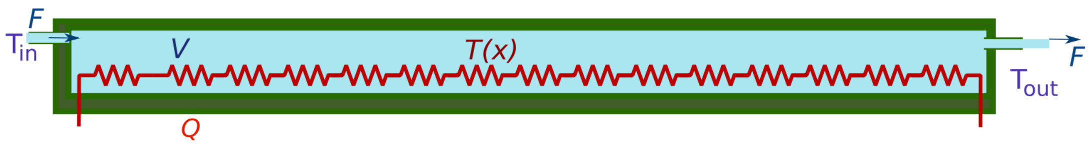
Modelling of a tubular heating as Partial Differential Equations (PDE)

Presentation in video: http://personales.upv.es/asala/YT/V/termedpEN.html

Objetive: Modelling with "finite elements" the dynamics of a tubular heater with a resistor along it heating an incompressible fluid, and end up with a Partial Differential Equation (PDE) when the elements become infinitesimal.



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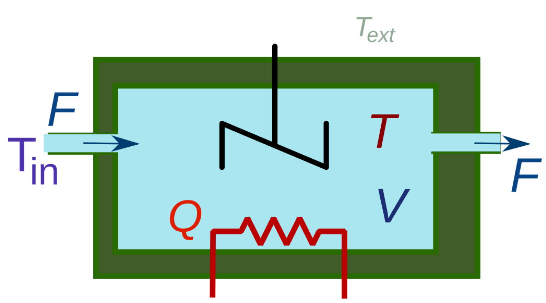
First-principle model

We'll consider a resistor (heating power Q, known, input signal to the system) which heats a liquid flowing into (and out) of a tank of a given volume. A perfect stirrer will very quickly (supposedly) make temperature to be "uniform" in all volume so a 1st-order setup will be enough for the moment being.



- Inputs:

```
syms F real %Input and output flow (incompressible fluid)
syms Tin real %Input temperature
syms Q real %Resistor's heating power
```

- Constant parameters:

```
syms V real%Tank volume
syms rho real %density
syms kappa real %thermal losses through tank's walls
```

We'll assume outside temperature equal to zero, constant, to avoid needing it in the model (there is no loss of generality as long as it is constant); We'll assume that kappa does not change with F

```
syms c real %Specific heat (mass, in say W/Kg/K)
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- State variable:

```
syms T  real % temperature of the liquid inside the tank (equal to the output temperatu
syms dTdt real % time derivative of the temperature (state variable)
```

Power (rate of change of energy) balance is

" *rate of change of energy inside control volume* $\left(\frac{dE}{dt}=Mc\frac{dT}{dt}\right)$, mass is constant inside the control volume so dM/dt is not considered inside the volume.

= *net heat power exchange with the outside environment* $(Q-\kappa T)$

+ *total energy entering the control volume per unit time due to incoming fluid* $+\frac{dM_{in}}{dt}cT_{in}=(+F\rho cT_{in})$

- *total energy per unit time leaving the control volume due to outgoing fluid* $-\frac{dM_{out}}{dt}cT_{out}=(-F\rho cT)$"

which is written as:

$$\underbrace{V\rho}\,c\frac{dT}{dt}=\underline{F\rho}cT_{in}-\underline{F\rho}cT-\kappa T+Q$$

Nota: $V\rho=Mass$, $F\rho=$mass flow $\dot{m}$; the term $\dot{m}cT$ has dimensions of power (enthalpy flow rate), we have incoming power $\dot{m}cT_{in}$ and outgoing power $\dot{m}cT$.

For simplicity, heat transfer around the tank's boundary is modelled as a constant times temperature (actually temperature increment with respect to outside one), but maybe there are convection coefficients wich might depend on flow F, say, a first approximation such as $\kappa=\kappa_0+\kappa_1F$. In our case κ_1 is neglected.

At the end, we have a model with a single equation. If we enter it in the Symbolic toolbox:

```
Model=  V*rho*c*dTdt == F*rho*c*Tin - F*rho*c*T - kappa*T+ Q;
```

Normalised Internal state-space representation amounts to solving for the time derivative of the state:

```
dTdt_sym=simplify(solve(Model,dTdt),50)
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dTdt_sym =
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$$\frac{Q - T\,\kappa}{V\,c\,\rho} - \frac{F\,(T - \mathrm{Tin})}{V}$$

We may write it as

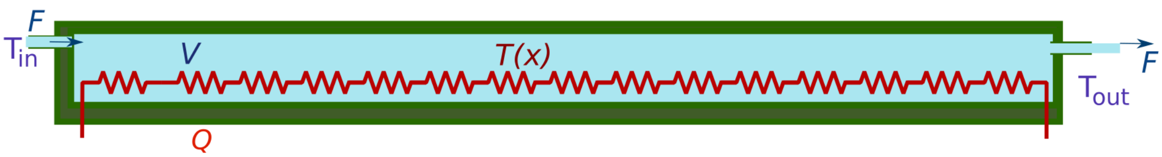
$$\frac{dT}{dt} = -\frac{F}{V} \cdot (T - T_{in}) - aT + bQ,$$

