Structural properties of positive periodic discrete-time linear systems: canonical forms

Rafael Bru [*], Sergio Romero, Elena Sánchez

Departament de Matemàtica Aplicada, Universitat Politècnica de València, Camí de Vera 14, 46022 València, Spain

Abstract

The reachability and controllability properties of positive periodic discrete-time linear systems are studied. Using the directed graph of the state matrices and the concepts of colored vertices and colored union of directed graphs, a characterization of these properties is established and hence, canonical forms are deduced.
© 2003 Elsevier Inc. All rights reserved.

Keywords: Positive N-periodic linear dynamic systems; Reachability; Controllability; Colored vertices; Canonical forms

1. Introduction

The reachability and controllability properties have been presented in most books of control linear system theory, for instance, [7,12] and [8]. We focus our attention on positive N-periodic discrete-time linear systems. Many authors have treated positive discrete-time linear systems in the invariant case, see [4,6,10,11], among others, and other authors in the periodic case, see [1,2]. In [2], the reachability property of a positive N-periodic system is characterized by means of the reachability property of the positive invariant cyclically

---

[*] Corresponding author.
  E-mail addresses: rbru@mat.upv.es (R. Bru), sromero@mat.upv.es (S. Romero), esanchezj@mat.upv.es (E. Sánchez).

augmented system associated with it. Moreover, characterizations in terms of the directed graph of the state matrix and canonical forms of the structural properties of positive invariant systems are established in [3] and [5].

The purpose of this paper is to obtain characterizations and canonical forms of the reachability and controllability properties of positive N-periodic discrete-time linear systems. In fact, we characterize the reachability and controllability properties by means of the corresponding canonical forms. These canonical forms have themself interest in the study of the reachability and controllability properties, and they can be the point of departure for obtaining other representations associated with reachability indices of any positive system. For systems without restrictions, that forms are well known (see [8,15]). A way to deal with periodic systems is to work with the associated invariant cyclically augmented system. However, the direct approach does not yield a cyclic canonical system, as we explain at the end of Section 2. To this end, we shall introduce the concepts of colored vertex and of colored union of directed graphs.

The paper has been structured as follows. In Section 2, we shall consider a positive N-periodic discrete-time linear system and the positive invariant cyclically augmented discrete-time linear system associated with it. In that section, the concepts of the combinatorial theory and the results over structural properties used throughout the paper will be introduced. Moreover, we shall define the deterministic paths and circuits belonging to the colored union of directed graphs of the state matrices of the N-periodic system. Hence, in Sections 3–5, characterizations and canonical forms for the reachability and controllability properties will be given.

2. Preliminaries and definitions

We consider a *positive N-periodic discrete-time linear control system* given by

$$x(k + 1) = F(k)x(k) + G(k)u(k), \quad k \in \mathbb{Z}_+ \tag{1}$$

where $F(k)$ and $G(k)$ are periodic matrices of period $N \in \mathbb{N}$, with nonnegative entries, i.e., $F(k) = F(k + N) \in \mathbb{R}_+^{n \times n}$, $G(k) = G(k + N) \in \mathbb{R}_+^{n \times m}$, $x(k) \in \mathbb{R}_+^n$ is the nonnegative *state* vector and $u(k) \in \mathbb{R}_+^m$ is the nonnegative *control* or *input* vector. We denote this system by $(F(\cdot), G(\cdot))_N \geqslant 0$.

The positive N-periodic system (1) is related to a *positive invariant cyclically augmented system*, which was used by Park and Verriest (see [9]) and Van Dooren (see [13]) and is given by

$$z(k + 1) = F_e z(k) + G_e u_e(k) \tag{2}$$

where $F_e \in \mathbb{R}_+^{nN \times nN}$ is weakly cyclic of index N (see [14]), that is,

$$F_e = \begin{bmatrix} O & F(0) \\ F & 0 \end{bmatrix}$$

with $F = \operatorname{diag}[F(1), \ldots, F(N-1)]$ and $G_e = \operatorname{diag}[G(0), G(1), \ldots, G(N-1)] \in \mathbb{R}_+^{nN \times mN}$. Moreover, the state vector and the input vector of system (2) are associated with the stacked vectors of the inputs and the states of (1), $\hat{x}(k) = \operatorname{col}[x(k), x(k+1), \ldots, x(k+N-1)]$ and $\hat{u}(k) = \operatorname{col}[u(k), u(k+1), \ldots, u(k+N-1)]$, by means of the following relations:

$$z(k) = M_n^{k-1} \hat{x}(k), \quad u_e(k) = M_m^k \hat{u}(k)$$

where

$$M_j = \begin{bmatrix} O & I_j \\ I_{(N-1)j} & 0 \end{bmatrix}$$

and I_q is the identity matrix of order q. We denote the invariant system given in (2) by (F_e, G_e).

Considering basic concepts of combinatorial theory of a matrix F, we assume $F = [f_{ij}] \in \mathbb{R}^{n \times n}$. We denote by $\Gamma(F)$ the corresponding *directed graph* consisting of a set of vertices $V = \{1, 2, \ldots, n\}$, and a set of arcs E. An arc (i, j) is in E if and only if $f_{ji} \neq 0$, and it is said that there is an arc from i to j. A *path from the vertex i to the vertex j* with $i, j \in V$, denoted by P_{ij}, is a sequence of arcs $(i, k_1), (k_1, k_2), \ldots, (k_{r-1}, j)$. In this case, the *length* of the path is r, length $(P_{ij}) = r$. A *deterministic* path is a path such that each vertex has no more than one outgoing arc, except possibly from the last vertex. And, a *deterministic circuit* is a closed deterministic path from i to i.

We consider the matrices $\{F(j)\}_{j=0}^{N-1}$ of order n of the positive N-periodic system $(F(\cdot), G(\cdot))_N \geqslant 0$ given in (1) and the directed graphs $\{\Gamma(F(j))\}_{j=0}^{N-1} = \{V(j), E(j)\}_{j=0}^{N-1}$, associated with the matrices $\{F(j)\}_{j=0}^{N-1}$.

By construction of the state matrix $F_e \in \mathbb{R}_+^{nN \times nN}$ of the invariant system given in (2), the corresponding directed graph $\Gamma(F_e) = \{V(F_e), E(F_e)\}$ can be identified with the directed graphs of the state matrices $\{F(j)\}_{j=0}^{N-1}$, as follows.

The vertex $k \in V(j)$ is identified with the vertex $jn + k \in V(F_e)$ and denoted by k^j, and it is said to be a *vertex of color j*. Moreover, an arc $(k, l) \in E(j)$ is identified with the arc

$$((j-1)n + k, jn + l) \in E(F_e) \text{ and denoted by } (k^{j-1}, l^j) \text{ if } j \neq 0 \quad (3)$$

or

$$((N-1)n + k, l) \in E(F_e) \text{ and denoted by } (k^{N-1}, l^0) \text{ if } j = 0 \quad (4)$$

In accordance with the above identification, the directed graph associated with the matrix F_e is called *colored union of directed graphs* and denoted by Λ. Thus, Λ is the pair constituted by the set of colored vertices and the corresponding set of arcs, $\Lambda = \{V, E\}$, where

$$V = \{1^0, \ldots, n^0, 1^1, \ldots, n^1, \ldots, 1^{N-1}, \ldots, n^{N-1}\}$$

and E is the set of suitable arcs leaving a vertex of color j and reaching a vertex of color $j + 1 \pmod N$. Note that in the colored union of directed graphs Λ, a deterministic path of length r from a vertex i of color $\alpha - 1$ to a vertex j of color $\alpha + r - 1$, for some $\alpha = 1, 2, \ldots, N$, is given by

$$(i^{\alpha-1}, k_1^\alpha), (k_1^\alpha, k_2^{\alpha+1}), \ldots, (k_{r-1}^{\alpha+r-2}, j^{\alpha+r-1})$$

Note that the length of a deterministic circuit of Λ must be a multiple of the period N.

