Existence of hypercyclic polynomials on complex Fréchet spaces

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Abstract

We show that every complex separable infinite dimensional Fréchet space admits hypercyclic polynomials of any degree. This result complements the analogous one for the linear case, due to Ansari, Bernal, Bonet and Peris.

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A continuous and linear operator T on a Banach space X is said to be hypercyclic if there exists a vector $x \in X$ whose orbit $\{x, Tx, T^2x, \dots\}$ is dense in X. Independently, S. Ansari [1] and L. Bernal-González [6] proved that every separable infinite dimensional Banach space admits a hypercyclic operator, answering thus a question posed by Rolewicz [26] in the late sixties. Later, J. Bonet and A. Peris [12] were able to prove it for Fréchet spaces (complete and metric locally convex spaces).

Concerning the hypercyclicity of polynomials, N. Bernardes [8] showed that for $d > 1$ there are no continuous d-homogeneous hypercyclic polynomials on any Banach space. On the other hand, it is known that there are chaotic d-homogeneous polynomials on the countable product of lines $\omega := \mathbb{C}^{\mathbb{N}}$, endowed with the product topology (a separable Fréchet space); and also there are chaotic non-homogeneous polynomials on the sequence Banach space ℓ^p (see [24] and [25]). Some examples of homogeneous hypercyclic and chaotic polynomials on spaces of continuous functions can be found in [2]. It is the aim of this note to show that hypercyclic polynomials of any degree greater than 1 may be defined on any complex separable infinite dimensional Fréchet space.

Contrary to the linear case, in which the are "computable" criteria for hypercyclicity and chaos [19, 9, 7, 5, 16, 14], mixing [15, 4], frequent hypercyclicity [3, 21, 13] and disjoint hypercyclicity [10], no criteria is available for the hypercyclic or chaotic behaviour of polynomials. This fact shows the difficulties to obtain hypercyclic polynomials on Banach or Fréchet spaces.

Our construction is partially based in the ideas of [12], in which one needs certain biorthogonal sequence $(x_i, f_i)_{i\in\mathbb{N}} \subset E \times E'$ to construct the desired polynomial.

Our notation is standard; we refer to the monograph [23] for Fréchet spaces and to [18] for polynomials on locally convex spaces.

Theorem. *Every separable infinite dimensional Fréchet space E admits a hypercyclic polynomial of arbitrary degree.*

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Proof. We know the result is true for ω [24]; assume thus $E \neq \omega$. In this case case there are a bounded sequence $(x_i)_i \subset E$ and an equicontinuous sequence $(f_i)_i \subset E'$ such that $\overline{\operatorname{span}\{x_i \ / \ i \in \mathbb{N}\}} = E$, $\langle x_i, f_j \rangle = 0$ if $i \neq j$, and $(\alpha_i)_i := (\langle x_i, f_i \rangle)_i$ is a decreasing sequence in $]0, 1[$ (see [12]).

Now, given a positive integer d and a decreasing sequence of positive weights $(v_i)_i$ such that

$$\sum_{i=1}^{\infty} \frac{v_{i+1}}{v_i} \frac{1}{\alpha_{i+1}^{d+1}} < \infty,$$

we define the polynomial on E

$$P(x) := x + \sum_{i=1}^{\infty} x_i \frac{v_{i+1}}{v_i} p_i(x),$$

where

$$p_i(x) := \left(\alpha_{i+1}^{-1} \langle x, f_{i+1} \rangle + v_i \alpha_i^{-1} \alpha_{i+1}^{-1} \langle x, f_i \rangle \langle x, f_{i+1} \rangle \right) \left(\sum_{j=0}^{d-1} (v_{i+1} \alpha_{i+1}^{-1} \langle x, f_{i+1} \rangle + 1)^j \right).$$

It is clear that

$$p_i(x) = \sum_{j=1}^{d+1} q_{i,j}(x)$$

where $q_{i,j} : E \to \mathbb{C}$ is a continuous j-homogeneous polynomial and, by the equicontinuity of $(f_i)_i$ and the properties of $(\alpha_i)_i$, there exists a 0-neighbourhood U such that $\left| q_{i,j}(x) \right| \leq \alpha_{i+1}^{-j}$ for all $x \in U$ and for every $i \in \mathbb{N}$, $j \in \{1, \ldots, d+1\}$. The selection of $(x_i)_i$ and $(v_i)_i$ shows that

$$Q_j(x) := \sum_{i=1}^{\infty} x_i \frac{v_{i+1}}{v_i} q_{i,j}(x), \ x \in E$$

defines a continuous j-homogeneous polynomial $Q_j : E \to E$ for $j = 1, \ldots d+1$. Therefore

$$P(x) = x + \sum_{j=1}^{d+1} Q_j(x)$$

is a polynomial of degree $d+1$ on E.

