}=\rho_a$, then it is expected that $\hat{T}_{00}^{cyl}$ be the isotropic term of an arbitrarily shaped cylinder with area A_{eff}, speed of sound c_{eff}, and density ρ_{eff}. In other words, it is possible to write Eq. (A16) as

$$\frac{iA_{eff}}{4}\left[\frac{1}{\bar{\rho}_{eff}\bar{c}_{eff}^2} - 1\right] = N\frac{i\pi R_a^2}{4}\left[\frac{1}{\bar{\rho}_a\bar{c}_a^2} - 1\right]. \tag{25}$$

By introducing the filling fraction f as the fraction of volume occupied by the N cylinders inside the area A_{eff} defined by the homogenized cluster,

$$f \equiv \frac{N\pi R_a^2}{A_{eff}}, \tag{26}$$

and by using the bulk modulus $B=\rho c^2$, Eq. (25) can be cast in the form

$$\frac{1}{B_{eff}} = \frac{f}{B_a} + \frac{1-f}{B_b}. \tag{27}$$

This equation recovers the well-known Wood's law[29] for the bulk modulus.

B. ρ_{eff} and c_{eff}

Up to now the method has been employed to clusters with arbitrary external shape. Hereafter, the analysis is focused to circular shaped clusters, so that the elements $\hat{T}_{sq}^{cyl}$ in Eqs. (A3a) and (A3b) will give the corresponding elements $\hat{T}_{sq}^{eff}$ by making the appropriated replacements; i.e., $R_{cyl}=R_{eff}$, $c_{cyl}=c_{eff}$, and $\rho_{cyl}=\rho_{eff}$.

The isotropic term has been previously used to determine B_{eff}, the next diagonal term in the power expansion is used below to determine ρ_{eff}.

From Eqs. (20) and (A3b):

$$\hat{T}_{11}^{eff} = \frac{i\pi R_{eff}^2}{4}\frac{\bar{\rho}_{eff}-1}{\bar{\rho}_{eff}+1}. \tag{28}$$

This term can be also obtained from Eq. (A20), which for $q=r=1$,

$$\hat{T}_{11}^{\mathrm{eff}} = \hat{T}_{11}\sum_{\alpha,\beta}(\hat{M}_{\alpha\beta}^{-1})_{11}. \tag{29}$$

The definition of f [see Eq. (26)] allows one to merge the last two equations in

$$\frac{\bar{\rho}_a - 1}{\bar{\rho}_a + 1}\frac{f}{\Delta} = \frac{\bar{\rho}_{eff}-1}{\bar{\rho}_{eff}+1}, \tag{30}$$

where the factor Δ is defined by

$$\frac{1}{\Delta} \equiv \frac{1}{N}\sum_{\alpha,\beta}(\hat{M}_{\alpha\beta}^{-1})_{11}. \tag{31}$$

Solving for ρ_{eff}, the result in absolute units is

$$\rho_{eff} = \frac{\rho_a(\Delta+f)+\rho_b(\Delta-f)}{\rho_a(\Delta-f)+\rho_b(\Delta+f)}\rho_b. \tag{32}$$

Now, c_{eff} can be easily obtained from this equation and from Eq. (27) throughout the definition of bulk modulus,

$$\frac{1}{c_{eff}^2} = \left(\frac{f}{\rho_a c_a^2} + \frac{1-f}{\rho_b c_b^2}\right)\frac{\rho_a(\Delta+f)+\rho_b(\Delta-f)}{\rho_a(\Delta-f)+\rho_b(\Delta+f)}\rho_b. \tag{33}$$

Note that ρ_{eff} and c_{eff} depend not only on f but also on the internal structure of the cluster throughout the Δ factor. This factor only depends of the relative positions of the cylinders in the cluster and of the properties of these cylinders (its section and material composition). As it is shown later, this factor Δ can be used to calculate the effective parameters for both ordered and disordered lattices.