We illustrate the above definitions with the following example.

Example 1. Consider a positive 3-periodic discrete-time linear system with a state-space of dimension $n = 5$, given by

$$[F(0)|G(0)] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$[F(1)|G(1)] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right],$$

$$[F(2)|G(2)] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

and $F(k + 3) = F(k)$ and $G(k + 3) = G(k)$ $\forall k \in \mathbb{Z}_+$.

The directed graphs of these periodic matrices are given by

$$\Gamma(F(1)) = \{\{1, 2, 3, 4, 5\}, \{(1, 5), (2, 3), (3, 4), (4, 2), (5, 1)\}\},$$
$$\Gamma(F(2)) = \{\{1, 2, 3, 4, 5\}, \{(1, 2), (2, 5), (3, 1), (4, 3), (4, 4), (5, 1), (5, 4)\}\},$$
$$\Gamma(F(0)) = \{\{1, 2, 3, 4, 5\}, \{(1, 2), (2, 5), (5, 4)\}\}$$

This periodic system is associated with the cyclically augmented system (F_e, G_e), where

$$F_e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a weakly cyclic matrix of index 3 and $G_e = \mathrm{diag}[G(0), G(1), G(2)]$. Moreover,

$$\Gamma(F_e) = \{V(F_e), E(F_e)\}$$
$$= \Big\{ \{k\}_{k=1}^{15}, \{(1, 10), (2, 8), (3, 9), (4, 7), (5, 6), (6, 12), (7, 15), (8, 11),$$
$$(9, 13), (9, 14), (10, 11), (10, 14), (11, 2), (12, 5), (15, 4)\} \Big\}$$

Then, from the identification constructed before, we find that $\Lambda = \{V, E\}$ is given by the set of colored vertices

$$V = \{1^0, 2^0, 3^0, 4^0, 5^0, 1^1, 2^1, 3^1, 4^1, 5^1, 1^2, 2^2, 3^2, 4^2, 5^2\}$$

and the corresponding set of arcs

$$E = \big\{ (1^0, 5^1), (2^0, 3^1), (3^0, 4^1), (4^0, 2^1), (5^0, 1^1), (1^1, 2^2), (2^1, 5^2),$$
$$(3^1, 1^2), (4^1, 3^2), (4^1, 4^2), (5^1, 1^2), (5^1, 4^2), (1^2, 2^0), (2^2, 5^0), (5^2, 4^0) \big\}$$

For example, note that in Λ there is a deterministic path from 3^0 to 4^1 of length 1, that is,

$$3^0 \to 4^1$$

and 4^1 has access to vertices 3^2 and 4^2. Furthermore, a deterministic circuit of Λ is

$$1^2 \to 2^0 \to 3^1$$

In this paper, we study the structural properties of positive reachability and controllability for positive N-periodic linear systems. According to [6], where the definitions have been given for positive invariant linear systems, a positive N-periodic system $(F(\cdot), G(\cdot))_N$ is said to be

(a) *reachable at time s* (*from* 0) if, for any nonnegative state $x_f \in \mathbb{R}_+^n$, there exists a nonnegative input sequence transferring the state of the system from the origin at time s, $x(s) = 0$, to x_f in finite time. It is *reachable* if it is reachable at time s, for all $s \in \mathbb{Z}_+$.

(b) *null-controllable at time s* if, for any nonnegative state x_0, there exists some nonnegative input sequence transferring the state of the system from x_0 at time s, $x(s) = x_0$, to the origin in finite time. It is *null-controllable* if it is null-controllable at time s, for all $s \in \mathbb{Z}_+$.

(c) (*completely*) *controllable at time s* if, for any pair of nonnegative states x_0 and x_f, there exists a nonnegative input sequence transferring the state of the system from x_0 at time s, $x(s) = x_0$, to x_f in finite time. The system is *controllable* if it is controllable at time s, for all $s \in \mathbb{Z}_+$.

Note that if $N = 1$, we have the reachability and controllability concepts for the invariant case. It is worth noting that for positive systems, on the contrary to the general case, reachability from zero does not imply controllability to zero. Further, in this case, complete controllability is obtained only if one adds controllability to zero to reachability from zero (see [6]).

Now, we consider the N-periodic system $(F(\cdot), G(\cdot))_N$ and the invariant cyclically augmented system (F_e, G_e). It is clear that $(F(\cdot), G(\cdot))_N$ is positive if and only if (F_e, G_e) is positive. As indicated reported by [2], $(F(\cdot), G(\cdot))_N \geqslant 0$ is reachable if and only if $(F_e, G_e) \geqslant 0$ is reachable. Moreover, it can be proved that $(F(\cdot), G(\cdot))_N \geqslant 0$ is completely controllable if and only if $(F_e, G_e) \geqslant 0$ is completely controllable, which is equivalent to $(F_e, G_e) \geqslant 0$ being reachable and F_e being nilpotent.

When considering systems without nonnegative restrictions, we know that the reachability and complete controllability properties are transferred under similar transformations. However, in the positive case, because we have to preserve the positive restrictions, these properties can be transferred only under monomial matrices $M = DP$, where D is a diagonal matrix and P is a permutation matrix. We illustrate this fact in the following example.

Example 2. Consider a positive discrete-time linear system of period $N = 1$, with a state-space of dimension $n = 2$, given by

$$F = \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note that this system $(F, G) \geqslant 0$ is reachable since the reachability matrix $[G|FG]$ contains a monomial submatrix of order 2 (see this characterization in [6]). Consider a transformation matrix

$$T = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} 1/2 & 0 \\ 3/2 & 1 \end{bmatrix}$$

Thus,

$$T^{-1}FT = \begin{bmatrix} \dfrac{5}{2} & \dfrac{1}{2} \\ \dfrac{15}{2} & \dfrac{3}{2} \end{bmatrix} \text{ and } T^{-1}G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence, the positive similar system $(T^{-1}FT, T^{-1}G) \geqslant 0$ is not reachable. Therefore, the transformation matrix T maintains the positiveness of the system, but it does not transfer the reachability property.

Consequently, given two positive N-periodic systems $(\widehat{F}(\cdot), \widehat{G}(\cdot))_N \geqslant 0$ and $(F(\cdot), G(\cdot))_N \geqslant 0$ such that there exists a N-periodic collection of monomial matrices $M(j) = M(j + N), j \in \mathbb{Z}_+$, verifying

$$M^{-1}(j)F(j)M(j-1) = \widehat{F}(j)$$
$$M^{-1}(j)G(j) = \widehat{G}(j), \quad j \in \mathbb{Z}_+ \tag{5}$$

then $(\widehat{F}(\cdot), \widehat{G}(\cdot))_N$ is reachable (completely controllable) if and only if $(F(\cdot), G(\cdot))_N$ is reachable (completely controllable). Without loss of generality, canonical forms will be established under N-periodic collections of permutation matrices.

