

## Radial Wave Crystals: Radially Periodic Structures from Anisotropic Metamaterials for Engineering Acoustic or Electromagnetic Waves

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We demonstrate that metamaterials with anisotropic properties can be used to develop a new class of periodic structures that has been named radial wave crystals. They can be sonic or photonic, and wave propagation along the radial directions is obtained through Bloch states like in usual sonic or photonic crystals. The band structure of the proposed structures can be tailored in a large amount to get exciting novel wave phenomena. For example, it is shown that acoustical cavities based on radial sonic crystals can be employed as passive devices for beam forming or dynamically orientated antennas for sound localization.

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Metamaterials are composites made of subwavelength units whose macroscopic properties are controlled by the microdistributed units rather than the constituent materials. For example, acoustic metamaterials made of units with embedded resonances show amazing properties such as negative dynamical mass and/or negative bulk modulus [1]. However, these local-resonant based metamaterials present a drawback: Their properties are shown only into the narrow frequency region where the resonances are excited. On the other hand, nonresonant based acoustic metamaterials also exhibit unusual acoustical properties such as anisotropic mass density [2]. This property is obtained by using sonic crystals (SCs), which are artificial structures made of periodic distributions of sound scatterers in a gas or a fluid. The unusual acousticlike behavior of these microstructured systems appears for any frequency below a certain cutoff frequency defining their behavior as homogeneous materials [3].

The rich physical phenomena associated to the propagation of physical waves through periodic distributions of their corresponding scatterers are embedded in the band structure  $\omega(\mathbf{K})$ , which gives information about the frequencies  $\omega$  available for propagation for a given wave vector  $\mathbf{K}$ . Band structures have been studied for a wide variety of waves and crystals, such as electronic waves in semiconductors, electromagnetic (EM) waves in photonic crystals, or acoustic waves in sonic and phononic crystals. A common feature of the different type of crystals is the spatial periodicity of material distribution, which is defined by the lattice vectors  $\mathbf{R}_n$  put in rectangular coordinates. Radially periodic structures have been scarcely studied. For example, structures referred to as “circular photonic crystals” have been studied by Horiuchi and co-workers [4], but their properties have not been analyzed in the framework of Bloch’s theorem due to the fact that the associated wave equations in polar or spherical coordinates are not invariant under translations.

In this Letter, we demonstrate that anisotropic metamaterials allow the introduction of new radially periodic structures in two dimensions (2D) as well as in three dimensions (3D). These new structures have been named radial wave crystals (RWCs) because they can be defined either in the realm of EM waves as well as in the realm of acoustic waves. For simplification purposes, crystals in 2D are those mainly discussed here, and numerical results are reported for the case of acoustic waves, the associated systems being called radial sonic crystals (RSCs). However, we will also comment how the results obtained with RSCs can be easily extended to EM waves in 2D by introducing “radial photonic crystals” and to both types of waves in 3D.

Let us start by considering a SC in which the mass density  $\rho(r)$  and bulk modulus  $B(r)$  are both radially dependent. Let us also assume that the pressure field at an arbitrary point of the 2D space  $(r, \theta)$  takes the form  $P(r, \theta) = \sum_q P_q(r) e^{iq\theta}$ . After factorization, the radial part of the acoustic wave equation can be cast as

$$\mathcal{H}_q P_q(r) = \omega^2 P_q(r), \quad (1)$$

where

$$\mathcal{H}_q = -\frac{B(r)}{r} \frac{\partial}{\partial r} \frac{r}{\rho(r)} \frac{\partial}{\partial r} + q^2 \frac{B(r)}{r^2 \rho(r)}. \quad (2)$$

Note that  $\mathcal{H}_q$  would be invariant under translations of the form  $r \rightarrow r + nd$  ( $n$  being an integer), only if the coefficients of the partial derivatives were periodic with periodicity  $d$ . Unfortunately, such a condition cannot be accomplished by any choice of  $\rho(r)$  and  $B(r)$  because the terms  $r/\rho(r)$  and  $r\rho(r)$  cannot be made simultaneously periodic.

