

Relaxed LMI conditions for closed-loop fuzzy systems with tensor product structure

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Abstract

Current fuzzy control research tries to obtain the less conservative conditions to prove stability and performance of fuzzy control systems. In many fuzzy models, membership functions with multiple arguments are defined as the product of simpler ones, where all possible combinations of such products conform a fuzzy partition. In particular, such situation arises with widely-used fuzzy modelling techniques for non-linear systems. These type of fuzzy models will be denoted as tensor-product fuzzy systems, because its expressions can be understood as operations on multi-dimensional arrays. This paper discusses the generalisation to tensor-product fuzzy systems of the results in [5,18]. The procedures here will allow to set up LMI conditions which are less conservative than the cited ones, by exploiting the tensor-product structure of the membership functions. A numerical example illustrates the achieved improvement.

Key words: fuzzy control, linear matrix inequalities, fuzzy modelling

1 Introduction

Fuzzy control started as an heuristic methodology in the 1970's, coding by hand control rules provided by experts to control nonlinear systems. However, in three decades, state-of-the-art research has become more and more formal and rigorous using advanced mathematical tools, in order to guarantee control specifications expressed in terms of stability, performance, robustness to modelling errors, *etc.*

Nowadays, Linear Matrix Inequality (LMI) techniques have become the tool of choice in order to design fuzzy controllers in most application areas where a fuzzy model of the process is available in the Takagi-Sugeno form [9] ($\dot{x} = \sum_{i=1}^r \mu_i f_i$, $x_{k+1} = \sum_{i=1}^r \mu_i f_i$ with f_i linear). Such fuzzy models may come from nonlinear first-principle equations and from data. LMIs were introduced by [12] in the fuzzy com-

munity. The reader is referred to [7] for a review of the current trends and open issues in fuzzy modelling, identification and control.

Most LMI control design techniques are based on proving positiveness (or negativeness) of a so-called double fuzzy summation [11,5,4,14], in expressions such as $\sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j x^T Q_{ij} x > 0$, related to decrecence of an associated Lyapunov function¹.

Early sufficient conditions in literature for positivity of the above fuzzy summations were $Q_{ij} > 0$ (the most elementary ones), or $Q_{ii} > 0$, $Q_{ij} + Q_{ji} > 0$ [12]; they have been later improved, achieving less conservative results. In particular, nowadays, the most widely-used conditions are those in [18] (generalising [5]). These conditions will be later discussed in this paper, and a more powerful version of them will be stated for a particular class of fuzzy systems.

Note, importantly, that all the above cited conditions are independent of the membership functions: that fact is a source of conservativeness in some cases. For instance, the system $\dot{x} = \mu_1(z) \cdot x + (1 - \mu_1(z)) \cdot (-x)$ cannot be proved stable for an arbitrary μ_1 , $0 \leq \mu_1(z) \leq 1$ (it is unstable for $\mu_1(z) = 1$). However, it is stable for, say, $\mu_1 = 0.2 + 0.2\sin(x)$ as $\dot{x} = (-1 + 2\mu_1)x$ is, trivially, an exponentially stable first-order nonlinear system when $\mu_1 \leq b < 0.5$, $b \in \mathbb{R}$.

Another example of such a conservativeness, as this paper will show, occurs when the membership functions can be expressed as the “tensor product” of simpler partitions, so that the fuzzy system can be written as a multi-dimensional fuzzy summation, for instance $\dot{x} = \sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^r \mu_i \mu_j \mu_k (A_{ijk}x + B_{ijk}u)$. The tensor notation to be used in this paper is motivated by the use of multidimensional arrays to describe this class of fuzzy systems (see Appendix and [1]).

Removing part of the conservatism in current solutions for the tensor-product case above is indeed of interest; this product structure is often the case in many engineering applications of fuzzy control:

- in the systematic “sector nonlinearity” fuzzy modelling techniques reported in [12];
- in many man-made rulebases for multi-input fuzzy systems, where the rules are built via the *conjunction* of simpler concepts arising from fuzzy partitions on each of the input domains. A typical example are rulebases formed with rules in the form “if z_1 is *large* and z_2 is *small* and ... then ...”, “if z_1 is *medium* and z_2 is *small* and ... then ...”, *etc.*, with the antecedents covering all combinations of fuzzy sets on $z_1, z_2, \text{etc.}$.

¹ Other settings, such as fuzzy observers, descriptor systems and fuzzy Lyapunov functions, may require higher summation dimension, for instance triple fuzzy summations appearing in [12,10].

- in approximate interpolation and model reduction techniques based in gridding and tensor-SVD algebra in [1].

These settings will give rise to a particular class of fuzzy models which will be denoted, following the nomenclature in [1], as *tensor-product* (**TP**) fuzzy systems. The reader is referred to the above references and later sections in this paper for a more precise definition of TP fuzzy systems. In particular, a tensor-product structure of Takagi-Sugeno fuzzy systems will be the object of study, denoted as tensor-product Takagi-Sugeno fuzzy systems (TPTS).

In summary, the objective of this contribution is defining and analysing the tensor-product fuzzy systems, presenting fuzzy control design tools for them which explicitly use the tensor-product structure. The study of the properties of this class of systems is very relevant, in the authors' opinion, as most of the fuzzy systems in nontrivial engineering applications of fuzzy control belong to this class, as above discussed.

In particular, a generalisation of Theorem 2 in [18], exploiting the particular structure of the TPTS systems will be presented. The result provides less conservative conditions than other approaches in literature in closed-loop analysis and controller design problems. A numerical example will illustrate the achieved improvement.

The structure of the paper is as follows: next section discusses preliminary concepts, in particular the well-known double fuzzy summations, arising from closed-loop fuzzy systems. In addition to that, the section reviews the literature results which will be later extended. Tensor product fuzzy systems are defined in Section 3. Section 4 generalises the results in Section 2 to this class of fuzzy systems. Section 5 presents an example of the proposed methodology, showing that significant improvements are possible. A conclusion section closes the paper. Tensor notation and properties are discussed in an appendix.

2 Preliminaries

Let us consider a Takagi-Sugeno (TS) fuzzy system [9], expressed in rule-based form as a set of r rules, being rule i , $1 \leq i \leq r$ stated as:

$$\text{IF } z \text{ is } M_i \text{ then } \dot{x} = A_i x + B_i u$$

where z is a suitable set of variables to describe the system's nonlinearity, x is the process state (a vector with length n) and u is the process input (a vector with length w). The variables z may include some (or all) of the components of x , u . Denoting by $\mu_i(z)$ the membership function of the fuzzy set M_i , the above rule base

is interpreted as the expression [12]:

$$\dot{x} = \sum_{i=1}^r \mu_i(z)(A_i x + B_i u) \quad (1)$$

On the following, $\mu_i(z)$ will be assumed to belong to a fuzzy partition $\{\mu_1(z), \mu_2(z), \dots, \mu_r(z)\}$, *i.e.*, fulfilling

$$\sum_{i=1}^r \mu_i(z) = 1 \quad 0 \leq \mu_i(z) \leq 1 \quad (2)$$

Shorthand μ_i denoting $\mu_i(z)$ will be used in the sequel.