In [3], canonical forms of the structural properties under permutation matrices were found for invariant systems. Two permutation matrices $\widehat{P}$ and $\widehat{Q}$ can them be found such that the similar system $(\widehat{P}^T F_e \widehat{P}, \widehat{P}^T G_e \widehat{Q})$ has the special structure given in [3]. Note that the new matrix $\widehat{P}^T F_e \widehat{P}$ is not a cyclic matrix and therefore, can not be associated directly with a N-periodic system (see [2]). To avoid this problem, we shall work as follows.

Given a positive N-periodic system $(F(\cdot), G(\cdot))_N$, the purpose of our study is to obtain a similar positive N-periodic system $(\widehat{F}(\cdot), \widehat{G}(\cdot))_N$ with a canonical structure given through two N-periodic collections of permutation matrices $P(j) = P(j + N), j \in \mathbb{Z}_+$, and $Q(j) = Q(j + N), j \in \mathbb{Z}_+$. That is, $(\widehat{F}(j), \widehat{G}(j))_N = (P^T(j)F(j)P(j-1), P^T(j)G(j)Q(j))_N$. From these N-periodic collections of permutation matrices, we can construct $P_e = \text{diag}[P(0), P(1), \ldots, P(N-1)]$ and $Q_e = \text{diag}[Q(0), Q(1), \ldots, Q(N-1)]$, and from these matrices, we obtain a new system, $(P_e^T F_e P_e, P_e^T G_e Q_e) \geqslant 0$, (similar to $(F_e, G_e) \geqslant 0$), where $P_e^T F_e P_e$ is structured as a cyclic matrix (as F_e) and $P_e^T G_e Q_e$ is a block diagonal matrix (as G_e).

3. Reachability of $(F(\cdot), G(\cdot))_N \geqslant 0$

In order to characterize the reachability property of a positive N-periodic system $(F(\cdot), G(\cdot))_N \geqslant 0$, we construct a partition in the colored union of directed graphs $\Lambda = \{V, E\}$.

Given $(F(\cdot), G(\cdot))_N \geqslant 0$, for each $j = 0, 1, \ldots, N - 1$, suppose that in the matrix $G(j)$ there exist r_j different monomial columns corresponding to

different unit vectors $\{e_{i_1^j}, \ldots, e_{i_{r_j}^j}\}$, with $0 \leqslant r_j \leqslant n$. Consider the set of vertices $\mathscr{I} = \bigcup_{j=0}^{N-1} \{i_1^j, \ldots, i_{r_j}^j\} \subseteq V$.
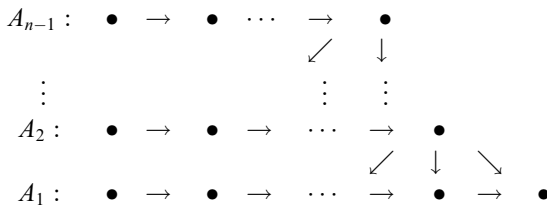
Now, consider the deterministic paths of Λ starting from vertices of $\mathscr{I}$, i.e., for each $j = 0, \ldots, N - 1$, that is,

$$
\begin{array}{ccccccc}
i_1^j & \to & i_1^{j+1} & \to & \cdots & \to & i_1^{j+k_1} \\
\vdots & & \vdots & \vdots & & & \vdots \\
i_{r_j}^j & \to & i_{r_j}^{j+1} & \to & \cdots & \to & i_{r_j}^{j+k_{r_j}}
\end{array}
\tag{6}
$$

In what follows, we assume that there are no repeated deterministic paths of Λ (in any other case, we eliminate them). We define A as *the set of all vertices in the deterministic paths* of (6). We construct, depending on the different relations among the deterministic paths given in (6), the following disjoint subsets of vertices of A:

1. The *subset of level 1*, A_1, formed by the vertices belonging to A which are in some deterministic path given in (6) whose last vertex does not have any outgoing arc.
2. The *subset of level 2*, A_2, formed by the vertices belonging to $A \setminus \{A_1\}$ which are in some deterministic path given in (6) whose last vertex has access only to vertices in A_1 (at least it has access to one of them). In general, we consider
3. The *subset of level h*, A_h, $h = 1, \ldots, n - 1$, formed by the vertices belonging to $A \setminus \{A_1 \cup \cdots \cup A_{h-1}\}$ which are in some deterministic path given in (6) whose last vertex have access only to vertices in $A_1 \cup \cdots \cup A_{h-1}$ (at least it has access to one vertex in A_{h-1}).

The following scheme shows the relationship between the structure of the deterministic paths with vertices in different levels:

$$
\begin{array}{cccccccccc}
A_{n-1}: & \bullet & \to & \bullet & \cdots & \to & & \bullet & & \\
& & & & & & \nearrow & \downarrow & & \\
\vdots & & & & & \vdots & & \vdots & & \\
A_2: & \bullet & \to & \bullet & \to & \cdots & \to & \bullet & & \\
& & & & & & \nearrow & \downarrow & \searrow & \\
A_1: & \bullet & \to & \bullet & \to & \cdots & \to & \bullet & \to & \bullet
\end{array}
$$

From these subsets contained in A, we establish the following definition.

Definition 1. We say that a *deterministic path* given in (6) is of *type (I)* if all their vertices belong to the set $A^{\mathrm{I}} = A_1 \cup \cdots \cup A_{n-1}$. Moreover, we say that a *deterministic path* of Λ is *of level k* if all their vertices belong to the subset of level k, A_k.

We illustrate these concepts in the following example.

Example 3. Consider the positive 3-periodic system given in the Example 1.
Take the different monomial vectors in each one of the matrices $G(j)$, $j = 0$, 1, 2. Thus,

$$\mathscr{I} = \{1^0, 3^0\} \cup \emptyset \cup \{2^2, 3^2, 4^2\}$$

Hence, the deterministic paths in Λ starting from the vertices in $\mathscr{I}$ are the following

$$1^0 \rightarrow 5^1$$
$$3^0 \rightarrow 4^1$$
$$2^2 \rightarrow 5^0 \rightarrow 1^1$$
$$3^2$$
$$4^2$$

Then, $A = \{1^0, 3^0, 5^0, 1^1, 4^1, 5^1, 2^2, 3^2, 4^2\}$.
The different paths of type (I), pointing to the corresponding levels, are:

$$3^0 \rightarrow 4^1 \quad \text{(path of level 2)}$$
$$3^2 \qquad \text{(path of level 1)}$$
$$4^2 \qquad \text{(path of level 1)}$$

because the last vertex of the first path has access only to vertices included in
paths of level 1 and the last vertex of the remaining paths has no outgoing arc.
Then, $A^{\text{I}} = A_1 \cup A_2 = \{3^2, 4^2\} \cup \{3^0, 4^1\}$.

In Example 7, the corresponding paths of type (I) for another colored union
of directed graphs Λ are also clarified.
Now, we consider the second type of deterministic paths given in (6).