By considering acoustic metamaterials with a tensorial dynamical mass density, a generalized radial wave equa-

tion can be introduced:

$$\left[ -\frac{B(r)}{r} \frac{\partial}{\partial r} \frac{r}{\rho_r(r)} \frac{\partial}{\partial r} + q^2 \frac{B(r)}{r^2 \rho_\theta(r)} \right] P_q(r) = \omega^2 P_q(r), \quad (3)$$

where  $\rho_r(r)$  and  $\rho_\theta(r)$  are the components of the mass density tensor. The new differential operator can now be made invariant under translations since the coefficients  $r/\rho_r(r)$ ,  $B(r)/r$ , and  $r\rho_\theta(r)$  can be made simultaneously periodic. Now Bloch's theorem can be applied to obtain a radial band structure defined by a relationship of the form  $\omega = \omega(K)$ . The previous conditions on the acoustic parameters define the so-called "radial sonic crystal."

The simplest example of RSCs in 2D consists of two alternating metamaterials  $a$  and  $b$  with constant thicknesses  $d_a$  and  $d_b$  along the radial direction. Let us introduce the vector  $\mathbf{X}(r) \equiv (\rho_r(r), \rho_\theta^{-1}(r), B(r))$  and also consider that the acoustic parameters are

$$\mathbf{X}(r) = \begin{cases} r\hat{\mathbf{X}}_a & \text{if } (n-1)d < r < (n-1)d + d_a, \\ r\hat{\mathbf{X}}_b & \text{if } (n-1)d + d_a < r < nd, \end{cases} \quad (4)$$

where  $d = d_a + d_b$ ,  $n$  is an integer that takes values  $n = 1, 2, \dots, \infty$ , and  $\hat{\mathbf{X}}_i$  (for  $i = a, b$ ) are vectors whose com-

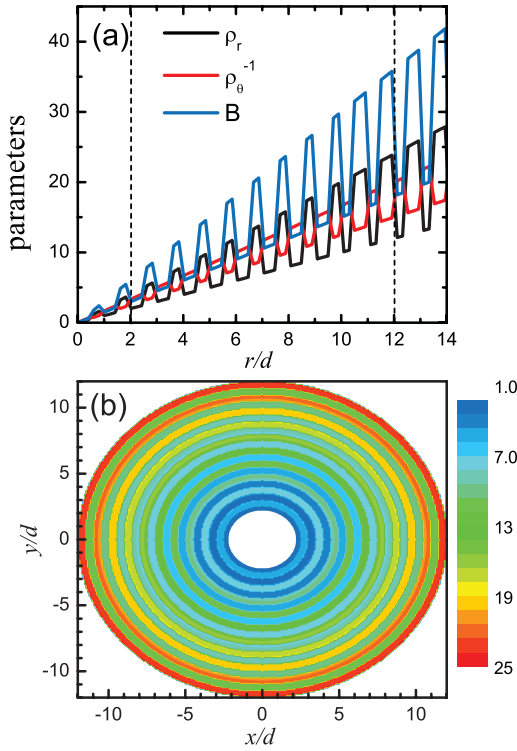


FIG. 1 (color). (a) Acoustic parameters of a radial sonic crystal consisting of two alternating anisotropic metamaterials  $a$  and  $b$ , with parameters  $(\rho_{ar}, \rho_{a\theta}^{-1}, B_a) = (2r, 1.25r, 3r)$  and  $(\rho_{br}, \rho_{b\theta}^{-1}, B_b) = (r, 1.67r, 1.5r)$ , respectively. They are depicted as a function of the radial coordinate normalized to  $d$ , the lattice period. (b) A radial-sonic-crystal shell whose dimension and parameters' values are defined by the vertical dashed lines shown in (a). The color scale corresponds to values taken by  $\rho_r$ .

ponents are real numbers giving the slope of the linearly dependent acoustic parameters, i.e.,  $\hat{\mathbf{X}}_i = (\hat{\rho}_{ir}, \hat{\rho}_{i\theta}^{-1}, \hat{B}_i)$ .