Widely-used controllers for TS systems (when z is measurable) are the so-called *parallel distributed compensators* (PDC) defined by:

$$u = - \sum_{k=1}^r \mu_k F_k x \quad (3)$$

which yield a closed-loop [13] given by:

$$\dot{x} = \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j (A_i - B_i F_j) x \quad (4)$$

A simple condition to ensure closed-loop stability of (4) can be derived from a quadratic Lyapunov function ($V = x^T P x$) as shown in [17,12], based on the positivity of V and $-\dot{V}$, *i.e.*,

$$-\dot{V} = \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j x^T (-A_i^T P - P A_i + P B_i F_j + F_j^T B_i^T P) x > 0 \quad (5)$$

After a standard change of variable $\psi = P^{-1} x$, the result is that stability (moreover, decay rate performance α) is proved [12] if:

$$\sum_{i=1}^r \sum_{j=1}^r -\mu_i \mu_j \psi^T (A_i X + X A_i^T - B_i M_j - M_j^T B_i^T + 2\alpha X) \psi > 0 \quad (6)$$

for $\psi \neq 0$, where $P^{-1} = X > 0$ and $M_i = F_i X$ are LMI decision variables and α is a user-defined decay-rate parameter.

This is the simplest example of a class of widely-used conditions for stability or performance of a closed-loop fuzzy control system. These conditions may be expressed, for some matrices Q_{ij} , in the form

$$\Xi(t) = \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j x^T Q_{ij} x \geq 0 \quad (7)$$

The left-hand term of expression (7) will be denoted as *double fuzzy summation*. For instance, in the above decay-rate fuzzy control problem,

$$Q_{ij} = -(A_i X + X A_i^T - B_i M_j - M_j^T B_i^T + 2\alpha X) \quad (8)$$

Note, importantly, that if Q_{ij} are linear in some matrix unknowns, then linear matrix inequality (LMI) techniques [12] may be used to check condition (7) by restating it as the requirement of positive-definiteness of the matrix $\sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j Q_{ij}$.

Example 1 Another example of performance-related condition uses²:

$$Q_{ij} = \begin{pmatrix} P A_i^T + R_j^T B_{2i}^T + A_i P + B_{2i} R_j & B_{1i} & P C_i^T + R_j^T D_{12i}^T \\ B_{1i}^T & -\gamma I & D_{11i}^T \\ C_i P + D_{12i} R_j & D_{11i} & -\gamma I \end{pmatrix} \quad (9)$$

in order to prove that the H_∞ norm (i.e., \mathcal{L}_2 to \mathcal{L}_2 induced norm) of a TS fuzzy system given by:

$$\dot{x} = \sum_{i=1}^r \mu_i(z) (A_i x + B_{1i} v + B_{2i} u) \quad (10)$$

$$y = \sum_{i=1}^r \mu_i(z) (C_i x + D_{11i} v + D_{12i} u) \quad (11)$$

is lower than γ . The reader is referred to [16] for details on how (9) is obtained.

Other well-known performance and robustness requirements for fuzzy systems can also be cast as similar expressions, as well as conditions for discrete-time TS systems $x_{n+1} = \sum_{i=1}^r \mu_i(A_i x_n + B_i u_n)$. The reader is referred to [11,6,12], etc. for details.

2.1 Sufficient Positivity Conditions

Sufficient conditions for positivity of Ξ in (7) are discussed in [11,5,18]. For convenience, some of them are reviewed below.

Lemma 1 If there exist matrices $X_{ij} = X_{ji}^T$ such that:

$$X_{ii} \leq Q_{ii} \quad (12)$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} \quad i < j \quad (13)$$

² In this case, x in (7) does not represent the state vector; it must be understood as a vector of artificial variables arising from Schur complements [2].

defining

$$\Theta(t) = \sum_{i=1}^r \sum_{j=1}^r \mu_i(z(t)) \mu_j(z(t)) x(t)^T X_{ij} x(t) \quad (14)$$

then $\Xi(t)$ in (7) fulfills

$$\Xi(t) \geq \Theta(t) \quad (15)$$

Proof: The proof is evident after reordering (7) and (14) as

$$\Xi(t) = \sum_{i=1}^r \mu_i^2 x^T Q_{ii} x + \sum_{i=1}^r \sum_{j=i+1}^r \mu_i \mu_j x^T (Q_{ij} + Q_{ji}) x \quad (16)$$

$$\Theta(t) = \sum_{i=1}^r \mu_i^2 x^T X_{ii} x + \sum_{i=1}^r \sum_{j=i+1}^r \mu_i \mu_j x^T (X_{ij} + X_{ji}) x \quad (17)$$

respectively. In this way, (12) and (13) indicate that each term in the summations in Ξ in (16) is larger than the corresponding one in the reordered Θ in (17). \square

Note that, in addition to an expression in the form (17), another expression for Θ is:

$$\Theta(t) = (\mu_1 x^T \ \mu_2 x^T \ \dots \ \mu_n x^T) \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} \begin{pmatrix} \mu_1 x \\ \mu_2 x \\ \vdots \\ \mu_n x \end{pmatrix} \quad (18)$$

which yields the well-known result below.

Theorem 1 [18]. *Expression (7) under fuzzy partition condition holds if there exist matrices $X_{ij} = X_{ji}^T$ such that:*

$$X_{ii} \leq Q_{ii} \quad (19)$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} \quad i \neq j \quad (20)$$

$$Y = \begin{pmatrix} X_{11} & \dots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \dots & X_{nn} \end{pmatrix} \quad Y > 0 \quad (21)$$

Note: In [5], all X_{ij} are forced to be symmetric (*i.e.*, $X_{ij} = X_{ji} = X_{ij}^T$). The authors in [18] realised that only symmetry of the larger matrix in (21) is needed ($X_{ij} = X_{ji}^T$ stated above); such minor amendment provided significantly less conservative conditions than in the earlier reference.

3 Tensor-product fuzzy systems

This section will first present the fuzzy systems and fuzzy summations in Section 2 with tensor notation, and then generalise the expressions via a new definition, which will encompass widely used classes of fuzzy systems. Basically, a so-called rank- p tensor will denote a p -dimensional array of real numbers. The reader is referred to the Appendix for tensor definitions, notation and operations with them.

3.1 Tensor expression for fuzzy systems

Note that the Takagi-Sugeno system (1) may be considered, by juxtaposing A_i and B_i as a matrix with size $n \times (n + w)$, as:

$$\dot{x} = \sum_{i=1}^r \mu_i(A_i B_i) \begin{pmatrix} x \\ u \end{pmatrix} \quad (22)$$

Consider now the one-dimensional array of matrices $(A_i B_i)$ to be the components of a suitably defined rank-3 tensor S so that the element s_{ijk} is the element (j, k) of the matrix $(A_i B_i)$, for $j = 1, \dots, n, k = 1, \dots, (n + w)$. Consider also the membership functions to be arranged as a vector (rank-1 tensor). Then, (22) may be written as a tensor product

$$\dot{x} = (\mu \cdot_1 S) \begin{pmatrix} x \\ u \end{pmatrix} \quad (23)$$

because the tensor product $\mu \cdot_1 S$ produces the so-called system matrix (rank-2 tensor):

$$\mu \cdot_1 S = \sum_{i=1}^r \mu_i(A_i B_i) \quad (24)$$

As the memberships are a rank-1 tensor, the above fuzzy systems will be also denoted as rank-1 fuzzy systems. The case of higher dimensionality (higher tensor rank) will be discussed later in this section.

It's also straightforward to check that the double fuzzy summations in (7) may also be expressed as:

$$\Xi = (\mu \otimes \mu \otimes x \otimes x) \cdot_4 Q \quad (25)$$

where Q is a rank-4 tensor (a “matrix” of matrices Q_{ij}), *i.e.*, element q_{ijkl} is equal to the element at position (k, l) of the matrix Q_{ij} . Note that Ξ is a scalar.