Definition 2. We say that a *deterministic path* given in (6) is *of type (II)* if its last
vertex has access only to vertices in paths of type (I) (at least it has access to
one of them) and to a vertex in a deterministic circuit of Λ whose vertices are
not in A. This deterministic circuit is called a *deterministic circuit of type B*.
Hence, we denote by A^{II} the subset of vertices in $A \setminus \{A^{\text{I}}\}$ which are in some
deterministic path of type (II), and we denote by B the subset of vertices in
$V \setminus \{A\}$ which are in some deterministic circuit of type B.

In the case of the existence of two or more deterministic paths of type (II)
associated with one of the deterministic circuits, we may choose any of them.
The scheme in this case is given as follows:

$$A^{II}$$
$$\bullet \to \cdots \to \bullet \to \text{vertices in paths of type (I)}$$
$$\searrow$$
$$\bullet \to \cdots \to \bullet$$ \hspace{3cm} (7)
$$B$$
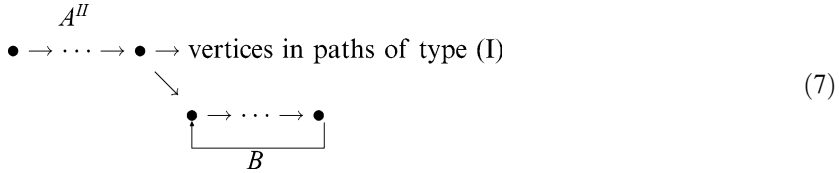
Finally, in A we have the resulting set:

Definition 3. We say that a *deterministic path* given in (6) is of *type (III)* if it is neither of type (I) nor type (II). Moreover, we denote $A^{III} = A \setminus \{A^I \cup A^{II}\}$.

Example 4. Note that for Example 3, we find the following deterministic paths of type (II) in Λ,

$$1^0 \to 5^1$$

because from the last vertex of this path, there exists an outgoing arc leading to the vertex $4^2 \in A^I$, and moreover, there exists another arc leading to a deterministic circuit, that is,

$$1^0 \to 5^1 \to 4^2$$
$$\searrow$$
$$1^2 \to 2^0 \to 3^1$$
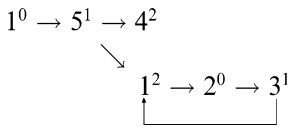
then $B = \{1^2, 2^0, 3^1\}$ and $A^{II} = \{1^0, 5^1\}$.

Therefore, the precedent provides the unique deterministic path of the type (III) in Λ:

$$2^2 \to 5^0 \to 1^1$$

and thus, $A^{III} = \{2^2, 5^0, 1^1\}$.

Remark 1. Note that the length of the deterministic paths of types (I)–(III), in general, is not a multiple of N. Therefore, they do not contain the same number of vertices of each color.

In the colored union of directed graphs Λ, other deterministic paths can be considered if we focus our attention on some special columns of the matrices $G(j)$, with $j = 0, \ldots, N - 1$. We consider nonmonomial columns of the matrices $G(j)$, which can be written as

$$\mathrm{col}\,G(j) = \tau e_{\gamma^j} + w, \quad \text{with } \gamma^j \notin A \cup B, \ \tau > 0 \hspace{2cm} (8)$$

and $w \neq 0$ where $w_{\gamma^j} = 0$ and w_{k^j} could be nonzero only if $k^j \in A^I$.

Definition 4. We say that a *deterministic circuit is of type C* if it contains some vertex γ^j such that condition (8) is held for a certain column of G. We denote by

C the subset of vertices in $V \setminus \{A \cup B\}$, which are in some deterministic circuit of type C.
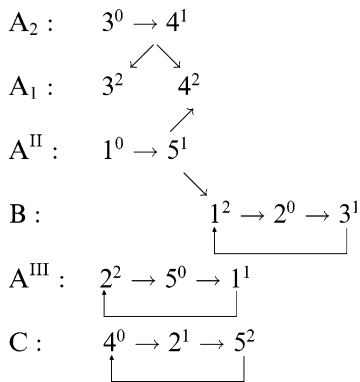
Remark 2. Note that the length of the deterministic circuits of types B and C is a multiple k of the period N. Moreover, each one of them contains k vertices of color j, for all $j = 0, 1, \ldots, N - 1$.

Example 5. Finally, for Example 1, we select the special columns in the matrices $G(j)$, $j = 0, 1, 2$, to obtain the vertices of the set C. We take col $G(0) = g_1^0 = \tau e_{4^0} + w$ with $\tau = 1 > 0$, $4^0 \notin A \cup B$ and $w_{k^0} = 0$ except to $k^0 = 3^0 \in A^I$. Hence, we may choose the deterministic circuit containing the vertex 4^0 which is given by

$$4^0 \to 2^1 \to 5^2$$

Therefore, $C = \{4^0, 2^1, 5^2\}$. Following Remark 2, note that the length of the deterministic circuits of the sets B and C is a multiple of the period $N = 3$. And, in accordance with Remark 1, for example, the path $1^0 \to 5^1$ does not contain a vertex of color 2.

Moreover, note that, from Examples 3–5, we can display the colored union of directed graphs of the positive 3-periodic system in Example 1 as follows:

$$
\begin{array}{ll}
A_2: & 3^0 \to 4^1 \\
 & \quad\swarrow\ \searrow \\
A_1: & 3^2 \qquad 4^2 \\
 & \qquad\quad\nearrow \\
A^{II}: & 1^0 \to 5^1 \\
 & \qquad\qquad\searrow \\
B: & \qquad\qquad 1^2 \to 2^0 \to 3^1 \\
 & \qquad\qquad\lfloor_____\rfloor \\
A^{III}: & 2^2 \to 5^0 \to 1^1 \\
 & \quad\lfloor_____\rfloor \\
C: & 4^0 \to 2^1 \to 5^2 \\
 & \quad\lfloor_____\rfloor
\end{array}
$$

Remark 3. Note that if the period N is equal to 1, the system $(F(\cdot), G(\cdot))_N$ is an invariant system (F, G), so the previous sets of vertices of the colored union of the directed graphs Λ, are equal to the sets of vertices of the directed graph of F, $\Gamma(F)$, defined in [3].

Using the definitions established in this section, we obtain the following characterization of the reachability property.

Theorem 1. *Let* $(F(\cdot), G(\cdot))_N \geqslant 0$ *be such that in matrix* $G(j)$, *there exist* r_j *different monomial columns corresponding to different unit vectors* $\{e_{i_1^j}, \ldots, e_{i_{r_j}^j}\}$, *with* $0 \leqslant r_j \leqslant n$, $j = 0, \ldots, N-1$. *Let* $\mathscr{I} = \bigcup_{j=0}^{N-1} \{i_1^j, \ldots, i_{r_j}^j\}$.