Figure 1(a) plots the radially dependent acoustic parameters for a RSC made of two metamaterials  $a$  and  $b$  with values  $\hat{\mathbf{X}}_a = (2.0, 1.25, 3.0)$  and  $\hat{\mathbf{X}}_b = (1.0, 1.67, 1.5)$ , respectively.

By introducing  $\hat{\mathbf{X}}(r)$  in Eq. (3), we arrive at the wave equations of the respective 2D homogeneous systems:

$$\frac{\partial^2 P_q(r)}{\partial r^2} + \left[ \omega^2 \frac{\hat{\rho}_{ir}}{\hat{B}_i} - q^2 \frac{\hat{\rho}_{ir}}{\hat{\rho}_{i\theta}} \right] P_q(r) = 0, \quad i = a, b. \quad (5)$$

These equations have plane-wave solutions with a dispersion relation given by

$$k_{iq}^2 = \omega^2 \frac{\hat{\rho}_{ir}}{\hat{B}_i} - q^2 \frac{\hat{\rho}_{ir}}{\hat{\rho}_{i\theta}}, \quad i = a, b. \quad (6)$$

The acoustic band structure of RSCs with parameters in (4) is easily obtained by following a method similar to that already employed for multilayered systems [5]. The final expression is

$$\cos Kd = \cos k_{aq} d_a \cos k_{bq} d_b - \frac{1}{2} \left[ \frac{\hat{\rho}_{ar} k_{bq}}{\hat{\rho}_{br} k_{aq}} + \frac{\hat{\rho}_{br} k_{aq}}{\hat{\rho}_{ar} k_{bq}} \right] \sin k_{aq} d_a \sin k_{bq} d_b, \quad (7)$$

where  $K$  is the Bloch wave number.

Figure 2(a) plots the acoustic band structure for the RSCs with parameters described in Fig. 1(a). Curves of the same color define branches with equal  $q$  values. Note

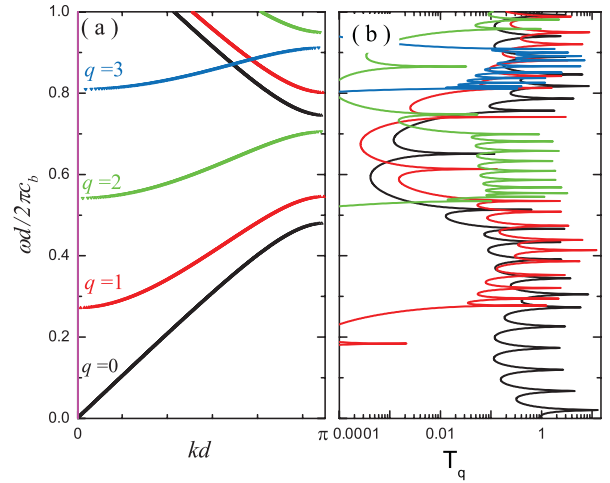


FIG. 2 (color). (a) Acoustic band structure of a radial sonic crystal with parameters depicted in Fig. 1(a). Only the dispersion relations corresponding to the modes  $q = 0, 1, \text{ and } 3$  are shown. The modes  $q = 0$  (black lines) follow a typical 1D band structure.  $q = 1$  modes (red lines),  $q = 2$  modes (green lines), and  $q = 3$  modes (blue lines) present a low band gap from  $\omega = 0$  to some  $\omega = q\omega_{cq}$  (see text). (b) Scattering transmission amplitude  $T_q$  as a function of the reduced frequency for the ten-layer radial-sonic-crystal shell shown in Fig. 1(b).

that branches with  $q \neq 0$  always have a low frequency band gap. Let us remark that the chosen parameters created a band structure such that the 1st branch with  $q = 2$  is totally included in the first band gap of branch  $q = 0$ . Besides, note that the 1st branch of  $q = 1$  contains modes that are also in the first band gap of branch  $q = 0$ .