Numerical simulations have been performed for a case consisting of 151 fluid cylinders embedded in nonviscous fluid or gas of different acoustic parameters. The cylinders are arranged in a hexagonal lattice with lattice parameter a. Besides, the cylinders form a cluster whose external shape is nearly circular. Since the filling fraction of the hexagonal lattice is $f_{hex}=\frac{2\pi}{\sqrt{3}}\left(\frac{R_a}{a}\right)^2$, results have been obtained under the assumption that the effective radius of the cluster in the homogenization limit is determined by the condition of filling fraction conservation. In other words, $f=f_{hex}$, and therefore

$$R_{eff} = \sqrt{\frac{N\sqrt{3}}{2\pi}}a. \tag{34}$$

Figure 2 shows the effective parameters calculated for the following systems: water (W) cylinders in air ($\rho_W/\rho_{air}=769$,

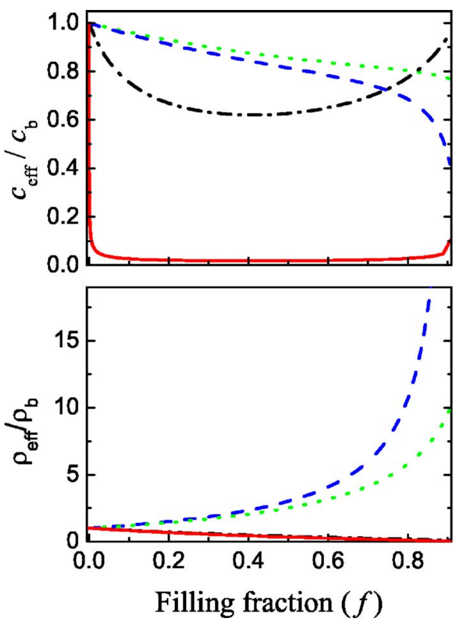

FIG. 2. (Color online) Effective acoustic parameters (relative to the embedded medium) for a cluster made of 151 cylinders of air in water (red continuous lines), water in air (blue dashed lines), mercury in water (green dotted lines), and water in mercury (black dashed-dotted lines).

c_W/c_{air}=5.15), air cylinders in water (ρ_{air}/ρ_W=0.001, c_{air}/c_W=0.19), mercury (Hg) cylinders in water (ρ_{Hg}/ρ_W =13.16, c_{Hg}/c_W=0.947), and water cylinders in mercury (ρ_W/ρ_{Hg}=0.076, c_W/c_{Hg}=1.056). Results in Fig. 2 show that these systems can be classified in two groups. The first group corresponds to systems in which the scatterers have acoustic impedance larger than that of the background like, for example, the case of water cylinders in air. In the second group the opposite occurs; i.e., the background has larger acoustic impedance than the scatterers. For the first group the behavior of c_{eff} and ρ_{eff} is similar to the case previously reported for rigid cylinders in air:[4,12] the effective sound speed decreases and the effective density increases with increasing filling fraction. The behavior of ρ_{eff} can also be reproduced by using the simplified approach of Berryman;[17] who developed an analytical expression that corresponds to that in Eq. (32) for $\Delta=1$. Note that the effective parameters corresponding to the close-packing (CP) condition (f_{hex}^{CP}=0.906) do not converge to those of the fluid inside the cylinders. This is because the close-packing condition does not imply a full filling of the available space by the circular cylinders; only cylinders with a square section will accomplish such convergence property.[30] For the systems of the second group (scatterers of lower density than the background) the behavior of c_{eff} is similar to the one reported for the 3D system consisting of air bubbles in water;[31,32] the sound speed starts decreasing for increasing (low) filling fractions but after some critical value it raises and increases with increasing volume fraction. The reader is addressed to Refs. 31 and 32 for a discussion of this phenomenon.

The fluid-fluid systems studied above are obviously very difficult (even impossible) to construct. Therefore from here onwards we will analyze a composite system consisting of cylinders made of an impenetrable fluid surrounded by air. This case represents actual systems where $\rho_{cyl}/\rho_b \gg 1$; for example, solid cylinders made of steel, aluminum, or wood embedded in air accomplish fairly well such a condition and can be simulated by the equations obtained under that approach ($\rho_{cyl}=\infty$). Let us point out that results obtained under the rigidity condition could be different from those obtained under the alternative simplification condition: fluid cylinders with extremely low sound velocity ($c_{cyl}\rightarrow 0$) or solid cylinders with velocities, longitudinal and transversal, both very small. Homogenization of clusters made of solid (elastic) cylinders is a interesting problem that is out of the scope of the present work, the corresponding results will be published elsewhere.