Then, $(F(\cdot), G(\cdot))_N$ *is reachable if and only if*

(i) *For all* $j = 0, 1, \ldots, N-1$, $[F(j)|G(j)]$ *has a monomial submatrix of order n.*

(ii) *The colored union of directed graphs* Λ *only contains deterministic paths of types* (I)–(III), *and deterministic circuits of types B and C.*

(iii) *The above paths and circuits cover the set of vertices V.*

Proof. We know that a positive N-periodic discrete-time linear system $(F(\cdot), G(\cdot))_N$ is reachable if and only if the invariant cyclically augmented system (F_e, G_e) is also reachable. Since (F_e, G_e) is an invariant system, using the characterization established in [3], it is satisfied that (F_e, G_e) is reachable if and only if $[F_e|G_e]$ has a monomial submatrix of order nN and the set of vertices $\{1, \ldots, nN\}$ of the directed graph of F_e is covered by the vertices of the sets A_e, B_e and C_e, that is $A_e \cup B_e \cup C_e = \{1, \ldots, nN\}$.

By constructing the matrices F_e and G_e from the matrices $F(j)$ and $G(j)$, $j = 0, 1, \ldots, N-1$, it is clear that the matrices $[F(j)|G(j)]$ for all $j = 1, \ldots, N-1$, have a monomial submatrix of order n if and only if the matrix $[F_e|G_e]$ has a monomial submatrix of order nN. Moreover, by the identification of vertices and arcs carried out in (3) and (4), the deterministic paths of type (I), (II) and (III) and the deterministic circuits of type B and C of the colored union of directed graphs Λ correspond to the deterministic paths and circuits of the directed graph of F_e, whose vertices are in the sets A_e, B_e and C_e. Thus, conditions (ii) and (iii) of this theorem hold if and only if $A_e \cup B_e \cup C_e = \{1, \ldots, nN\}$ hold too. $\square$

We shall now construct a canonical form of the reachability property under N-periodic collections of permutation matrices.

4. Canonical form of reachability

In this section, a canonical form of reachability shall be constructed. Given a reachable positive N-periodic system $(F(\cdot), G(\cdot))_N$, we shall find a positive similar system, $(\widehat{F}(\cdot), \widehat{G}(\cdot))_N$, under N-periodic collections of permutation matrices, $P(j)$ and $Q(j)$, $j \in \mathbb{Z}_+$, whose matrices $\widehat{F}(j)$ and $\widehat{G}(j)$, $j \in \mathbb{Z}_+$ have a special structure. Note that this last system is also reachable (see (5)).

We choose $P(j)$, for all $j = 0, 1, \ldots, N-1$, establishing a suitable relabelling of the vertices in $V = \bigcup_{j=0}^{N-1} \{1^j, \ldots, n^j\}$. Then, since the deterministic paths of Λ of types (I)–(III) and the deterministic circuits of Λ of types B and C cover the set of vertices V, we choose a relabelling of the vertices in V, as follows.

For setting a block superior triangular structure of the state matrix, first, we relabel the vertices which are in the paths of type (III), consecutively the vertices included in the paths of type (II) and type (I) (from the greater to the inferior level). Afterwards, we relabel the vertices in the deterministic circuits of type B and finally, the vertices in the deterministic circuits of type C. Moreover, in each kind of path (or circuit), we start to relabel the vertices in the longest deterministic paths (or deterministic circuits) and finish with the vertices in the shortest deterministic paths (or deterministic circuits).

We always relabel the vertices taking color into account. Each vertex of color j is relabelled in decreasing order in the set $\{n^j, \ldots, 1^j\}$. That is, the new label of each vertex of color j is the greater vertex of color j which has not yet been used. This relabelling provides the permutation matrices $P(j)$, $j = 0, 1, \ldots, N - 1$.

		RELABELLING		
	TYPE	LENGTH	COLOR	
PATH	(III) (II) (I), level $n-1$ $\vdots$ (I), level 1	In any case, take orderly deterministic paths and circuits from greater to inferior length	Each vertex k of color j is relabelled in decreasing order in the set $\{1^j, \ldots, n^j\}$ for all j	(9)
CIRCUIT	B C			

Moreover, we take the permutation matrices $Q(j)$, $j = 0, 1, \ldots, N - 1$, such that each $Q(j)$ places, as the first columns, the monomial columns of the matrix $G(j)$ in decreasing order, and next, the nonmonomial columns of the matrix $G(j)$ associated with the vertices of set C.

Hence, we construct two N-periodic collections of permutation matrices extending periodically the above permutation matrices, that is, $\{P(j) = P(j + N), j \in \mathbb{Z}_+\}$, $\{Q(j) = Q(j + N), j \in \mathbb{Z}_+\}$. Then, we consider the similar reachable positive N-periodic system $(\widehat{F}(\cdot), \widehat{G}(\cdot))_N$ given by

$$\widehat{F}(j) = P^{\mathrm{T}}(j)F(j)P(j - 1)$$
$$\widehat{G}(j) = P^{\mathrm{T}}(j)G(j)Q(j), \quad j \in \mathbb{Z}_+$$

which has the following structure:

$$[P^T(j)F(j)P(j-1)|P^T(j)G(j)Q(j)] = \left[\begin{array}{ccccc|c} \mathscr{C}_j & O & O & O & \Delta_j & \\ O & \mathscr{B}_j & O & \Sigma_j & \Delta_j & \\ O & O & \mathscr{A}_j^I & \Delta_j & \Delta_j & \mathscr{G}_j \\ O & O & O & \mathscr{A}_j^{II} & \Delta_j & \\ O & O & O & O & \mathscr{A}_j^{III} & \end{array}\right], \tag{10}$$

for all $j = 0, \ldots, N - 1$, where the blocks $\mathscr{A}_j^I$, $\mathscr{A}_j^{II}$, $\mathscr{A}_j^{III}$, $\mathscr{B}_j$, $\mathscr{C}_j$, are associated with the deterministic paths or circuits of type (I), (II), (III), B and C, respectively, and the blocks Δ_j and Σ_j are associated with the connections among the different deterministic paths or circuits.

Remark 4. Consider j, $j = 0, 1, \ldots, N - 1$. Each one of the blocks $\mathscr{A}_j^{\mathrm{II}}$, $\mathscr{A}_j^{\mathrm{III}}$, $\mathscr{B}_j$ and $\mathscr{C}_j$ has, at the same time, a structure of blocks, where each diagonal block corresponds to a deterministic path or circuit of types (II), (III), B and C, respectively, and the remaining blocks correspond to the relations among the deterministic paths or circuits of the same type.

However, in block $\mathscr{A}_j^{\mathrm{I}}$, the kth diagonal block, for all $k = 1, \ldots, n$, corresponds to all the paths of type (I) of level k, and the remaining blocks correspond to the relations among the paths of type (I) of the different levels. Then, the kth diagonal block of $\mathscr{A}_j^{\mathrm{I}}$ has a structure of blocks, where each diagonal block corresponds to a deterministic path of type (I) of level k.

Remark 5. The corresponding block to a deterministic path or circuit of types (I)–(III), B or C, has as many columns (rows) as vertices of color $j - 1$ (j) contains such path or circuit. Then, following Remark 1, each block of $\widehat{F}(j)$ associated with a path of types (I)–(III) is not necessarily a square block. Even in some cases, these blocks do not appear. However, following Remark 2, each block of $\widehat{F}(j)$ associated with a deterministic circuit of type B and C is a square block.

Remark 6. Given an index j, $j = 0, 1, \ldots, N - 1$, each arc (k^{j-1}, l^j) contained in a deterministic path or circuit of Λ (of a certain type), provides a l-monomial vector in the kth column of the corresponding block.