Infinite RSCs are not feasible in practical applications; finite structures have to be used instead. Particularly, we analyze here the finite RSC defined by the vertical dashed lines in Fig. 1(a); it is a ten-layer RSC shell that occupies the region  $2d \leq r \leq 12d$ . The medium at the cavity defined by the shell ( $r < 2d$ ) and the medium outside the shell ( $r > 2d$ ) are considered the same homogeneous and isotropic acoustic medium.

First, let us analyze the behavior of the RSC shell when a sound source  $P^0$  is put inside the cavity defined by the shell [see Fig. 1(b)]. In this region, the general expression for the exciting field is

$$P^0 = \sum_q A_q^0 H_q(k_0 r) e^{iq\theta}, \quad (8)$$

where  $H_q$  are the Hankel functions,  $k_0 = \omega/c_0$ , and  $c_0$  is the speed of sound inside the cavity.

As a response to the exciting field, a reflected field  $P^r$  will appear inside the cavity ( $r < 2d$ ):

$$P^r(r, \theta) = \sum_q A_q^0 R_q J_q(k_0 r) e^{iq\theta}, \quad (9)$$

where  $J_q$  are the Bessel functions. The transmitted field  $P^t$  will appear at the region outside the shell ( $r > 12d$ ):

$$P^t(r, \theta) = \sum_q A_q^0 T_q H_q(k_0 r) e^{iq\theta}, \quad (10)$$

where coefficients  $T_q$  give information about the interaction between acoustic waves and the RSC. Coefficients  $T_q$  of the RSC shell under study are calculated by a method slightly modified from that reported by Cai and a co-worker [6,7]. Figure 2(b) shows the results plotted in the logarithmic scale. Note that coefficients  $T_q$  are very small at the corresponding band gap regions. Also note that Fabry-Perot-like oscillations appear in frequency regions where the band structure is almost linear. The peaks with frequencies in the band gaps indicate the existence of localized states inside the cavity shell.

Coefficients  $T_q$  are also plotted in Fig. 3(a) in a linear scale and are compared with those calculated for the case of a “homogeneous multilayer” (HML) that appears in Fig. 3(b). The HML shell is made of two alternating layers of homogeneous and isotropic material  $a$  and  $b$  with the same thicknesses as those in the RSC. Results for the HML shell show that a band gap, corresponding to Bragg reflection, is also present, but this band gap is the same for all of the  $q$  modes. It is possible to observe that Fabry-Perot oscillations also appear, but they are strongly mixed with all of the  $q$  modes. Instead, for the RSC shell it is possible to see that the oscillations have high values of  $T_q$  at their

peak maximum and the peaks are also very thin, both features that make the quality factor  $Q$  of the resonances be huge. It is also remarkable that there are frequency regions where modes of only a certain  $q$  are present; in these regions the cavity field would oscillate like a  $q$  pole. From this comparison it is concluded that the RSC shell is a “highly ordered” system that can be used to produce interesting physical phenomena as shown below.

RSC shells can be used like  $q$ -pole resonators. For example, Fig. 3(b) shows that at  $\omega d/2\pi c_0 = 0.608$  (black dot) there is a quadrupolelike resonance ( $q = 2$  mode) that has a calculated quality factor  $Q = \omega_0/\Delta\omega \approx 2000$ . Then, if a 2D punctual source of the type  $P \approx H_0(kr)$  is placed inside the cavity slightly separated from the origin, the field inside the cavity is

$$P^0 \approx \sum_q A_q^0 H_q(k_0 r) e^{iq\theta}. \quad (11)$$

Because of the quadrupolelike resonance, outside the cavity the field is

$$P^t \approx T_2 H_2(k_0 r) \cos(2\theta). \quad (12)$$

The corresponding field distribution appears in Fig. 4(a), which shows how a 0-pole (monopole) source generates a 2-pole (quadrupole) oscillation. This is an example of a RSC shell acting as a passive device for beam forming.