Under the simplified assumption of $\rho_a \approx \infty$, it is possible to obtain the following expressions:

$$B_{eff}=\frac{B_b}{1-f}, \tag{35a}$$

$$\rho_{eff}=\frac{\Delta+f}{\Delta-f}\rho_b, \tag{35b}$$

$$c_{eff}=\sqrt{\frac{\Delta-f}{1-f}}\frac{c_b}{\sqrt{\Delta+f}}. \tag{35c}$$

For low f, the contribution to $\hat{\mathcal{M}}$ of matrix $\hat{\mathcal{G}}$ can be neglected. Then, matrix $\hat{\mathcal{M}}$ is simply the identity, so that $(\hat{\mathcal{M}}_{\alpha\beta}^{-1})_{11}\approx\delta_{\alpha\beta}$, and, moreover, $\Delta=1$. Finally, B_{eff} remains the same but not ρ_{eff} and c_{eff} that now are

$$\rho_{eff}=\frac{1+f}{1-f}\rho_b, \tag{36a}$$

$$c_{eff}=\frac{c_b}{\sqrt{1+f}}. \tag{36b}$$

At this point it is interesting to stress that Eq. (36a) recovers the Berryman's effective density[17] and Eq. (36b) recovers the heuristic model reported in Ref. 4 for c_{eff}. Moreover, these expressions have also been found in studying the refraction of water waves by the CPA method.[14]

Continuous lines in Figs. 2(a) and 2(b) represent the effective parameters calculated by using Eqs. (35b) and (35c). The result for c_{eff} fairly agrees with that calculated in Ref. 12 for an infinite medium by a plane wave expansion method. The red dotted lines in those figures define the approximated values for c_{eff} and ρ_{eff} given by Eqs. (36a) and (36b). As it is shown, the approximated model is valid over a wide range of filling fractions.

The symbols in Fig. 2 represent the parameters experimentally determined in Ref. 16 by using 151 wooden cylinders in air. The validity of our homogenization approach is confirmed by the measurements, which established a lowering of the effective sound speed and an increasing of the effective density for the cluster analyzed. Of course, further experimental work should be performed in order to support the theoretical predictions in the full range of f.

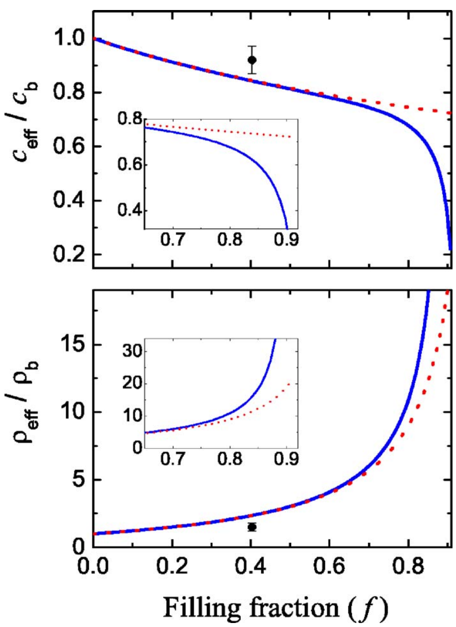

FIG. 3. (Color online) Effective acoustic parameters relative to the embedded medium calculated (blue continuous lines) for a homogenized cluster of 151 rigid cylinders. The red-dotted lines represent the values obtained by using the approach in Eqs. (36a) and (36b). The black dots with error bars define the data reported in Ref. 30.

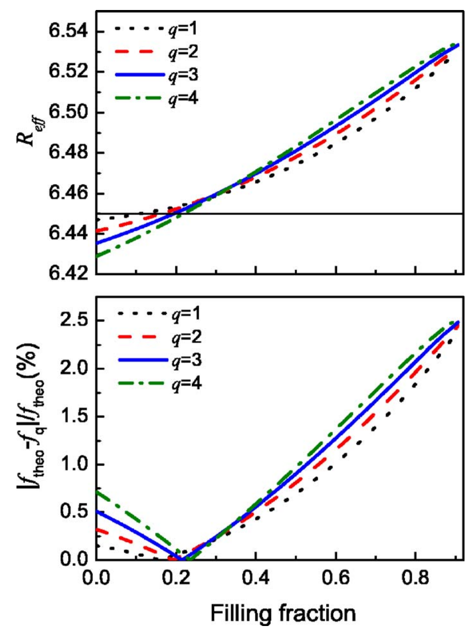

FIG. 4. (Color online) (a) Effective radius plotted as a function of the filling fraction calculated from Eq. (37) for four different q values. The horizontal line defines the value given by Eq. (34). (b) Relative error of the theoretically assumed filling fraction, compared with the ones obtained from the T matrix [see Eq. (37)] for four different q values.