We claim that $P : E \to E$ is hypercyclic. In order to prove this, we first consider the polynomial

$$Q : \ell^1(1/v) \to \ell^1(1/v) \ : \ (z_i)_i \mapsto ((z_i + 1)(z_{i+1} + 1)^d - 1)_i,$$

where $\ell^1(1/v)$ is the weighted ℓ^1-space associated to $1/v := (1/v_i)_i$. Let B be the backward shift $B((x_i)_i) := (x_{i+1})_i$ defined on $\ell^1(1/v)$. Observe that the decreasing condition imposed to $(v_i)_i$ ensures continuity of B and Q. We may consider the following diagram

$$\begin{array}{ccc} \ell^1(1/v) & \xrightarrow{T} & \ell^1(1/v) \\ \varphi \big\downarrow & & \varphi \big\downarrow \\ \ell^1(1/v) & \xrightarrow{Q} & \ell^1(1/v) \end{array}$$

where $T := I + dB$ and $\varphi((z_i)_i) := (e^{z_i} - 1)_i$. We have $Q \circ \varphi = \varphi \circ T$, T is hypercyclic [27], and φ is a continuous map with dense range. Then Q is hypercyclic on $\ell^1(1/v)$ (see [22, Lemma 2.1]).

We now define the polynomial $\overline{Q} := D_v^{-1} \circ Q \circ D_v : \ell^1 \to \ell^1$, where $D_v : \ell^1 \to \ell^1(1/v)$ is the usual diagonal isomorphism $(z_i)_i \mapsto (v_i z_i)_i$. An easy computation shows that

$$\overline{Q}((z_i)_i) = \left((z_i + \frac{1}{v_i})(v_{i+1} z_{i+1} + 1)^d - \frac{1}{v_i} \right)_i$$

and, obviously, it is also hypercyclic on ℓ^1.

Next, we set $\phi : \ell^1 \to E$ given by $(z_i)_i \mapsto \sum_{i=1}^{\infty} z_i x_i$ and we easily get that ϕ is a well-defined operator with dense range. Finally, to show the hypercyclicity of P we check that the equality $P \circ \phi = \phi \circ \overline{Q}$ holds. Consider $(z_i)_i \in \ell^1$ and let $x := \sum_{i=1}^{\infty} z_i x_i$. On one hand we have

$$\begin{aligned}
P(\phi((z_i)_i)) &= P(\sum_{i=1}^{\infty} z_i x_i) = \sum_{i=1}^{\infty} z_i x_i + \sum_{i=1}^{\infty} x_i \frac{v_{i+1}}{v_i} p_i(x) \\
&= \sum_{i=1}^{\infty} z_i x_i + \sum_{i=1}^{\infty} x_i \frac{v_{i+1}}{v_i} (z_{i+1} + v_i z_i z_{i+1}) \left(\sum_{j=0}^{d-1} (v_{i+1} z_{i+1} + 1)^j \right) \\
&= \sum_{i=1}^{\infty} \left(z_i + \left(z_i + \frac{1}{v_i} \right) ((v_{i+1} z_{i+1} + 1) - 1) \left(\sum_{j=0}^{d-1} (v_{i+1} z_{i+1} + 1)^j \right) \right) x_i \\
&= \sum_{i=1}^{\infty} \left(z_i + \left(z_i + \frac{1}{v_i} \right) \left((v_{i+1} z_{i+1} + 1)^d - 1 \right) \right) x_i = \sum_{i=1}^{\infty} \left(\left(z_i + \frac{1}{v_i} \right) (v_{i+1} z_{i+1} + 1)^d - \frac{1}{v_i} \right) x_i.
\end{aligned}$$

On the other hand

$$\phi(\overline{Q}((z_i)_i)) = \sum_{i=1}^{\infty} \left((z_i + \frac{1}{v_i})(v_{i+1} z_{i+1} + 1)^d - \frac{1}{v_i} \right) x_i = P(\phi((z_i)_i)).$$

This completes the proof $\square$

We would like to make the following observation concerning the dynamics of P: We showed that the hypercyclicity of P is deduced, by a commutative diagram, from the one of the linear operator $T = I + dB$ on $\ell^1(1/v)$ by Salas's result [27]. Actually, Grivaux showed that $I + dB$ is mixing [20, Lemma 2.3]. Therefore P is also mixing, and we obtain that every separable infinite dimensional Fréchet space E admits a mixing polynomial of arbitrary degree.

According to Devaney, a continuous map on a metric space is said to be chaotic if it is topologically transitive and admits a dense set of periodic points [17]. Transitivity is equivalent to hypercyclicity in our setting, but our ideas in the previous proof fail to provide chaotic polynomials since compact perturbations of the identity are never chaotic [22]. In this situation the following question arises:

Problem: Does every separable infinite dimensional Fréchet space admit a chaotic polynomial of degree $d > 1$?

The corresponding question for operators has a negative answer: There are separable infinite dimensional Banach spaces that admit no chaotic operator [11]. Our conjecture is that the answer to the above question is also negative, in general.

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