Moreover, if a deterministic path or circuit of Λ (of a certain type) finishes in a vertex t^{j-1}, then the tth column of the corresponding block is a nonnegative vector. Note that each positive entry of this vector is associated with an arc from t^{j-1} to a suitable vertex of color j.

Remark 7. Following these remarks, note that each deterministic path or circuit of certain type has a block associated of the following kind. If it does not finish in a vertex of color $j - 1$ then the associated block is a monomial matrix and if it finishes in a vertex of color $j - 1$, then the associated block is given by $[v|M]$ where v is a nonnegative vector and M is a monomial matrix.

Next, we analyze every block appeared in (10).

(i) It is known that the deterministic circuits of type B (C) are not mutually accessible, then following Remark 4, $\mathscr{B}_j$ ($\mathscr{C}_j$) has a block diagonal structure. Moreover, following Remark 2 and the relabelling given in (9), the vertices in a deterministic circuit of type B (C) are relabelled in a consecutive way. For example, for period $N = 3$ and length of deterministic circuit $2N = 6$,

$$k^h \to k^{h+1} \to k^{h+2} \to (k-1)^h \to (k-1)^{h+1} \to (k-1)^{h+2} \tag{11}$$

Therefore, we find that the structure of the diagonal blocks of $\mathscr{B}_j$ ($\mathscr{C}_j$) depending on whether this deterministic circuit starts in a vertex of color j or not. If the deterministic circuit of type B (C) starts in a vertex of color j, then the block has associated a cyclic irreducible matrix, that is,

$$
\begin{bmatrix}
0 & + & 0 & \ldots & 0 \\
0 & 0 & + & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & + \\
+ & 0 & 0 & \ldots & 0
\end{bmatrix}
\tag{12}
$$

where $+$ denotes a strictly positive entry. If the deterministic circuit does not start in a vertex of color j, then the associated block is a diagonal matrix with strictly positive entries on the diagonal.

(ii) Following Remark 4 and since the path of type (I) of level k can has access only to paths of type (I) of a level less than k, then the matrix $\mathscr{A}_j^{\mathrm{I}}$ has an upper triangular block structure. That is,

$$
\mathscr{A}_j^{\mathrm{I}} =
\begin{bmatrix}
\mathscr{A}_{1j} & \Delta_{2j} & \cdots & \Delta_{n-1j} \\
O & \mathscr{A}_{2j} & \cdots & \Delta_{n-1j} \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & \mathscr{A}_{n-1j}
\end{bmatrix}
\tag{13}
$$

Moreover, following Remark 4, each block $\mathscr{A}_{kj}^{\mathrm{I}}$ has a block diagonal structure, since there are not connections among the different paths of type (I) in the same level. Following Remark 7, each diagonal block of $\mathscr{A}_{kj}^{\mathrm{I}}$ is a monomial matrix, M, if the corresponding path does not finish in a vertex of color $j-1$, and is $[O|M]$, where O represents a zero vector and M, a monomial matrix, if it finishes in a vertex of color $j-1$.

Note that each block Δ_{kj} is given by

$$
[D|D|\cdots|D]
\tag{14}
$$

where each block D corresponds to a path of type (I) of level k. Further, D is a zero matrix if the corresponding path does not finish in a vertex of color $j-1$ and it is $[v|O]$, where v is a nonnegative vector, if it finishes in a vertex of color $j-1$.

(iii) Following Remark 4 and since the paths of type (II) do not have mutual access, then the matrix $\mathscr{A}_j^{\mathrm{II}}$ has a block diagonal structure. Moreover, following Remark 7, each diagonal block is a monomial matrix, M, if the corresponding path of type (II) does not finish in a vertex of color $j-1$, and it is $[O|M]$, where O represents a zero vector and M, a monomial matrix, if it finishes in a vertex of color $j-1$.

The matrix Δ_j (Σ_j) of the canonical form, in the same block column as $\mathscr{A}_j^{\mathrm{II}}$, shows the relationship between paths of type (II) and the others of type (I) (deterministic circuits of type B).

Moreover, note that each block Δ_j (Σ_j) is given in (14), where D is a zero matrix if the corresponding path does not finish in a vertex of color $j - 1$ and it is $[v|O]$, where v is a nonnegative vector (monomial vector), if it finishes in a vertex of color $j - 1$.

(iv) Following remark, $\mathscr{A}_j^{\mathrm{III}}$ is structured by blocks. Following Remark 7, the kth diagonal block (a block in the kth block column out of the diagonal) is a monomial matrix, M (a zero matrix) if the corresponding path of type (III) does not finish in a vertex of color $j - 1$, and it is $[v|M]$ ($[v|O]$), where v represents a nonnegative vector and M a monomial matrix, if it finishes in a vertex of color $j - 1$.

A matrix Δ_j of the canonical form, in the same block column as $\mathscr{A}_j^{\mathrm{III}}$, shows the relationship between the paths of type (III) and the other type of deterministic paths or circuits. Moreover, note that Δ_j is given in (14), where D is a zero matrix if the corresponding path does not finish in a vertex of color $j - 1$ and it is $[v|O]$, where v is a nonnegative vector, if it finishes in a vertex of color $j - 1$.

(v) Finally, considering the permutation matrices $Q(j)$, $j = 0, 1, \ldots, N - 1$, in such a way that the first columns of the matrix $\mathscr{G}_j$ are, in decreasing order, the monomial columns of the matrix $G(j)$ and the following columns are associated with the deterministic circuits of type C (the position in the matrix depends on the relabelling carried out).

Moreover, it is easy to verify that the similar system $(\widehat{F}(\cdot), \widehat{G}(\cdot))_N$, where $[\widehat{F}(j)|\widehat{G}(j)]$ is given in (10), also is reachable.

To summarize, we establish the following theorem.

Theorem 2. *The system $(F(\cdot), G(\cdot))_N \geqslant 0$ is reachable if and only if there exist two N-periodic collections of permutation matrices $\{P(j), j \in \mathbb{Z}_+\}$ and $\{Q(j), j \in \mathbb{Z}_+\}$, such that the matrix $[P(j)^{\mathrm{T}}F(j)P(j-1)|P^{\mathrm{T}}(j)G(j)Q(j)]$ for each $j \in \mathbb{Z}_+$ is given in (10), where the blocks $\mathscr{C}_j$, $\mathscr{B}_j$, $\mathscr{A}_j^{\mathrm{II}}$, Δ_j, Σ_j, $\mathscr{A}_j^{\mathrm{I}}$, $\mathscr{A}_j^{\mathrm{III}}$ and $\mathscr{G}_j$ are given in* (i)–(v).

Example 6. Consider the positive $N = 3$-periodic system $(F(\cdot), G(\cdot))_N$ given in Example 1. We construct the permutation matrices $\{P(0), P(1), P(2)\}$, $\{Q(0), Q(1), Q(2)\}$ and then we extend them periodically which provide the structure of the canonical form of reachability of this periodic system.