In summary, our homogenization theory [Eqs. (35)] predicts for the case of infinite-density cylinders in a fluid (Fig. 3) that $\bar{c}_{eff}$ increases and $\bar{\rho}_{eff}$ decreases for increasing filling fraction. Moreover, when f increases, the graphs show that the parameters of the cluster converge to those of the cylinders. For this case in which the cylinders are considered rigid (infinite density) the speed of sound goes to zero because the sound cannot propagate inside the cylinder.

C. Effective radius

In the previous section, the simulations were performed under the simplified assumption that the radius of the homogenized cluster is determined by the condition of filling fraction conservation. For the cluster studied ($N=151$) that condition gives $R_{eff}=6.452a$, which fairly agrees with the physical dimensions of the cluster. Here, we analyzed the error associated to such an assumption by using the properties of the T matrix.

Effectively, the accuracy of R_{eff} can be studied by calculating the ratio between diagonal terms of the T matrix. In fact, from Eq. (A3b)

$$\frac{\hat{T}^{cyl}_{q+1q+1}}{\hat{T}^{cyl}_{qq}} = \frac{1}{4q(q+1)}R^2_{cyl}; \quad q \neq 0. \tag{37}$$

It is expected that the same identity be satisfied for the effective matrix resulting from the homogenized cluster [see Eq. (A20)]. Since it is impossible to get an exactly circular cluster, different values for the effective radius will be obtained for different values of q. Then, the variations observed in R_{eff} will provide an account of how the cluster is different from a circular shape.

In Fig. 4(a) the values of R_{eff} obtained for four different values of q have been plotted. It is shown that deviations from the imposed value (horizontal line) occur, but these variations are not very important. The relevance of these variations can be better understood if we represent a comparison of the associated filling fractions. On the one hand, for a given q, the value $R_{eff}(q)$ given by Eq. (A20) is employed in calculating the filling fraction f_q of the lattice, then

$$f_q = N\left(\frac{R_a}{R_{eff}(q)}\right)^2. \tag{38}$$

On the other hand, the theoretical assumed filling fraction is defined as usual, $f_{theo}=f_{hex}$. The relative error $err(\%)=100\times|f_{theo}-f_q|/f_{theo}$, is shown in Fig. 4(b). Notice that the error is always smaller than 2.5%, which supports the validity of Eq. (34) to determine the effective radius of a homogenized cluster with external circular shape.

V. DISORDERED LATTICES

When disorder is introduced in a lattice of sonic or elastic scatterers many interesting phenomena might appear. Perhaps the more fundamental is related with the problem of localization. Acoustic localization has been predicted in sound propagation through liquid media containing air-filled bubbles.[33] More recently, widening of phononic band gaps has been found in certain disordered phononic systems due to strong Anderson localization.[34]

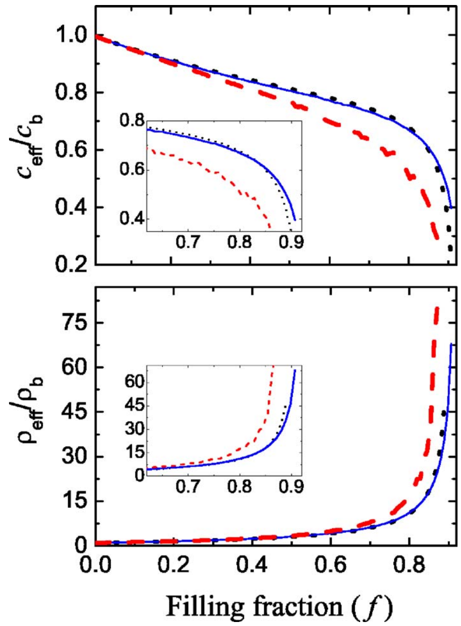

FIG. 5. (Color online) Effective acoustic parameters (relative to the embedded medium) defining a homogenized cluster of 151 rigid cylinders with positional disorder. The blue continuous lines define results corresponding to the "weakly disordered" case (see text). The red dashed lines corresspond to the case randomly disordered. The parameters of the perfectly ordered case (black dotted lines) also appear for comparison purposes.