3.2 Multi-dimensional tensor-product fuzzy systems

In many applications, membership functions in a multi-input fuzzy model are chosen to be the product of simpler memberships with a linguistic interpretation, and all the possible products of such simpler memberships appear as rule antecedents³. Let us discuss a couple of simple motivating examples.

Example 2 Consider a so-called fuzzy-PD regulator built by setting up a fuzzy partition on an “error (e)” variable (say, a partition with 5 sets given by {negative large, negative, zero, positive, positive large}), and another (different) partition in the “error derivative (de)” (say, a partition with 3 sets {negative, zero, positive}).

For convenience, the membership functions on the error partition will be denoted by $(\mu_{11}(e), \dots, \mu_{15}(e))$, respectively, and those on the error derivative, by $(\mu_{21}(de), \mu_{22}(de), \mu_{23}(de))$. The partitions are assumed to verify $\sum_{i=1}^5 \mu_{1i} = 1$, $\sum_{i=1}^3 \mu_{2i} = 1$.

Once such partitions have been defined, rules are stated in a form such as:

IF e is negative large **and** de is positive **THEN** $u = u_{13}$
IF e is negative **and** de is zero **THEN** $u = u_{22}$
 \vdots

In this example, the total number of rules is $5 \times 3 = 15$. If the conjunction is interpreted as the algebraic product, the output of the controller may be expressed as:

$$u = \sum_{i_1=1}^5 \sum_{i_2=1}^3 \mu_{1i_1}(e) \mu_{2i_2}(de) u_{i_1 i_2} \quad (26)$$

Now, consider the tensor outer product of the vectors (i.e., rank-1 tensors) $\mu_1 = (\mu_{11}(e), \dots, \mu_{15}(e))$ and $\mu_2 = (\mu_{21}(de), \mu_{22}(de), \mu_{23}(de))$. Then, considering the following “membership tensor”,

$$\mu_1(e) \otimes \mu_2(de) = \begin{pmatrix} \mu_{11}(e)\mu_{21}(de) & \mu_{11}(e)\mu_{22}(de) & \mu_{11}(e)\mu_{23}(de) \\ \mu_{12}(e)\mu_{21}(de) & \mu_{12}(e)\mu_{22}(de) & \mu_{12}(e)\mu_{23}(de) \\ \mu_{13}(e)\mu_{21}(de) & \mu_{13}(e)\mu_{22}(de) & \mu_{13}(e)\mu_{23}(de) \\ \mu_{14}(e)\mu_{21}(de) & \mu_{14}(e)\mu_{22}(de) & \mu_{14}(e)\mu_{23}(de) \\ \mu_{15}(e)\mu_{21}(de) & \mu_{15}(e)\mu_{22}(de) & \mu_{15}(e)\mu_{23}(de) \end{pmatrix}$$

³ Such simpler functions usually refer to a reduced number of input variables (but the definitions later in this section allow for any set of variables in any membership).

it's easy to see that (26) may be expressed as an inner product of two tensors:

$$u = (\mu_1 \otimes \mu_2) \cdot_2 U \quad (27)$$

for a suitably crafted matrix (rank-2 tensor) U of size 5×3 whose elements are the corresponding rule consequents u_{ij} for $i = 1, \dots, 5$, $j = 1, 2, 3$.

Example 3 Consider a nonlinear model $\dot{x} = A(x)x + B(x)u$ where

$$A(x) = 0.75x - 2.25\sin(x) + \sin(x)x - 2.5 \quad (28)$$

$$B(x) = 0.42x + 1.25\sin(x) - 0.42\sin(x)x - 0.25 \quad (29)$$

for which a fuzzy model is to be set up for $x \in [-\pi, \pi]$. In this case, x may be written as $x = \sum_{i=1}^2 v_i p_i$, and $\sin(x)$ as $\sin(x) = \sum_{i=1}^2 \eta_i q_i$, with:

$$x = v_1(x) \cdot \pi + v_2(x) \cdot (-\pi), \quad \sin(x) = \eta_1(x) \cdot 1 + \eta_2(x) \cdot (-1)$$

where membership functions are $v_1 = \frac{1}{2\pi}(x + \pi)$, $v_2 = 1 - \mu_1$, $\eta_1 = \frac{1}{2}(\sin(x) + 1)$, $\eta_2 = 1 - \eta_1$, resulting in

$$A(x) = 0.75 \sum_{i=1}^2 v_i p_i - 2.25 \sum_{i=1}^2 \eta_i q_i + \left(\sum_{i=1}^2 \eta_i q_i \right) \left(\sum_{i=1}^2 v_i p_i \right) - 2.5$$

$$A(x) = \sum_{i=1}^2 \sum_{j=1}^2 v_i \eta_j (0.75 p_i - 2.25 q_j + p_i q_j - 2.5) = \sum_{i=1}^2 \sum_{j=1}^2 v_i \eta_j a_{ij}$$

where

$$a_{11} = 0.748, a_{12} = -1.035, a_{21} = -10.247, a_{22} = 0.536$$

and similarly

$$B(x) = \sum_{i=1}^2 \sum_{j=1}^2 v_i \eta_j (0.42 p_i + 1.25 q_j - 0.42 p_i q_j - 0.25) = \sum_{i=1}^2 \sum_{j=1}^2 v_i \eta_j b_{ij}$$

where:

$$b_{11} = 1, b_{12} = 1.139, b_{21} = 1, b_{22} = -4.139$$

Hence, the fuzzy system can be expressed as:

$$\dot{x} = \sum_{i=1}^2 \sum_{j=1}^2 v_i \eta_j (a_{ij}x + b_{ij}u) = \sum_{\mathbf{i} \in \mathbb{I}_2} \tilde{\mu}_{\mathbf{i}} (a_{\mathbf{i}}x + b_{\mathbf{i}}u) \quad (30)$$

where \mathbf{i} is a two-dimensional index variable (i_1, i_2) taking values in the set $\mathbb{I}_2 = \{1, 2\} \times \{1, 2\}$, and $\tilde{\mu}_{\mathbf{i}} = v_{i_1} \eta_{i_2}$, using the multiindex notation in the Appendix. In an analogous way to (26) in the previous example, a tensor notation can be thought of (see below).

Motivated by the above examples, let us consider now a definition for a general tensor-product fuzzy model in the Takagi-Sugeno (TS) framework (TS fuzzy systems are the most frequently used process model for fuzzy control in current literature), in order give a compact notation to fuzzy systems whose expression is a multi-dimensional sum, as in the above examples.