A remarkable phenomenon appears when the  $q$ -pole resonances are excited by sound sources outside the RSC shell. In Fig. 4(b) is depicted the field map generated by a punctual omnidirectional (monopole) sound source placed outside two identical RSC shells and that oscillates at a frequency  $\omega d/2\pi c_0 = 0.414$ . This frequency corresponds

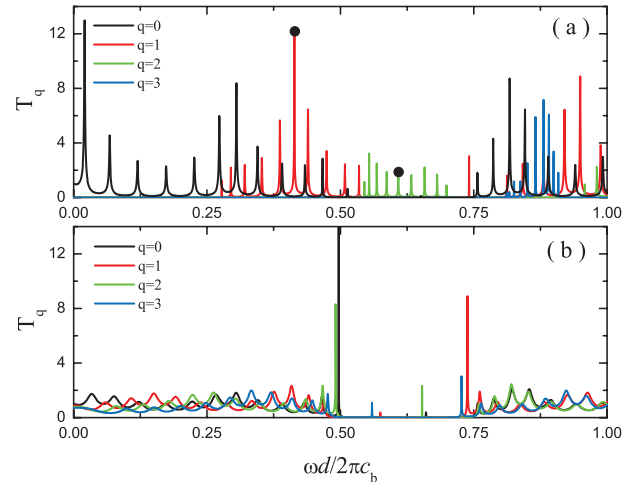


FIG. 3 (color). (a) Transmission coefficients for the radial-sonic-crystal shell described in Fig. 1(b). (b) Coefficients for a multilayer cylindrical shell made of two homogeneous and isotropic layers with the same thicknesses and constant parameters  $(\rho_a, B_a) = (2.0, 3.0)$  and  $(\rho_b, B_b) = (1.0, 1.5)$ , respectively. Note that Fabry-Perot resonances of the radial-sonic-crystal shell are thinner and higher than that in the homogeneous multilayer shell.

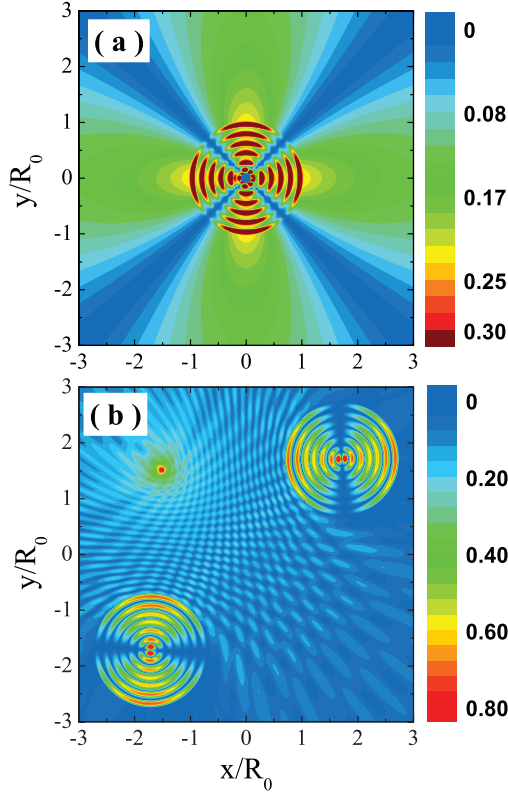


FIG. 4 (color). (a) Pressure field generated by a punctual sound source placed inside the cavity RSC shell, at a position  $r \neq 0$ . The source oscillates at  $\omega d/2\pi c = 0.608$ , which corresponds to a quadrupolelike resonance. (b) Pressure field generated by a punctual sound source placed at the position  $(-1.5, 1.5)$  outside two RSC shells. The frequency of the exciting field is  $\omega d/2\pi c = 0.414$ , which corresponds to a dipolelike resonance ( $q = 1$ ) of the RSC shell. Note how the dipolelike fields are orientated along the direction pointing to the sound source.