The purpose of this section is to study how c_{eff} and ρ_{eff} change when the condition of ideal cluster is released. First, the possibility of having cylinders put at positions different to the ones of the hexagonal lattice (positional disorder) is analyzed. Second, cylinders with different radii are considered (structural disorder) in order to show their implications over the effective parameters.

A. Positional disorder

Equations (35b) and (35c) give ρ_{eff} and c_{eff} as a function of f and Δ. The last parameter can be calculated for any arbitrary structure, ordered or not. Then, it is possible to study, for example, the homogenization of clusters of cylinders with positional disorder inside a certain region and to determine its effective parameters by calculating the corresponding value Δ.

In Fig. 5 the effective acoustic parameters for the homogenization of 151 cylinders with equal radii put inside an imaginary circle have been calculated for two different cases of positional disorder. We first studied the "weakly disordered" case, which corresponds to puting the cylinders at random inside each one of the 151 hexagonal unit cells defined inside the circle. In other words, every cylinder in the ordered lattice is randomly moved inside the unit cell. The resulting effective parameters are represented by blue continuous lines in Fig. 5 and compared with the perfectly ordered case, which is given by the black dotted lines. Notice that no appreciable difference appears between them because, in fact, both distributions of cylinders are equivalent on the average.

The second case studied corresponds to the "fully disordered" case; that is, inside the imaginary circle (the one defined by the ideal cluster) all the cylinders have been put at random. The red dashed lines in Fig. 5 define the results obtained after averaging over ten different configurations. The amount of configurations employed is enough to guarantee that the standard deviation is lower than 10% even in the case of large filling fractions. Now, it is shown that deviations from the ideal case appear even at small filling fractions. The simulations predict effective parameters such that c_{eff} (ρ_{eff}) is always smaller (greater) than that of the corresponding ideal cluster. Note that for large filling fraction, the parameters of the disordered case do not converge to those of the ideal one though the touching of cylinders is achieved. The explanation for that disagreement is simple: the structure of voids between touching cylinders is different in the two configurations.

B. Structural disorder

The case of a cluster made of cylinders that are different can be easily treated by taking into account that the diagonal terms are now

$$\hat{T}_{00}^{eff} = \sum_{\alpha} (\tilde{T}_{\alpha})_{00}, \tag{39}$$

$$\hat{T}_{11}^{eff} = \sum_{\alpha,\beta} (\hat{M}_{\alpha\beta}^{-1})_{11} (\hat{T}_{\beta})_{11}. \tag{40}$$

When the cluster is made of several i-types of cylinders equally distributed, the partial filling fraction f_i can be defined as

$$f_i \equiv N_i \left(\frac{R_i}{R_{eff}} \right)^2, \tag{41}$$

where N_i is the total number of cylinders of material i and radius R_i. The following expressions are easily obtained for the effective parameters

$$\frac{1}{B_{eff}} = \frac{1-f}{B_b} + \sum_i \frac{f_i}{B_i}, \tag{42}$$

$$\rho_{eff} = \frac{1 + \sum_i f_i \frac{\rho_i - \rho_b}{\rho_i + \rho_b}}{1 - \sum_i f_i \frac{\rho_i - \rho_b}{\rho_i + \rho_b}} \rho_b, \tag{43}$$

where B_i and ρ_i are, respectively, the bulk modulus and density of cylinders made of material i. The effective speed of sound could be obtained from the previous expressions as usual (see Sec. IV). The results above for B were also obtained in Ref. 17 for a 3D case, where cylinders are replaced by spheres.