Definition 1 (tensor-product Takagi-Sugeno fuzzy systems.) *Consider a vector of measurable variables, z , in an universe of discourse Z . Consider also p fuzzy partitions defined on Z , each of them with n_1, \dots, n_p fuzzy sets, respectively.*

The fuzzy sets will be assumed to have linguistic labels denoted by M_{1i_1} , $i_1 = 1, \dots, n_1$ for the first partition, M_{2i_2} , $i_2 = 1, \dots, n_2$ for the second partition, etc. and membership functions arranged in rank-1 tensors:

$$\begin{aligned}\mu_1 &= (\mu_{11}(z) \ \mu_{12}(z) \ \dots \ \mu_{1n_1}(z)) \\ \mu_2 &= (\mu_{21}(z) \ \mu_{22}(z) \ \dots \ \mu_{2n_2}(z)) \\ &\vdots \\ \mu_p &= (\mu_{p1}(z) \ \mu_{p2}(z) \ \dots \ \mu_{pn_p}(z))\end{aligned}\tag{31}$$

fulfilling

$$\sum_{k=1}^{n_l} \mu_{lk} = 1 \quad 0 \leq \mu_{lk} \leq 1 \quad l = 1, \dots, p$$

Then, a rank- p continuous-time tensor-product Takagi-Sugeno fuzzy system (TPTS) built on the above fuzzy sets will be defined as the one described by the rules⁴:

$$\text{IF } z \text{ is } (M_{1i_1} \text{ and } M_{2i_2} \text{ and } \dots \text{ and } M_{pi_p}) \text{ THEN } \dot{x} = A_{i_1i_2\dots i_p}x + B_{i_1i_2\dots i_p}u$$

being its output evaluated with:

$$\dot{x} = \sum_{\mathbf{i} \in \mathbb{I}_p} \tilde{\mu}_{\mathbf{i}} (A_{\mathbf{i}}x + B_{\mathbf{i}}u)\tag{32}$$

where x and u are the TPTS state and input variables, respectively, $\mathbf{i} = i_1i_2\dots i_p$, and

$$\tilde{\mu}_{\mathbf{i}} = \prod_{k=1}^p \mu_{ki_k}\tag{33}$$

⁴ In many applications, such as the one in Example 2, the rules have the form:

$$\text{IF } z_1 \text{ is } M_{1i_1} \text{ and } z_2 \text{ is } M_{2i_2} \text{ and } \dots \text{ and } z_p \text{ is } M_{pi_p} \text{ THEN } \dot{x} = A_{i_1i_2\dots i_p}x + B_{i_1i_2\dots i_p}u$$

i.e., fuzzy partitions are defined over universes of discourse of smaller dimension, so that $Z = Z_1 \times Z_2 \times \dots \times Z_p$. However, that's not necessary, in principle, for the results in this paper to apply. For instance if Z is \mathbb{R}^2 , we could have $p = 3$, with three fuzzy partitions defined on, say, $z_1 + z_2$, $z_1 - \sqrt{z_2}$ and $(\sin(z_1) + 1)/(\cos(z_2) + 1)$. Hence, the rules above in this footnote are a particular case of the ones in Definition 1.

Remark: Analogous definitions may be cast for discrete-time TPTS systems and also for systems incorporating output equations, but they are omitted for brevity.

Using tensor notation, the following definition for TPTS systems is equivalent to the previous one (proof is omitted as it is just an issue of notation).

Definition 2 Consider a state vector x with dimension d , and an input vector u with dimension w , and form a vector of dimension $n + w$ by juxtaposing x and u . Consider a set of p fuzzy partitions defined on a universe Z , each of them arranged as a rank-1 tensor μ_i , $i = 1, \dots, p$, i.e., as in (31) above. Then, a TPTS fuzzy system is described by:

$$\dot{x} = (\tilde{\mu} \cdot_p S) \begin{pmatrix} x \\ u \end{pmatrix} \quad (34)$$

where S is a tensor with rank $p + 2$ and dimensions $n_1, n_2, \dots, n_p, n_{p+1} = d, n_{p+2} = d + w$ and

$$\tilde{\mu} = \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_p$$

is a tensor with dimensions n_1, n_2, \dots, n_p (whose elements are, evidently, given by (33)), denoted as membership tensor. S will be denoted as consequent tensor⁵.

An example of a membership tensor element is, for instance $\mu_{3,4,1,1} = \mu_{13}\mu_{24}\mu_{31}\mu_{41}$, which will denote a particular rule in a rank-4 TPTS fuzzy system.

Note that $\tilde{\mu} \cdot_p S$ is a rank-2 tensor (i.e., a matrix which multiplies the state-input vector with the ordinary matrix-vector multiplication).

Obviously, the notations (32) and (34) are equivalent to an expression such as:

$$\dot{x} = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \dots \sum_{i_p=1}^{n_p} \mu_{1i_1} \mu_{2i_2} \dots \mu_{pi_p} (A_{i_1 i_2 \dots i_p} x + B_{i_1 i_2 \dots i_p} u) \quad (35)$$

For instance, the fuzzy system (30) may be considered a rank-2 TPTS one.

Remarks on TPTS modelling: Many fuzzy systems in practice have the tensor-product structure:

- Example 2 shows how they naturally arise from man-made rules.
- Another paradigmatic example is the “sector nonlinearity” modelling methodology in [12]; Example 3 in this work is one of the simplest cases of the referred modelling technique. The reader is also referred to Example 3, in section 2.2.1 of

⁵ Notation in (34) is somehow different from that in [1], but equivalent. We wanted to emphasise the concept of “membership tensor” (generated via an outer product) whereas Baranyi used n -mode products [3] for subsequent singular-value-related computations.

the referred book, which results in a 16-rule model TPTS described by a membership tensor of dimensions $2 \times 2 \times 2 \times 2$ (of course, the authors there do not use the notation introduced here), *i.e.*, a rank-4 TPTS system.

- Last, [1] proposes a tensor-product based methodology to approximate functions of multiple variables via Takagi-Sugeno fuzzy systems. The procedure, instead of being based on the previously-discussed sector-nonlinearity approach, is based on multi-dimensional gridding, lookup and interpolation. A subsequent step of complexity reduction based on higher-order singular value decomposition [3] is needed in order to get a reduced number of rules.

In fact, fuzzy system without a TSTP structure are seldom present in applications, except in the simplest cases (even some first-order single-input TS systems can be better modelled as TSTP, by using the sector-nonlinearity methodology above cited, as demonstrated in Example 3 in this work).

Proposition 1 *Standard TS fuzzy systems are rank-1 TPTS fuzzy systems. Conversely, TPTS systems are a subclass of standard TS fuzzy systems.*

Proof: The first affirmation is evident from the definitions, and it has already been discussed in Section 3.1. Regarding the second one, consider the well-known identity

$$\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_p=1}^{n_p} \mu_{1i_1} \cdots \mu_{pi_p} = 1 \quad (36)$$

It shows that the tensor product conforms a fuzzy partition composed of $q = n_1 \times n_2 \times \cdots \times n_p$ fuzzy sets. Such partition is given by the rank-1 membership functions obtained by unfolding (flattening) the tensor $\tilde{\mu}$ onto a vector. The idea can be formalised by using proposition 3, as:

$$\dot{x} = (\tilde{\mu} \cdot_p S) \begin{pmatrix} x \\ u \end{pmatrix} = (fl_{1 \leftarrow \cdots \leftarrow p} \tilde{\mu} \cdot_1 fl_{1 \leftarrow \cdots \leftarrow p} S) \begin{pmatrix} x \\ u \end{pmatrix} \quad (37)$$

Hence, the original TPTS fuzzy system is expressed as a standard TS one because the membership tensor has been unfolded onto a vector, and the consequent tensor S has been suitably rearranged by the fl operator as a rank 3 tensor. Such rank-3 tensor $fl_{1 \leftarrow \cdots \leftarrow p} S$ produces an ordinary matrix when subject to the product with the unfolded $fl_{1 \leftarrow \cdots \leftarrow p} \tilde{\mu}$ (rank 1). \square