to a resonance of the  $q = 1$  mode marked with a black dot in Fig. 3(b); this dipolelike mode has a  $Q$  factor close to 900. It is observed how the localized dipolelike modes are excited by the source outside the RSC shells. Note that the dipoles are oriented in the direction to the punctual source, which makes these systems potentially useful as dynamically orientable antennas for sound source detectors.

The  $Q$  factors and the resonant phenomena reported above are associated to Fabry-Perot-like resonances. Our numerical simulations also indicate the existence of modes with frequencies in the band gaps whose  $Q$  factors are much larger than that of Fabry-Perot-like modes. The analysis of these modes strongly localized inside the cavity defined by the RSC shell is out of the scope of the present short Letter and will be reported elsewhere.

RSCs can be also proposed in 3D, where the acoustic wave equation for radially symmetric problems is

$$\left[ -\frac{B(r)}{r^2} \frac{\partial}{\partial r} \frac{r^2}{\rho_r(r)} \frac{\partial}{\partial r} + l(l+1) \frac{B(r)}{r^2 \rho_\phi(r)} \right] P_{\text{ln}} = \omega^2 P_{\text{ln}}, \quad (13)$$

where it has been assumed that  $\rho_\theta = \rho_\phi$ . Now, to obtain

periodic coefficients, the quantities  $r^2/\rho_r(r)$ ,  $B(r)/r^2$ , and  $\rho_\phi(r)$  must be simultaneously periodic. From this point onwards, the same procedure already reported for the 2D case applies.

For EM waves, radial photonic crystals can be also proposed in 2D and 3D. In 2D, the governing wave equation can be decoupled in TM and TE modes, where the  $z$  component of the corresponding field modifies Eq. (3), in which the components  $(\rho_r, \rho_\theta, B)$  are replaced by  $(\mu_\theta, \mu_r, \varepsilon_z^{-1})$  for the TE modes and by  $(\varepsilon_\theta, \varepsilon_r, \mu_z^{-1})$  for the TM modes. Thus the periodic conditions derived for the 2D acoustic case can be applied for the EM case with the corresponding change of variables.

In 3D there is not an equivalence between the acoustic and the EM wave equations. For this case it can be shown that the wave equation is

$$\left[ -\frac{1}{p_\phi q_\phi} \frac{\partial}{\partial r} \frac{1}{p_\phi} \frac{\partial}{\partial r} p_\phi + l(l+1) \frac{1}{q_\phi p_r r^2} \right] \psi = \omega^2 \psi, \quad (14)$$

where  $(p_r, p_\phi, q_r, q_\phi)$  are  $(\varepsilon_r, \varepsilon_\phi, \mu_r, \mu_\phi)$  for TE modes and  $(\mu_r, \mu_\phi, \varepsilon_r, \varepsilon_\phi)$  for TM modes. Now the requirements of periodicity applies to magnitudes  $\varepsilon_\phi(r)$ ,  $\mu_\phi(r)$ , and  $r^2 \varepsilon_r(r)$  for the case of TE modes and to  $\mu_\phi(r)$ ,  $\varepsilon_\phi(r)$ , and  $r^2 \mu_r(r)$  for TM modes.

The parameters required for building the proposed RWCs are not available by any materials existing in nature. However, they can be engineered by using the metamaterial concept. For example, we have demonstrated that mass anisotropy can be made feasible by using sonic crystals based on nonsymmetric lattices [2,8]. For EM it has been recently demonstrated that tapered waveguides can be used to develop anisotropic EM metamaterials [9].

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