In the most general case, when all the cylinders are different, if the homogenized cluster is still considered as homogeneous and isotropic, its corresponding effective parameters are given by

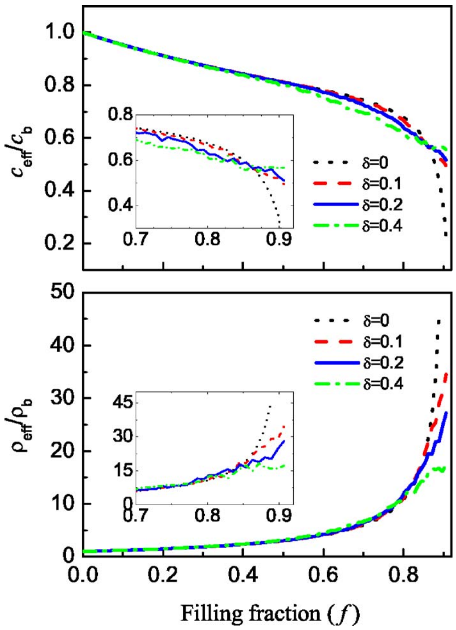

FIG. 6. (Color online) Effective parameters of a homogenized cluster of 151 rigid cylinders enclosed in a circle of radius $R_{eff} = 6.452a$. The values are calculated under the assumption that the lattice of cylinders has structural disorder (see Sec. V B). Three different degrees of disorder have been considered. The ideal case is represented by the lines $\delta=0$.

$$\frac{1}{B_{eff}} = \frac{1+\epsilon}{B_b}, \tag{44}$$

$$\rho_{eff} = \frac{1+\mu}{1-\mu}\rho_b, \tag{45}$$

where

$$\mu = -\frac{i4}{\pi R_{eff}^2}\hat{T}_{11}^{eff}, \tag{46}$$

$$\epsilon = -\frac{i4}{\pi R_{eff}^2}\hat{T}_{00}^{eff}. \tag{47}$$

These expressions have been used to study the case of a cluster made of 151 rigid cylinders having different radii. The radii of the cylinders are considered to follow a normal distribution centered in a certain radius R_0, which defines the filling fraction in the x-axis. The function that defines the distribution of cylinders' radii for a given R_0 is

$$P(R) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(R-R_0)^2/2\sigma^2}. \tag{48}$$

Results have been obtained for three different values of the variance σ. This value has been defined assuming that there is a relative error δ in the radius of the cylinder and that $R_0\delta \approx 3\sigma^2$.

Results for $\delta=0.1$, 0.2, and 0.4 are shown in Fig. 6 and compared with the case of an ideal cluster of identical cylinders ($\delta=0$). All the curves are obtained after averaging parameters of ten different configurations of cylinders. It can be concluded that structural disorder produces effective parameters that deviate from those obtained in clusters of identical cylinders only for large filling fractions.

VI. SUMMARY

Multiple scattering theory is used to develop a method allowing to study clusters of fluid cylinders in the homogenization limit. Exact formulas for the effective parameters have been found as a function of the individual properties and positions of the cylinders that form the cluster. Afterward, the formulas have been applied to obtain the effective parameters for several circular clusters of 151 fluid/gas cylinders in a different fluid/gas background. The case of a cluster of rigid cylinders in air has been deeply analyzed obtaining that the resulting effective speed of sound is in agreement with previous theories[12] based on a plane wave expansion. Moreover, the effective density found also agrees with value obtained by using statistical theories of the T-matrix.[17] Also, approximated expressions of the exact formulas valid at low fraction of volume occupied by scatterers have been obtained. The method has also been applied to study disordering effects in the lattice, showing that the effective parameters deviated from that of the ordered lattice only for large filling fractions. It can be concluded that the method here reported is a useful tool to design acoustic metamaterials with prefixed acoustic parameters, which can be used to fabricate refractive devices or new structures able to demonstrate fundamental properties of composite fluids like acoustic Bloch oscillations.[35]