Example 4 *Consider a TP fuzzy system with $p = 2$, $n_1 = 2$, $n_2 = 3$. It may be equivalently considered as an “unfolded” fuzzy system with 6 membership functions, denoted as $\beta_k(z)$ given by:*

$$\begin{aligned}
k=1 \quad \beta_1(z) &= \mu_{11}(z)\mu_{21}(z) \\
k=2 \quad \beta_2(z) &= \mu_{11}(z)\mu_{22}(z) \\
k=3 \quad \beta_3(z) &= \mu_{11}(z)\mu_{23}(z) \\
k=4 \quad \beta_4(z) &= \mu_{12}(z)\mu_{21}(z) \\
k=5 \quad \beta_5(z) &= \mu_{12}(z)\mu_{22}(z) \\
k=6 \quad \beta_6(z) &= \mu_{12}(z)\mu_{23}(z)
\end{aligned}$$

As another example, consider the fuzzy model of Example 3. In the same way as above, if $\mu_1 = v_1\eta_1$, $\mu_2 = v_1\eta_2$, $\mu_3 = v_2\eta_1$ and $\mu_4 = v_2\eta_2$ were defined, a fuzzy TS with four models:

$$\dot{x} = \sum_{i=1}^4 \mu_i(a_i x + b_i u) \quad (38)$$

$$a_1 = 0.748, a_2 = -1.035, a_3 = -10.247, a_4 = 0.536 \quad (39)$$

$$b_1 = 1, b_2 = 1, b_3 = 1, b_4 = -4.139 \quad (40)$$

will exactly describe the nonlinear system under analysis for $x \in [-\pi, \pi]$.

The reader is referred to [12] for more examples of this tensor-product nonlinear modeling methodology (although tensor notation is not used and the final model is always unfolded).

Remark: Proposition (1) seems to make ill-fated any attempt to approach fuzzy control design for TPTS systems because TPTS are TS systems and vice-versa. However, a crucial fact is overlooked in this argumentation: most results for stability and performance of TS fuzzy systems are *independent* of the membership shapes – particularly those in [12,5,18]. However, an unfolded TPTS system does *not* sweep over all possible membership values⁶. Hence, such membership-independent stability and performance conditions are conservative in the case of TPTS systems. This is the key issue motivating the work in Section 4 in this paper.

3.3 Closed-loop tensor-product fuzzy systems

Definition 3 (tensor-product controller) Given a rank- p TPTS system (32), a controller in the form:

$$u = - \sum_{\mathbf{j} \in B_p} \tilde{\mu}_{\mathbf{j}}(z) F_{\mathbf{j}} x = -(\tilde{\mu} \cdot_p F) x \quad (41)$$

will be denoted as rank- p tensor product PDC controller (F is a rank- $(p+2)$ tensor formed by suitably arranging matrices $F_{\mathbf{j}}$).

⁶ for instance, it's impossible to have $\beta_1 = 0.1$ and β_2 or β_3 larger than 0.1 in Example 4.

By analogy with (4), it is straightforward to prove that, when a rank- p tensor-product PDC controller is used to control a rank- p system (32), the closed loop equations are given by:

$$\dot{x} = \sum_{\mathbf{i} \in B_p} \sum_{\mathbf{j} \in B_p} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} G_{\mathbf{ij}} = ((\tilde{\mu} \otimes \tilde{\mu}) \cdot_{2p} G)x \quad (42)$$

where $G_{\mathbf{ij}} = A_{\mathbf{i}} - B_{\mathbf{i}}F_{\mathbf{j}}$ defines a tensor G with rank $2p + 2$ (note that, for fixed \mathbf{i} and \mathbf{j} , $B_{\mathbf{i}}$ and $F_{\mathbf{j}}$ are rank-2 tensors, following notation (66), so the product is well defined, being the usual matrix product).

In general, analogously to (7), many stability and performance criteria for tensor-product closed-loop fuzzy systems can be expressed as requiring, for any $x \neq 0$:

$$\Theta = \sum_{\mathbf{i} \in B_p} \sum_{\mathbf{j} \in B_p} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} x^T Q_{\mathbf{ij}} x > 0 \quad (43)$$

For instance, it's almost evident to check that a condition for quadratic stability of a TPTS fuzzy system is (43) with $Q_{\mathbf{ij}}$ given by (6) but replacing the i and j with its boldfaced counterparts.

In tensor notation, stability and performance conditions (43) look like

$$\Theta = (\tilde{\mu} \otimes \tilde{\mu} \otimes x \otimes x) \cdot_{2p+2} Q > 0 \quad (44)$$

for a suitably defined tensor Q with rank $2p + 2$. Indeed,

$$\Theta = \sum_{\mathbf{i} \in B_p} \sum_{\mathbf{j} \in B_p} \sum_{k=1}^n \sum_{l=1}^n \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} x_k x_l Q_{\mathbf{ij}kl}$$

Unfolding to a TS system. A possibility to work with TPTS systems is considering them as ordinary TS systems (Proposition 1) and design fuzzy controllers for them. Indeed, this is the commonly considered option in literature which this paper seeks to improve.

The above argumentation may be equivalently stated by using Proposition 3 on (44), which results in stating:

$$\Theta = (fl_{1 \leftarrow \dots \leftarrow p} \tilde{\mu} \otimes fl_{1 \leftarrow \dots \leftarrow p} \tilde{\mu} \otimes x \otimes x) \cdot_4 fl_{1 \leftarrow \dots \leftarrow p} fl_{(p+1) \leftarrow \dots \leftarrow 2p} Q \quad (45)$$

where $fl_{1 \leftarrow \dots \leftarrow p} \tilde{\mu}$ is a rank-1 tensor (*i.e.*, the memberships of an ordinary TS system arranged as a vector, suitably ordered) so (45) may be written as (25), *i.e.*, (7). Hence, LMIs for such conditions can be applied, such as Theorem 1 (details are omitted for brevity).

The next section discusses an explicit use of the tensor-product form of the memberships in order to produce conditions less conservative than the “unfolding + Theorem 1” procedure used in literature.

4 Main Result: relaxed stability and performance conditions for TPTS fuzzy systems

Theorem 2 *Expression (43) (equiv. (44)) holds if there exists a rank- $(2p+2)$ tensor X such that the conditions stated below hold. For ease of notation, note that $X_{\mathbf{i}kj s}$, $\mathbf{i}, \mathbf{j} \in \mathbb{I}_{p-1}$, $k, s \in I_p$ is a rank-2 tensor (matrix), and the same applies to $Q_{\mathbf{i}kj s}$. The conditions are:*

$$X_{\mathbf{i}kj s} = X_{\mathbf{i}sj k}^T \quad (46)$$

$$X_{\mathbf{i}kj k} \leq Q_{\mathbf{i}kj k} \quad (47)$$

$$X_{\mathbf{i}kj s} + X_{\mathbf{i}sj k} \leq Q_{\mathbf{i}kj s} + Q_{\mathbf{i}sj k} \quad (48)$$

$$Y = fl_{p \leftarrow 2p+1} fl_{2p \leftarrow 2p+2} X, \text{ i.e.,}$$

$$Y_{\mathbf{ij}} = \begin{pmatrix} X_{\mathbf{i}1\mathbf{j}1} & \dots & X_{\mathbf{i}1\mathbf{j}n_p} \\ \vdots & \ddots & \vdots \\ X_{\mathbf{i}n_p\mathbf{j}1} & \dots & X_{\mathbf{i}n_p\mathbf{j}n_p} \end{pmatrix} \quad (49)$$

$$\sum_{\mathbf{i} \in B_{p-1}} \sum_{\mathbf{j} \in B_{p-1}} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} \xi(t)^T Y_{\mathbf{ij}} \xi(t) > 0 \quad (50)$$

where Y is a rank- $(2p)$ tensor (hence $Y_{\mathbf{ij}}$ is a matrix).