We obtain matrices $P(j)$, $j = 0, 1, 2$, by relabelling the vertices of V in the different deterministic paths or circuits of Λ. We start with vertices in paths of type (III), the unique path in this case is

$$2^2 \to 5^0 \to 1^1$$

then

$$2^2 \rightsquigarrow 5^2$$
$$5^0 \rightsquigarrow 5^0$$
$$1^1 \rightsquigarrow 5^1$$

Consecutively, vertices in deterministic paths of type (II),

$$1^0 \rightsquigarrow 4^0$$
$$5^1 \rightsquigarrow 4^1$$

For deterministic paths of type (I), we find the following vertices of the set V

$$3^0 \rightsquigarrow 3^0 \quad \text{(first, vertices in paths of longer length)}$$
$$4^1 \rightsquigarrow 3^1$$
$$3^2 \rightsquigarrow 4^2$$
$$4^2 \rightsquigarrow 3^2$$

Finally, we end with vertices in the deterministic circuits of types B and C, thus

$$1^2 \rightsquigarrow 2^2 \quad 4^0 \rightsquigarrow 1^0$$
$$2^0 \rightsquigarrow 2^0 \quad 2^1 \rightsquigarrow 1^1$$
$$3^1 \rightsquigarrow 2^1 \quad 5^2 \rightsquigarrow 1^2$$

Therefore, we know that

$$P(0) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad P(1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$P(2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Moreover, using

$$Q(0) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q(2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

we construct the canonical form associated with this system,

$$[\widehat{F}(1)|\widehat{G}(1)] = [P^T(1)F(1)P^T(0)|P^T(1)G(1)Q(1)]$$

$$
= \begin{bmatrix}
\mathscr{C}_1 & O & O & O & \Delta_1 \\
O & \mathscr{B}_1 & O & \Sigma_1 & \Delta_1 \\
O & O & \mathscr{A}_1^I & \Delta_1 & \Delta_1 \\
O & O & O & \mathscr{A}_1^{II} & \Delta_1 \\
O & O & O & O & \mathscr{A}_1^{III}
\end{bmatrix} \mathscr{G}_1 =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
$$

$$[\widehat{F}(2)|\widehat{G}(2)] = [P^T(2)F(2)P^T(1)|P^T(2)G(2)Q(2)]$$

$$
= \begin{bmatrix}
\mathscr{C}_2 & O & O & O & \Delta_2 \\
O & \mathscr{B}_2 & O & \Sigma_1 & \Delta_2 \\
O & O & \mathscr{A}_2^I & \Delta_1 & \Delta_2 \\
O & O & O & \mathscr{A}_2^{III}
\end{bmatrix} \mathscr{G}_2 =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
$$

$$[\widehat{F}(0)|\widehat{G}(0)] = [P^T(0)F(0)P^T(2)|P^T(0)G(0)Q(0)]$$

$$
= \begin{bmatrix}
\mathscr{C}_0 & O & O & \Delta_0 \\
O & \mathscr{B}_0 & O & \Delta_0 \\
O & O & \mathscr{A}^I & \Delta_0 \\
O & O & O & \Delta_0 \\
O & O & O & \mathscr{A}_0^{III}
\end{bmatrix} \mathscr{G}_0 =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

5. Complete controllability

Suppose that $(F(\cdot), G(\cdot))_N \geqslant 0$ is completely controllable. Thus, the positive invariant cyclically augmented system (F_e, G_e) is also completely controllable, that is, it is reachable and F_e is nilpotent. [3] proved that $(F_e, G_e) \geqslant 0$ is completely controllable if and only if the matrix $[F_e|G_e]$ has a monomial submatrix of order nN and the set A_e^1 is equal to the set $\{1, 2, \ldots, nN\}$. Hence, as in Theorem 1, we can obtain the following result.

Theorem 3. *Let* $(F(\cdot), G(\cdot))_N \geqslant 0$ *be such that in the matrix* $G(j)$, *there exist* r_j *different monomial columns corresponding to different unit vectors* $\{e_{i_1^j}, \ldots, e_{i_{r_j}^j}\}$, *with* $0 \leqslant r_j \leqslant n$, $j = 0, \ldots, N - 1$. *Let* $\mathscr{I} = \bigcup_{j=0}^{N-1} \{i_1^j, \ldots, i_{r_j}^j\}$. *Then,* $(F(\cdot), G(\cdot))_N$ *is completely controllable if and only if*

  (i) *For all* $j = 0, 1, \ldots, N - 1$, $[F(j)|G(j)]$ *has a monomial submatrix of order* n,
 (ii) *The colored union of directed graphs* Λ *contains only deterministic paths of type* (I),
(iii) *These paths cover the set of vertices* V.

Now, we construct a canonical form of complete controllability as follows.

Using Theorem 3, the colored union of directed graphs Λ is only formed by deterministic paths of type (I). Using the same method to relabel the vertices as in the proof of Theorem 2, we obtain two N-periodic collections of permutation matrices $\{P(j), j \in \mathbb{Z}_+\}$ and $\{Q(j), j \in \mathbb{Z}_+\}$, such that the similar system $(\widehat{F}(\cdot), \widehat{G}(\cdot))_N$ to $(F(\cdot), G(\cdot))_N$ given by

$$\widehat{F}(j) = P^{\mathrm{T}}(j)F(j)P(j-1),$$

$$\widehat{G}(j) = P^{\mathrm{T}}(j)G(j)Q(j), \; j \in \mathbb{Z}_+$$

is completely controllable and $\widehat{F}(j) = \mathscr{A}_j^{\mathrm{I}}$ where $\mathscr{A}_j^{\mathrm{I}}$ is the block matrix associated with the path of type (I), given in (13). Then, we establish the following result.

Theorem 4. *The system $(F(\cdot), G(\cdot))_N \geqslant 0$ is completely controllable if and only if there exist two N-periodic collections of permutation matrices $\{P(j), j \in \mathbb{Z}_+\}$ and $\{Q(j), j \in \mathbb{Z}_+\}$, such that for each $j \in \mathbb{Z}_+$,*

$$[P^{\mathrm{T}}(j)F(j)P(j-1)|P^{\mathrm{T}}(j)G(j)Q(j)] = \left[\mathscr{A}_j^{\mathrm{I}}|\mathscr{G}_j\right] \tag{15}$$

where $\mathscr{A}_j^{\mathrm{I}}$ and $\mathscr{G}_j$ are explained as in (iii) and (v), respectively.

**Example 7.** Consider a positive 3-periodic system given by

$$[F(0)|G(0)] = \left[\begin{array}{cccccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$[F(1)|G(1)] = \left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$[F(2)|G(2)] = \left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

and $F(k+3) = F(k)$ and $G(k+3) = G(k)$, $\forall k \in \mathbb{Z}_+$.

By definition, we have $\mathscr{I} = \{1^0, 2^0, 3^0, 5^0\} \cup \{1^1, 4^1, 5^1\} \cup \{1^2, 5^2\}$. Moreover, we know that the colored union of directed graphs Λ is given by
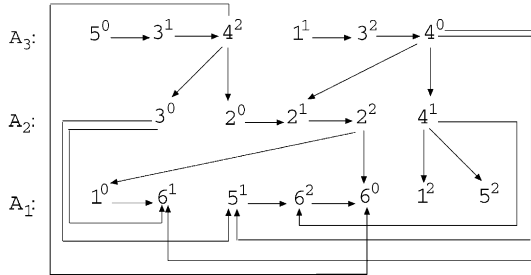
By definition, it is easy to check that all of them are deterministic paths of type (I). In this case, the subset of level 1 is $A_1 = \{1^0, 6^0, 5^1, 6^1, 1^2, 5^2, 6^2\}$. The subset of level 2, A_2, is formed by vertices in deterministic paths such that from the last vertex all their arcs extend to vertices in level 1, then $A_2 = \{2^0, 3^0, 2^1, 4^1, 2^2\}$. Thus, the subset of level 3 is $A_3 = \{4^0, 5^0, 1^1, 3^1, 3^2, 4^2\}$. Moreover, $A_4 = A_5 = \emptyset$. Thus, $A^{\mathrm{I}} = A_1 \cup A_2 \cup A_3$.