with $a = \frac{\kappa}{V\rho c}, \quad b = \frac{1}{V\rho c}.$

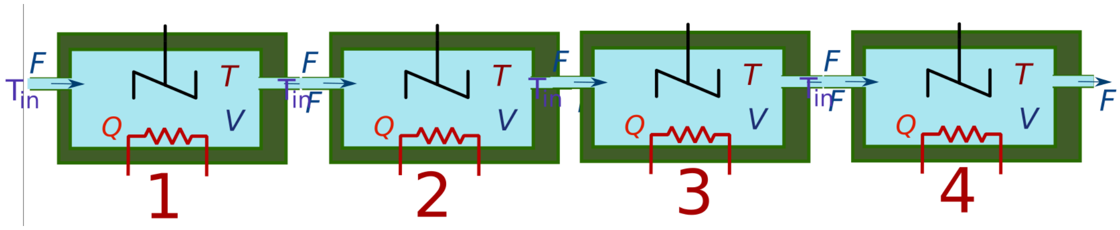
V/F is the so-called "turnover or flushing time", equal in a perfectly stirred tank or reactor to the so called "mean residence time" of the outgoing flow

Multi-Element model

We will understand a tubular heater:



as the "series" interconnection of many single-element heaters (a total of N, later on letting $N \rightarrow \infty$)... with $N = 4$, this would be:



We'll split the total heater volume and the total heating power equally for each of the elements:

Flow at element i is $F_i = F_{i-1} = F$ (common to all elements), volume $V_i = V/N$, heating power by resistor $Q_i = Q/N$, heat transfer coefficient to outside environment $\kappa_i = \kappa/N$.

Input temperature to i-th element $T_{in,i}$ will be the output temperature of the previous element: $T_{in,i} = T_{i-1}$; except for $i = 1$, of course, where $T_{in,1} \equiv T_{in}$ will be an arbitrary input signal.

The overall dynamic model will compute $\frac{dT_i}{dt}$ in its state equation; the outlet temperature will be T_N.

Infinitesimal elements: partial differential equations model

If each element's equation $\frac{dT}{dt} = -\frac{1}{V}F(T - T_{in}) - \frac{\kappa}{V\rho c}T + \frac{Q}{V\rho c}$ is rewritten thinking on $T(x, t)$, so that at each element its "left" end is at position x, and its "right" end is at position $x + dx$, we get:

$$T_{in} = T(x, t),\ T = T(x + dx, t),\ Q = \bar{Q}dx,\ V = Sdx,\ \kappa = \bar{\kappa}dx,$$

being $\bar{Q}$ the heating power per unit length generated by the resistor, $\bar{\kappa}$ the heat transfer coefficient per unit length, and S the cross-section area of the tubular heater.

Writhing the first-order dynamical model of the element between x and $x + dx$ results in:

$$\frac{\partial T(x + dx, t)}{\partial t} = -\frac{1}{Sdx}F \cdot (T(x + dx, t) - T(x, t)) - \frac{\bar{\kappa}dx}{Sdx\rho c}T(x + dx, t) + \frac{\bar{Q}dx}{Sdx\rho c}$$

simplifying dx, where possible, we end up with:

$$\frac{\partial T(x + dx, t)}{\partial t} = -\frac{1}{S}F \cdot \frac{T(x + dx, t) - T(x, t)}{dx} - \frac{\bar{\kappa}}{S\rho c}T(x + dx, t) + \frac{\bar{Q}}{S\rho c}$$

so, now, making dx tend to zero when the number of elements N tends to infinity, we reach the final PDE written as:

$$\frac{\partial T}{\partial t} = -\frac{F}{S} \cdot \frac{\partial T}{\partial x} - \frac{\bar{\kappa}}{S\rho c}T + \frac{\bar{Q}}{S\rho c}$$

Interpretation: PDE that relates the temporal derivatives $\frac{\partial T(x, t)}{\partial t}$ and spatial derivatives $\frac{\partial T(x, t)}{\partial x}$ of the temperature with inputs F and $\bar{Q}$.

*****Input T_{in} has apparently disappeared from the equation, but it's still there: it has transformed onto a "boundary condition", indicating that $T(0,t) = T_{in}(t)$ must be enforced.

Particular cases:

Transport delay

Under no heating $\bar{Q} = 0$ and perfect insulation $\bar{\kappa} = 0$ we get the PDE of a "transport delay", convective transport $\frac{\partial T}{\partial t} = -v\frac{\partial T}{\partial x}$, being $v = \frac{F}{S}$ the linear transport velocity.

Washout time is *Volume / Volumetric_flow = Length / linear_fluid_speed*; this will be the actual value of the "*delay*" between T_{in} and the output temperature T_{out} at the right-hand side of the pipe.

Steady-state (thermal equilibrium)

Another well-studied equation is the stationary (equilibrium) case with no heating $\bar{Q} = 0$. Indeed, assuming equilibrium amounts to assuming no variation in time $\frac{\partial T}{\partial t} = 0$. That equilibrium solution will be named as $T_{eq}(x)$, resulting in the 1st-order expression $\frac{\partial T_{eq}}{\partial x} = -\frac{\bar{\kappa}}{F\rho C_e}T_{eq}$, which is an ordinary differntial equation (ODE) in the "spatial" variable x, giving the popular exponential formula for steady-state heat exchangers:

$$T_{eq}(x) = T_{eq}(0) \cdot e^{\frac{-\bar{\kappa}}{F\rho C_e}\cdot x} = T_{in} \cdot e^{\frac{-\bar{\kappa}}{F\rho C_e}\cdot x}$$