If (50) can be proved by a set of LMI sufficient conditions, then such conditions jointly with (47)–(49) are still an LMI problem stating sufficient conditions for (43).

Proof. Note that, for $\mathbf{i} \in \mathbb{I}_{p-1}$, $k \in I_p$, for any tensor T of rank greater than p :

$$\sum_{\mathbf{h} \in \mathbb{I}_p} \tilde{\mu}_{\mathbf{h}} T_{\mathbf{h}} = \sum_{\mathbf{i} \in \mathbb{I}_{p-1}} \sum_{k=1}^{n_p} \tilde{\mu}_{\mathbf{i}k} T_{\mathbf{i}k} = \sum_{\mathbf{i} \in \mathbb{I}_{p-1}} \tilde{\mu}_{\mathbf{i}} \sum_{k=1}^{n_p} \mu_{pk} Q_{\mathbf{i}k}$$

Similarly, (43) may be written as:

$$\Theta = \sum_{\mathbf{i} \in \mathbb{I}_{p-1}} \sum_{\mathbf{j} \in \mathbb{I}_{p-1}} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} \sum_{k=1}^{n_p} \sum_{s=1}^{n_p} \mu_{pk} \mu_{ps} x^T Q_{\mathbf{i}kj s} x \quad (51)$$

Then, Lemma 1 can be applied to

$$\delta_{\mathbf{ij}} = \sum_{k=1}^{n_p} \sum_{s=1}^{n_p} \mu_{pk} \mu_{ps} x^T Q_{\mathbf{i}kj s} x \quad (52)$$

considering \mathbf{i} and \mathbf{j} as fixed, so that, if (47), (48) hold, then (considering the analogous formulas to (16) and (17)):

$$\delta_{\mathbf{ij}} \geq p_{\mathbf{ij}} = \sum_{k=1}^{n_p} \sum_{s=1}^{n_p} \mu_{pk} \mu_{ps} x^T X_{\mathbf{ikj}s} x \quad (53)$$

and, hence, building the matrix $Y_{\mathbf{ij}}$ in (49),

$$p_{\mathbf{ij}} = \xi^T Y_{\mathbf{ij}} \xi \quad (54)$$

where $\xi = fl_{1 \leftarrow 2}(\mu_p \otimes x) = (\mu_{p1x_1} \dots \mu_{p1x_n} \mu_{p2x_1} \dots \mu_{pn_p x_n})$ expressed as a column vector. As the elements of the membership tensors are all positive, we have

$$\Theta \geq \sum_{\mathbf{i} \in \mathbb{I}_{p-1}} \sum_{\mathbf{j} \in \mathbb{J}_{p-1}} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} p_{\mathbf{ij}} \quad (55)$$

and the proof is complete. \square

The above theorem is a generalisation of Theorem 1. It provides a sufficient condition which transforms computation of positivity conditions for a “double p -dimensional sum” (43) into computations with a “double $(p-1)$ -dimensional sum” and larger matrices (the size of $Y_{\mathbf{ij}}$ is $(n_p \cdot n) \times (n_p \cdot n)$, where n is the size of the square matrices $Q_{\mathbf{ij}}$).

From a computational point of view, recursive application of the above theorem allows to reach $p = 1$, and directly applying Theorem 1 as a last step. Then, Theorem 2 allows to assert that (43) holds if a certain $nq \times nq$ matrix is positive definite.

Note that the size of the final matrix is the same as the one obtained by unfolding (43) and applying Theorem 1: the number of elements of tensors X , Y and Q are the same, but arranged differently. However, the larger number of relaxation variables X in Theorem 2, with various sizes, allows to produce less conservative results as the example in next section shows.

Recursive application of Theorem 2 for a rank- p TP fuzzy system needs $p-1$ tensors of decision variables (of rank $2p+2$, $2p$, \dots , 4). All of these tensors have the same number of elements as the original Q .

5 Example

The following example illustrates the effectiveness of the new stability condition (Theorem 2) compared to the usual approach in literature, *i.e.*, Theorem 1 applied after unfolding to a standard fuzzy system. Consider a continuous fuzzy plant composed of the following four rules:

R_{11} : **IF** x_1 is M_{11} **and** x_2 is M_{21} **THEN** $\dot{x} = A_{11}x + B_{11}u$
 R_{12} : **IF** x_1 is M_{11} **and** x_2 is M_{22} **THEN** $\dot{x} = A_{12}x + B_{12}u$
 R_{21} : **IF** x_1 is M_{12} **and** x_2 is M_{21} **THEN** $\dot{x} = A_{21}x + B_{21}u$
 R_{22} : **IF** x_1 is M_{12} **and** x_2 is M_{22} **THEN** $\dot{x} = A_{22}x + B_{22}u$

where

$$\begin{aligned}
A_{11} &= \begin{bmatrix} 0.5 & -0.05 \\ 0 & -5 \end{bmatrix}, B_{11} = \begin{bmatrix} a \\ 0.1 \end{bmatrix} \\
A_{12} &= \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}, B_{12} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} \\
A_{21} &= \begin{bmatrix} -1 & 0.1 \\ 0 & -2 \end{bmatrix}, B_{21} = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix} \\
A_{22} &= \begin{bmatrix} b & -0.01 \\ 0 & -3 \end{bmatrix}, B_{22} = \begin{bmatrix} 1 \\ 0.05 \end{bmatrix}
\end{aligned}$$

represented by the equations:

$$\dot{x} = \sum_{\mathbf{i} \in \mathbb{I}_2} \tilde{\mu}_{\mathbf{i}}(A_{\mathbf{i}}x + B_{\mathbf{i}}u) = \sum_{i_1=1}^2 \sum_{i_2=1}^2 \mu_{1i_1} \mu_{2i_2} (A_{i_1 i_2} x + B_{i_1 i_2} u) \quad (56)$$

where $\mathbb{I}_2 = \{1, 2\} \times \{1, 2\}$. Membership functions $\{\mu_{11}, \mu_{12}\}$ and $\{\mu_{21}, \mu_{22}\}$ are supposed to be fuzzy partitions on the domain of x_1 and x_2 respectively. Hence, the system conforms to the definition of a rank-2 TPTS one. The shape of each of the four membership functions is arbitrary as long as $\mu_{11} = 1 - \mu_{12}$ and $\mu_{21} = 1 - \mu_{22}$.

A stabilising PDC controller with 4 rules is to be designed, $u = -\sum_{\mathbf{i} \in \mathbb{I}_2} \tilde{\mu}_{\mathbf{i}} F_{\mathbf{i}} x$. The stabilization conditions expressed in the form (43) are obtained from (5) via a change of variable [12], resulting in:

$$Q_{\mathbf{ij}} = -ZA_{\mathbf{i}} - A_{\mathbf{i}}^T Z + B_{\mathbf{i}} N_{\mathbf{j}} + N_{\mathbf{j}}^T B_{\mathbf{i}}^T \quad (57)$$

where $\mathbf{i}, \mathbf{j} \in \mathbb{I}_2$, and $Z, N_{\mathbf{j}}$ are LMI decision variables. Z should be a symmetric positive-definite matrix, and the PDC controller is provided by $F_{\mathbf{j}} = N_{\mathbf{j}} Z^{-1}$.