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APPENDIX: LOWER ORDER ELEMENTS OF THE K-EXPANSION OF THE T MATRIX

For a cylinder of circular section, radius R_{cyl}, speed of sound c_{cyl}, and density ρ_{cyl}, the T matrix is diagonal and its elements are given by[20]

$$T_{sq} = -\frac{\rho_q J_q'(kR_{cyl}) - J_q(kR_{cyl})}{\rho_q H_q'(kR_{cyl}) - H_q(kR_{cyl})}\delta_{sq}, \tag{A1}$$

where

$$\rho_q \equiv \bar{\rho}_{cyl}\bar{v}_{cyl}\frac{J_q(kR_{cyl}/\bar{v}_{cyl})}{J_q'(kR_{cyl}/\bar{v}_{cyl})}, \tag{A2}$$

and $\bar{c}_{cyl} = c_{cyl}/c_b$, $\bar{\rho}_{cyl} = \rho_{cyl}/\rho_b$. By using the power expansion of Hankel and Bessel functions for small arguments, is easy to show that the lower order terms of this matrix are

$$\lim_{k\to 0} T_{00} \approx \frac{i\pi R_{cyl}^2}{4}\left[\frac{1}{\bar{\rho}_{cyl}\bar{v}_{cyl}^2} - 1\right]k^2 + \vartheta(k^4), \quad \text{(A3a)}$$

$$\lim_{k\to 0} T_{sq} \approx \frac{i\pi R_{cyl}^{2|q|}}{4^{|q|}} \frac{1}{(|q|-1)!|q|!} \frac{\bar{\rho}_{cyl}-1}{\bar{\rho}_{cyl}+1} k^{2|q|}\delta_{sq} + \vartheta(k^{2|q|+2}). \quad \text{(A3b)}$$

The k-independent factors of lower terms in the power expansion define the $\hat{T}_{sq}$ elements:

$$\hat{T}_{00} \equiv \lim_{k\to 0}\frac{T_{00}}{k^2}, \quad \text{(A4)}$$

$$\hat{T}_{sq} \equiv \lim_{k\to 0}\frac{T_{sq}}{k^{2|q|}}. \quad \text{(A5)}$$

For the effective T matrix defined in Eq. (15), lower order terms are obtained from the dominant terms of each of the three matrices which form it. The power expansion of the matrix $(J_\alpha)_{pq}$ becomes

$$(J_\alpha)_{pq} \approx (\hat{J}_\alpha)_{pq} k^{|p-q|}, \quad \text{(A6)}$$

where

$$(\hat{J}_\alpha)_{pq} = \frac{\sigma_{p-q}^{p-q}}{|p-q|!}\left[\frac{R_\alpha}{2}\right]^{|p-q|} e^{i(q-p)\Phi_\alpha} \quad \text{(A7)}$$

and

$$\sigma_n^m = \begin{cases} 1 & \text{if} \quad n \geqslant 0 \\ (-1)^m & \text{if} \quad n < 0. \end{cases} \quad \text{(A8)}$$

Though the case of clusters having cylinders with different radii are possible to deal with (see Sec. V B), hereafter all the cylinders are considered identical and with circular section. Therefore the matrix $(T_\beta)_{rt} = T_{rr}\delta_{rt}$ and, consequently, in Eq. (15) the addition $\Sigma_t (T_\beta)_{rt}(J_\beta)^*_{st} = T_{rr}(J_\beta)^*_{sr}$. The corresponding assymtotic form is

$$T_{rr}(J_\beta)^*_{sr} \approx \hat{T}_{rr}(\hat{J}_\beta)^*_{sr} k^{2|r|+2\delta_{r0}|2-r|}. \quad \text{(A9)}$$

In the expression above, we used that $T_{rr} \approx \hat{T}_{rr} k^{2|r|+2\delta_{r0}}$.

The asymptotic form of matrix $(M^{-1}_{\alpha\beta})_{qr}$ is more complex because there is not an explicit form for this matrix, but one can be obtained from its properties

$$\sum_r \boldsymbol{M}^{-1}_{qr}\boldsymbol{M}_{rs} = \delta_{qs}\boldsymbol{I}, \quad \text{(A10)}$$

$$\sum_r \boldsymbol{M}_{qr}\boldsymbol{M}^{-1}_{rs} = \delta_{qs}\boldsymbol{I}. \quad \text{(A11)}$$

In the relationships above, the greek subindexes have been omitted because they are not relevant in the discussion that follows.