Now, we construct the canonical form of controllability choosing suitable permutation matrices $\{P(0), P(1), P(2)\}$, $\{Q(0), Q(1), Q(2)\}$ and afterwards, we extend them periodically which provides the structure of the canonical form of reachability of this periodic system.

We take the matrices $P(j)$, $j = 0, 1, 2$, from the relabelling, in decreasing order, of the vertices of V, depending on the level in which they are found in the paths of type (I). We start with colored vertices in the higher level. For this system, in level 3, we have the following paths

$$5^0 \rightarrow 3^1 \rightarrow 4^2$$
$$1^1 \rightarrow 3^2 \rightarrow 4^0$$

then

$$5^0 \rightsquigarrow 6^0 \quad 1^1 \rightsquigarrow 5^2$$
$$3^1 \rightsquigarrow 6^1 \quad 3^2 \rightsquigarrow 5^2$$
$$4^2 \rightsquigarrow 6^2 \quad 4^0 \rightsquigarrow 5^0$$

In level 2, we find

$$2^0 \rightarrow 2^1 \rightarrow 2^2$$
$$3^0$$
$$4^1$$

Then, relabelling, taking the length of paths into account, we obtain

$$2^0 \rightsquigarrow 4^0 \quad 3^0 \rightsquigarrow 3^0$$
$$2^1 \rightsquigarrow 4^1 \quad 4^1 \rightsquigarrow 3^1$$
$$2^2 \rightsquigarrow 4^2$$

Finally, we relabel the vertices in paths of level 1. The deterministic paths of this level are

$$5^1 \rightarrow 6^2 \rightarrow 6^0$$
$$1^0 \rightarrow 6^1$$
$$1^2$$
$$5^2$$

Therefore,

$$5^1 \rightsquigarrow 2^1 \quad 6^1 \rightsquigarrow 1^1$$
$$6^2 \rightsquigarrow 3^2 \quad 1^2 \rightsquigarrow 2^2$$
$$6^0 \rightsquigarrow 2^2 \quad 5^2 \rightsquigarrow 1^2$$
$$1^0 \rightsquigarrow 1^0$$

By means of the above relabelling, we know that

$$P(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P(1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P(2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Moreover, with

$$Q(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad Q(1) = Q(2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we construct the canonical form of complete controllability associated with this system,

$$[\widehat{F}(0)|\widehat{G}(0)] = [P^T(0)F(0)P^T(2)|P^T(0)G(0)Q(0)] = \left[\mathscr{A}_0^I|\mathscr{G}_0\right]$$

$$= \left[\begin{array}{ccc|c|c}\mathscr{A}_{10} & \Delta_0 & \Delta_0 \\ O & \mathscr{A}_{20} & \Delta_0 \\ O & O & \mathscr{A}_{30}\end{array}\middle|\mathscr{G}_0\right] = \left[\begin{array}{ccc|c|cc|cccc}0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\end{array}\right]$$

$$[\widehat{F}(1)|\widehat{G}(1)] = [P^T(1)F(1)P^T(0)|P^T(1)G(1)Q(1)] = \left[\mathscr{A}_1^I|\mathscr{G}_1\right]$$

$$= \left[\begin{array}{c|c|c|c}\mathscr{A}_{11} & \Delta_1 & \Delta_1 \\ O & \mathscr{A}_{21} & \Delta_1 \\ O & O & \mathscr{A}_{31}\end{array}\middle|\mathscr{G}_1\right] = \left[\begin{array}{cc|cc|cc|cc|cc}1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\end{array}\right]$$

$$[\widehat{F}(2)|\widehat{G}(2)] = [P^T(2)F(2)P^T(1)|P^T(2)G(2)Q(2)] = \left[\mathscr{A}_2^I|\mathscr{G}_2\right]$$

$$= \left[\begin{array}{c|c|c|c}\mathscr{A}_{12} & \Delta_2 & \Delta_2 \\ O & \mathscr{A}_{22} & \Delta_2 \\ O & O & \mathscr{A}_{32}\end{array}\middle|\mathscr{G}_2\right] = \left[\begin{array}{cc|cc|cc|cccc}0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\end{array}\right]$$

with $\widehat{F}(j) = P^T(j)F(j)P(j-1)$ and $\widehat{G}(j) = P^T(j)G(j)Q(j)$ if $j = 0, 1, 2$.

Acknowledgements

References

[1] R. Bru, V. Hernández, Structural properties of discrete-time linear positive periodic systems, Linear Algebra and its Applications 121 (1989) 171–183.
[2] R. Bru, C. Coll, V. Hernández, E. Sánchez, Geometrical conditions for the reachability and realizability of positive periodic discrete systems, Linear Algebra and its Applications 256 (1997) 109–124.

[3] R. Bru, S. Romero, E. Sánchez, Canonical forms for positive discrete-time linear control systems, Linear Algebra and its Applications 310 (2000) 49–71.

[4] P.G. Coxson, L.C. Larson, H. Schneider, Monomial patterns in the sequence $A^k b$, Linear Algebra and its Applications 94 (1987) 89–101.

[5] L. Caccetta, V.G. Rumchev, Reachable discrete-time positive systems with minimal dimension control sets, Dynamics of Continuous, Discrete and Impulsive Systems 4 (1998) 539–552.

[6] P.G. Coxson, H. Shapiro, Positive input reachability and controllability of positive systems, Linear Algebra and its Applications 94 (1987) 35–53.

[7] D.F. Delchamps, State-Space and Input–Output Linear Systems, Springer Verlag, New York, 1988.

[8] T. Kailath, Linear Systems, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[9] B. Park, E.I. Verriest, Canonical forms on discrete linear periodically time-varying systems and a control application, in: Proceedings of the 28th IEEE Conference on Decision and Control, Tampa, Florida, 1989, pp. 1220–1225.

[10] V.G. Rumchev, D.J.G. James, Spectral characterization and pole assignment for positive linear discrete-time systems, International Journal Systems Science 26 (2) (1995) 295–312.

[11] V.G. Rumchev, D.J.G. James, Controllability of positive linear discrete-time systems, International Journal of control 50 (3) (1989) 845–857.

[12] E.D. Sontag, Mathematical Control Theory: Deterministic Finite Dimensional Systems, Springer Verlag, New York, 1991.

[13] J. Sreedhar, P. Van Dooren, B. Bamieh, Computing H_∞-norm of discrete-time periodic systems—a quadratically convergent algorithm, Proceedings of European Control Conference 6 (1997).

[14] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[15] I. Zaballa, Interlacing and majorization in invariant factor assignment problems, Linear Algebra and its Applications 122 (1989) 409–421.