The parameters a in B_{11} , and b in A_{22} , will take values in a prescribed grid, in order to check the feasibility of the associated fuzzy control synthesis problem under two different approaches.

Usual approach. A first possibility in order to design the above regulator would be considering the fuzzy system to be a four-rule standard one (unfolding), with $A_1 =$

$A_{11}, A_2 = A_{12}, A_3 = A_{21}$ and $A_4 = A_{22}$, using a similar notation for B , generating $Q_{ij}, i, j = 1, \dots, 4$.

This well-known approach has been compared to the one proposed in this work. Note that 16 Lyapunov matrices (57) are defined in both approaches, the only difference is how they are indexed (via two integer indices from 1 to 4, in the usual approach; via two rank-2 indices of size $\{1, 2\} \times \{1, 2\}$ in this work).

Proposed approach. Applying Theorem 2, expression (43) holds if there exist a rank-6 tensor X from which matrices X_{ikjs} can be extracted so that $X_{ikjs} = X_{isjk}^T$ for each $\mathbf{i}, \mathbf{j} \in \mathbb{I}_1, k, s \in I_1$ ($\mathbb{I}_1 = I_1 = \{1, 2\}$), and

$$X_{i1j1} \leq Q_{i1j1}, \quad X_{i2j2} \leq Q_{i2j2} \quad (58)$$

$$X_{i1j2} + X_{i2j1} \leq Q_{i1j2} + Q_{i2j1} \quad (59)$$

$$Y_{ij} = \begin{pmatrix} X_{i1j1} & X_{i1j2} \\ X_{i2j1} & X_{i2j2} \end{pmatrix} \quad (60)$$

$$\sum_{\mathbf{i} \in B_1} \sum_{\mathbf{j} \in B_1} \mu_i \mu_j \xi^T Y_{ij} \xi > 0 \quad (61)$$

Then, regarding the positivity of $\sum_{\mathbf{i} \in B_1} \sum_{\mathbf{j} \in B_1} \mu_i \mu_j \xi^T Y_{ij} \xi$ Theorem 1 is directly applied, because \mathbf{i} and \mathbf{j} are now one-dimensional indices: Theorem 1 requires the existence of matrices $W_{ij} = W_{ij}^T$ for each $i, j \in \{1, 2\}$, such that

$$W_{11} \leq Y_{11}, W_{22} \leq Y_{22} \quad (62)$$

$$W_{12} + W_{21} \leq Y_{12} + Y_{21} \quad (63)$$

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \geq 0 \quad (64)$$

Note that the set of conditions (57) jointly with (58)–(60), (62)–(64) are LMIs.

Results. Figure 1 shows the values of a and b where a stabilising controller is found, based on either Theorem 1 (after unfolding) or Theorem 2, using a suitable LMI solver.

In this figure, the \circ mark indicates the existence of feasible stabilising regulators proved by Theorem 1 (and, of course, also by Theorem 2); the \times mark indicates parameter values for which stabilizability is proved from Theorem 2, but not from Theorem 1. Hence, substantially better results are obtaining by exploiting the tensor-product structure of the four involved TS rules.

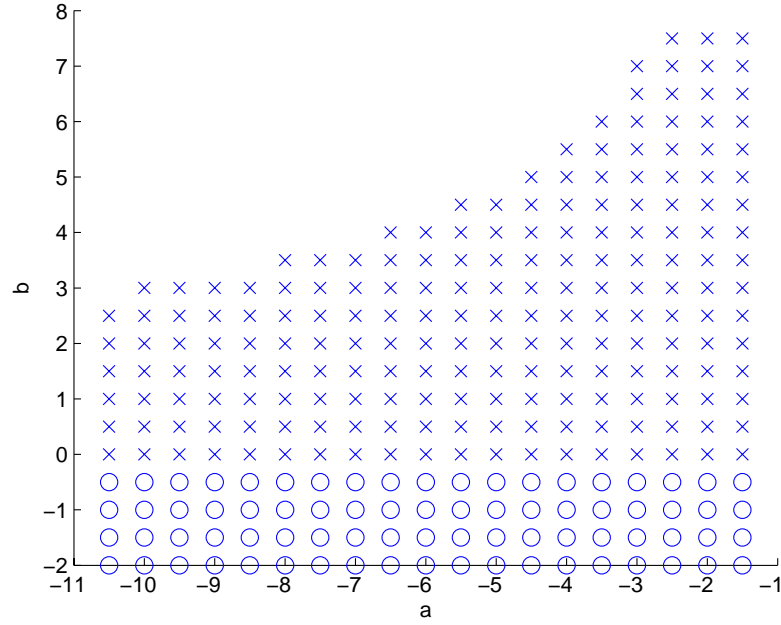


Fig. 1. Parameter values for which feasible stabilising regulators are found: unfolding + Theorem 1 (\circ); Theorem 2(\circ , \times).

Similar results are obtained when the methodology is applied to the nonlinear system in Example 3 expressed as a rank-2 TPTS fuzzy system: the usual approach does not find a stabilising controller, whereas the one proposed in this work does.

6 Conclusions

This paper has provided a generalisation of double-fuzzy summation results in literature to multiple summations with a tensor-product structure. Such structure is indeed common in many fuzzy models and, hence, this paper allows for less conservative results in fuzzy controller designs for such systems, as demonstrated in one numerical example. Although, for simplicity, the chosen example only considers stabilisation, the presented procedure applies to other more sophisticated performance/robustness requirements, by considering well-known different choices for Q_{ij} .

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Appendix: Tensor and multi-index notation

Tensor calculus originated in 19th century physics as a way of working with multilinear transformations, even in non-Euclidean geometries [8]. When the multilinear transformations have arguments in \mathbb{R}^n with the usual Euclidean metric and Hilbert space structure, tensors may be considered as multi-dimensional arrays. This is the case in this work.

In the definitions below, the notation I_q will refer to array index sets in the form $I_q = \{1, 2, \dots, n_q\}$ for some n_q . Several values of q will be used in defining multi-dimensional arrays.

Definition 4 *A tensor T is a multilinear application which can be represented as a multidimensional array $T \in \mathbb{R}^{I_1 \times \dots \times I_p}$ relative to the basis vectors being chosen on each array dimension. The number p is denoted as tensor rank. When the tensor structure is to be made explicit, the notation $T_{I_1 \times \dots \times I_p}$ will be used, or even $T_{n_1 \times n_2 \times \dots \times n_p}$ to describe both the rank and the sizes on each dimension. The tensor elements are real numbers, denoted by a lowercase symbol, indexed by a multi-dimensional index variable (to be denoted as multi-index):*

$$t_{i_1 i_2 \dots i_p} \quad 1 \leq i_q \leq n_q, \quad q = 1, \dots, p \quad (65)$$

Note that rank-1 tensors may be considered transpose-free *vectors* and rank-2 ones are *matrices*. In the same way that matrices can be considered as a collection of vectors, a tensor can be considered a collection of lower-rank ones. On the sequel, when a rank- p tensor $T \in \mathbb{R}^{I_1 \times \dots \times I_p}$ is indexed by an index with less than p components, the result will be a tensor (thus, denoted by uppercase), symbolised by the notation, for $q < p$:

$$T_{i_1 i_2 \dots i_q} \in \mathbb{R}^{I_{q+1} \times \dots \times I_p} \quad (66)$$

For instance a rank-5 tensor may be considered as a 3-dimensional array of matrices or a 4-dimensional array of vectors.