The lowest order term in the k-expansion of the M matrix is

$$\boldsymbol{M}_{rs} = \delta_{rs}\boldsymbol{I} - \hat{\boldsymbol{G}}_{rs} k^{2|r|+2\delta_{r0}-|r-s|} \quad \text{(A12)}$$

where

$$(\hat{G}_{\alpha\beta})_{rs} = \hat{T}_{rr}\sigma_{r-s}^{r-s}(|r-s|-1)!\frac{1}{i\pi}\left[\frac{2}{r_{\alpha\beta}}\right]^{|r-s|} \times e^{i(s-r)\theta_{\alpha\beta}}(1-\delta_{rs})(1-\delta_{\alpha\beta}). \quad \text{(A13)}$$

The δ_{rs} appears because when $r=s$, the asymptotic form of the G matrix has a factor of the form $\approx \boldsymbol{I}k^2 \ln k$, which is lower than the factor $\boldsymbol{I}$. With this definition, multiplyng both equations by $k^{|s|-|q|}$ and defining a new matrix

$$\hat{\boldsymbol{M}}^{-1}_{qr} \equiv \boldsymbol{M}^{-1}_{qr} k^{-|q|+|r|}. \quad \text{(A14)}$$

Equations (A10) and (A11) take the form

$$\sum_r \hat{\boldsymbol{M}}^{-1}_{qr}[\delta_{rs}\boldsymbol{I} - \hat{\boldsymbol{G}}_{rs} k^{|r|+|s|+2\delta_{r0}-|r-s|}] = \delta_{qs}\boldsymbol{I}, \quad \text{(A15a)}$$

$$\sum_r [\delta_{qr}\boldsymbol{I} - \hat{\boldsymbol{G}}_{qr} k^{|q|+|r|+2\delta_{q0}-|q-r|}]\hat{\boldsymbol{M}}^{-1}_{rs} = \delta_{qs}\boldsymbol{I}. \quad \text{(A15b)}$$

In these equations the power of k is always bigger than or equal to zero. Therefore, when $k \to 0$ all terms disappear except those that have k raised to zero. The inverse of the following matrix

$$\overset{o}{\boldsymbol{M}}_{qs} = \begin{cases} \delta_{qs}\boldsymbol{I} - \hat{\boldsymbol{G}}_{qs} & \text{if} \quad qs \leqslant 0 \wedge q \neq 0 \\ \delta_{qs}\boldsymbol{I} & \text{if} \quad qs > 0 \vee q = 0 \end{cases} \quad \text{(A16)}$$

is $\hat{\boldsymbol{M}}^{-1}_{qr}$. Note that both $\overset{o}{\boldsymbol{M}}_{qs}$ and $\hat{\boldsymbol{M}}^{-1}_{qr}$ are independent of k and then

$$\boldsymbol{M}^{-1}_{qr} = \hat{\boldsymbol{M}}^{-1}_{qr} k^{|q|-|r|} \quad \text{(A17)}$$

is the lower order term of the k-expansion of M^{-1}.

Finally, the effective T matrix is

$$T^{\text{eff}}_{ps} = \sum_{\alpha,\beta}\sum_{r,q} (\hat{J}_\alpha)_{pq}(\hat{M}^{-1}_{\alpha\beta})_{qr}\hat{T}_{rr}(\hat{J})^*_{sr} k^{|p-q|}k^{|q|-|r|}k^{2|r|+2\delta_{r0}+|s-r|}. \quad \text{(A18)}$$

The lower order terms of this equation will be those such that $|p-q|+|q|=|p|$ and $|r|+|s-r|+2\delta_{r0}=|s|+2\delta_{s0}$. These conditions binds the possible values which can take r and q in the definition of $\hat{T}_{ps}$. These values define, respectively, the sets $\mathcal{R}$ and $\mathcal{Q}$, and therefore

$$T_{ps}^{\mathrm{eff}} = \hat{T}_{ps}^{\mathrm{eff}} k^{|p|+|s|+2\delta_{s0}} \tag{A19}$$

where

$$\hat{T}_{ps}^{\mathrm{eff}} = \sum_{r,q\in\mathcal{R},\mathcal{Q}} \sum_{\alpha,\beta} (\hat{J}_{\alpha})_{pq} (\hat{M}_{\alpha\beta}^{-1})_{qr} \hat{T}_{rr} (\hat{J})_{sr}^{*}. \tag{A20}$$

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