Definition 5 (Outer tensor product) . *The outer tensor product of $U_{n_1 \times \dots \times n_p}$ and $T_{n'_1 \times \dots \times n'_s}$ is a tensor $V_{n_1 \times \dots \times n_{p+s}} = U \otimes T$, where $n_{p+q} \equiv n'_q$, $q = 1, \dots, s$. The elements of V are:*

$$v_{i_1 \dots i_p i_{p+1} \dots i_{p+s}} = u_{i_1 \dots i_p} t_{i_{p+1} \dots i_{p+s}} \quad (67)$$

Definition 6 (Multi-indices) *On the following, boldface symbols will denote multi-indices when its structure is clear from the context:*

$$\mathbf{i} = i_1 i_2 \dots i_p \quad 1 \leq i_q \leq n_q, \quad q = 1, \dots, p \quad (68)$$

and, similarly, the cartesian product of index sets will be referred to by the notation:

$$\mathbb{I}_p = I_1 \times \dots \times I_p \quad (69)$$

For instance, if either the dimensions have been suitably defined beforehand or they are not relevant to a particular discussion, the elements referred to in (65) will be denoted as t_i , $i \in \mathbb{I}_p$ for convenience. The multi-index will be said to have rank p , as the tensor it indexes.

Multi-indices of higher rank will also be represented by the juxtaposition of indices of smaller rank. For instance, the elements of the tensor resulting from the outer product in (67) will be denoted, when convenient, by $v_{ij} = u_i t_j$, for suitably defined $i \in \mathbb{I}_p$, $j \in \mathbb{I}'_s$.

The following definition extends the usual matrix product along p shared dimensions⁷.

Definition 7 (product) *The ordinary product of two tensors $U \in \mathbb{R}^{\mathbb{I}''_s \times \mathbb{I}_p}$ and $V \in \mathbb{R}^{\mathbb{I}_p \times \mathbb{I}'_q}$, which share the dimensions \mathbb{I}_p , is a tensor $T \in \mathbb{R}^{\mathbb{I}''_s \times \mathbb{I}'_q}$ which will be denoted as $T = U \cdot_p V$ whose elements are:*

$$t_{i''i'} = t_{i''_1 \dots i''_s i'_1 \dots i'_q} = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_p=1}^{n_p} u_{i'_1 \dots i''_s i_1 i_2 \dots i_p} v_{i_1 i_2 \dots i_p i'_1 \dots i'_q} = \sum_{i \in \mathbb{I}_p} u_{i''i} v_{ii'}$$

The notation $U \cdot V$ (or UV) will be used to represent the product with shared index of rank 1, i.e., $UV = U \cdot V = U \cdot_1 V$ (if U and V are rank-2 tensors, UV is the usual matrix product).

As an example of the use of the product notation, a quadratic form ($x^T Q x$ in matrix notation) may be expressed as $(x \otimes x) \cdot_2 Q$.

Proposition 2 *Given rank- p tensors A_1, A_2 , a rank- q tensor B and a rank- $(p+q)$ tensor C , the ordinary product and the outer product verify:*

$$A_1 \cdot_p A_2 = A_2 \cdot_p A_1 \quad (70)$$

$$(A \otimes B) \cdot_{p+q} C = A \cdot_p (C \cdot_q B) \quad (71)$$

Note that $A_1 \cdot_p A_2$ is a real number, which is the generalisation of the vector scalar product. For a rank-2 tensor (matrix) $\sqrt{A \cdot_2 A}$ is the Frobenius norm.

Definition 8 (unfolding) *The unfolding operation (“flattening”), denoted as $fl_{r \leftarrow q} V$ reduces the rank of a tensor $V \in \mathbb{R}^{I_1 \times \dots \times I_r \times \dots \times I_q \times \dots \times I_p}$ by one, converting it to a new tensor $U \in \mathbb{R}^{I_1 \times \dots \times I'_r \times \dots \times I_{q-1} \times I_{q+1} \times \dots \times I_p}$ whose elements are given by:*

$$u_{i_1 \dots i_r \dots i_p} = v_{i_1 \dots i_{r-1} j_r i_{r+1} \dots i_{q-1} j_q i_{q+1} \dots i_p} \quad (72)$$

⁷ There are other alternative definitions and notations for (inner) tensor products [3,1], as the number and position of the shared dimensions may vary. The one presented here has been adopted for convenience.

where $(i_r - 1) = (j_r - 1) * n_q + (j_q - 1)$, i.e., $j_r - 1$ is the integer part of the quotient $(i_r - 1)/n_q$, and $(j_q - 1)$ is the remainder.

As unfolding can be nested, successive applications of the operator can rearrange the tensor as a matrix or even as a vector. The notation

$$fl_{p \leftarrow q \leftarrow r \leftarrow \dots \leftarrow t \leftarrow s} = fl_{p \leftarrow q} fl_{q \leftarrow r} \dots fl_{t \leftarrow s} \quad p < q < r < \dots t < s$$

will be later used.

Example 5 Consider the tensor of rank 3, with $n_1 = 2$, $n_2 = 3$, $n_3 = 2$ given by: $t_{i_1 i_2 i_3} = 2^{i_1-1} 3^{i_2-1} 5^{i_3-1}$. Then,

$$fl_{2 \leftarrow 3} T = \begin{pmatrix} 1 & 5 & 3 & 15 & 9 & 45 \\ 2 & 10 & 6 & 30 & 18 & 90 \end{pmatrix}$$

and

$$fl_{1 \leftarrow 2 \leftarrow 3} T = fl_{1 \leftarrow 2} fl_{2 \leftarrow 3} T = (1 \ 5 \ 3 \ 15 \ 9 \ 45 \ 2 \ 10 \ 6 \ 30 \ 18 \ 90)$$

Example 6 Unfolding a rank-3 tensor T may produce 6 different matrices: $fl_{1 \leftarrow 2} T$, $fl_{1 \leftarrow 3} T$, $fl_{2 \leftarrow 1} T$, $fl_{2 \leftarrow 3} T$, $fl_{3 \leftarrow 1} T$ and $fl_{3 \leftarrow 2} T$. The n -mode matrix of a rank- p tensor T (Definition 4 in [1]) is, for $n > 2$, the transpose of the matrix resulting from the unfolding $fl_{2 \leftarrow 1 \leftarrow 3 \leftarrow 4 \leftarrow \dots \leftarrow n-1 \leftarrow n+1 \leftarrow p} T$.

As unfolding is just a reordering of the tensor elements, it's easy to prove the following proposition (details omitted for brevity).

Proposition 3 The inner product of tensors remains invariant under unfolding on any of the shared dimensions, i.e.,

$$(fl_{r \leftarrow q} U) \cdot_{p-1} (fl_{r \leftarrow q} V) = U \cdot_p V$$

In particular, the above proposition generalises the transformation from (17) to (18), which used the fact that

$$(\mu \otimes \mu \otimes x \otimes x) \cdot_4 X = fl_{1 \leftarrow 3} fl_{2 \leftarrow 4} (\mu \otimes \mu \otimes x \otimes x) \cdot_2 fl_{1 \leftarrow 3} fl_{2 \leftarrow 4} X$$

There are many other definitions in tensor algebra (n -mode tensor-matrix products [3,1], etc.) which are out of the scope of this paper. The reader is referred to the just cited works and textbooks [8,15] for further information about tensor algebra.