On the conservativeness of fuzzy and fuzzy-polynomial control of nonlinear systems

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Abstract

A fairly general class of nonlinear plants can be modeled as fuzzy systems, i.e., as a time-varying convex combination of “vertex” linear systems. As many linear LMI control results naturally generalize to such fuzzy systems, LMI formulations for fuzzy control became the tool of choice in the 1990s. Important results have since been obtained in the fuzzy arena, although significant sources of conservativeness remain. This paper reviews some of the sources of conservativeness of fuzzy control designs based on the linear vertex models instead of the original nonlinear equations. Then, ideas that may overcome some of the conservativeness issues (but increasing computational requirements) are discussed. Recently, the sum of squares paradigm extended some linear results to polynomial systems; this idea can be used for the so-called fuzzy polynomial systems that are also discussed in this work.

Key words: LMI, fuzzy control, relaxed stability conditions, polynomial fuzzy systems, nonlinear control, sum of squares

1 Introduction

Fuzzy control concepts were first proposed in the mid-1970’s, in an attempt to embed logical reasoning in the feedback control loop (Arzén, 1996). In the early days, control rules were basically set heuristically in so-called “expert control systems”. Indeed, the heuristic-based aspect of fuzzy logic is currently popular in control applications, and in other aspects of automation such as decision support or diagnosis (Filip, 2008; Sala, 2008).

However, these purely logical and heuristic designs have lost relevance in the current fuzzy control area. This is because one of the following two factors is usually present: (a) the rules show no fundamental differences with standard PID regulators, in the so-called fuzzy-PD, fuzzy-PI, and similar (Li and Gatland, 1996); or (b) heuristic designs, which fuzzify specialized operation rules for a complex plant, are revealed to be one-of-a-kind tailored developments of little interest to a broad research-focused audience. As a result, current fuzzy control research uses rigorous core control theory results to guarantee specifications expressed in terms of stability, performance, and robustness to modelling errors, etc.

This paper focuses on a class of nonlinear systems for which a systematic modeling methodology (sector nonlinearity (Tanaka and Wang, 2001)) can be used for transformation into the so-called fuzzy Takagi-Sugeno (TS) form (Takagi and Sugeno, 1985). This is a time-varying convex combination of linear models (denoted as consequent or vertex models).

The above class is quite general (all $\dot{x} = f(x)$, with $f$ being $C^1$), and the transformation is exact, meaning that the TS models are equivalent to the original nonlinear models (in many cases, only locally in a compact region of interest containing the origin). In this way, nonlinear systems may be “embedded” in a linear time-varying (LTV) dynamic. Hence, when designing a controller for a nonlinear system in this class, both fuzzy tools (Tanaka and Wang, 2001; Sala et al., 2005), and direct nonlinear tools (Khalil, 1996; Freeman and Kokotovic, 1996; Isidori, 1995) can be applied.

The main advantage of the fuzzy approach when compared to generic nonlinear control is that many linear, LTV, and LPV techniques can be almost directly applied to nonlinear plants in TS form - using efficient semidefinite-programming tools (LMI (Boyd et al., 1994; Boyd and Vandenberghe, 2004)) in a unified approach.
However, there are important drawbacks arising from focusing only on the vertex models to infer properties of the original nonlinear system.

The main goal of this paper is to review such conservativeness issues in fuzzy system analysis and controller design. The main sources of conservativeness in current TS fuzzy literature are: locality, shape-independence, sufficiency of the conditions and, lastly, restrictions on possible choices of Lyapunov functions. Further shortcomings appear when discussing observers, identification from data, and non-uniqueness of the TS representation; as well as the intrinsic computational complexity of some results. For other open issues and trends in fuzzy systems focusing only on the vertex models to infer properties of the system, the reader is referred to (Sala et al., 2005; Feng, 2006; Babuska and Verbruggen, 2003).

Recently, the advent of sum-of-squares tools (Prajna et al., 2004b) has spurred some analysis and controller design techniques for (non-fuzzy) polynomial systems. Such techniques are a natural generalization of LMI-based techniques for linear systems. Fuzzy control can also benefit from the idea, by generalizing some of the polynomial system ideas to the so-called fuzzy polynomial models. Such models can also be obtained by a variation of the sector-nonlinearity technique (Sala and Ariño, 2007). As puts $\dot{x} = f(x, u)$, under mild assumptions, and in the following form (left: continuous-time, right: discrete-time):

$$\dot{x} = \sum_{i=1}^{r} \mu_i(x, u) f_i(x, u), \quad x_{k+1} = \sum_{i=1}^{r} \mu_i(x, u) f_i(x, u)$$

(1)

with $f_i$ linear and $0 \leq \mu_i \leq 1$, $\sum_{i=1}^{r} \mu_i = 1$, and meaning that $\mu_i$ belongs to the so-called standard simplex. Such a form is denoted as the Takagi-Sugeno (TS) form (Takagi and Sugeno, 1985) in fuzzy literature, while the functions $\mu_i$ are denoted as membership functions. TS models are also discussed under the term of LTV embedding in some literature (Angeli et al., 2000).

Basically, if an $n$-th order nonlinear system with $p$ inputs $\dot{x} = f(x, u)$, with the standard equilibrium condition $f(0, 0) = 0$, can be expressed via algebraic transformations as:

$$\dot{x}_i = \sum_{j=1}^{n} \xi_j (x, u) x_j + \sum_{j=1}^{p} \gamma_j (x, u) u_j$$

(2)

then, considering a region of interest $\Omega$ (including the equilibrium $x = 0$ in its interior $^2$), each $\xi_i$ may be expressed as:

$$\xi_j = \mu_1^i (x, u) \xi_{j,1} + \mu_2^i (x, u) \xi_{j,2}$$

(3)

$$\mu_1^i (x, u) = \frac{\xi_j (x, u) - \xi_{j,2}}{\xi_{j,1} - \xi_{j,2}} \quad \mu_2^i = 1 - \mu_1^i$$

(4)

Furthermore, for any pair of constants $\xi_{j,k}$, so verifying:

$$\xi_{j,1} \geq \sup_{(x,u) \in \Omega} \xi_j, \quad \xi_{j,2} \leq \inf_{(x,u) \in \Omega} \xi_j$$

(5)

the obtained $\mu_1^i, \mu_2^i$ lie in the closed interval $[0,1]$. If $\Omega$ is compact and $\xi_j$ are continuous, such supremum and infimum are actually attained at minimum and maximum points. Operating in a similar way with $\gamma_j$, (2) may be exactly expressed as:

$$\dot{x}_i = \sum_{j=1}^{2} \mu_1^j \xi_{i,j} x_1 + \cdots + \sum_{j=1}^{2} \mu_2^n \xi_{i,n} x_n + \sum_{j=1}^{2} \mu_1^{n+1} \xi_{i,n+1} u_1 + \cdots + \sum_{j=1}^{2} \mu_2^{n+p} \xi_{i,p} u_p$$

(6)

Now, as $\sum_{j=1}^{2} \mu_1^j = 1$ for all $i$, the above expression can be written as (7) on page 16. The summations between brackets in (7) depict a linear state-space model (when all $i$ are considered) for each index vector $(j_1, j_2, \ldots, j_{n+p})$, as described in (8) on page 16. This is already a so-called Tensor-Product TS fuzzy system (Ariño and Sala, 2007).

As

$$\sum_{j=1}^{2} \sum_{j_1=1}^{2} \cdots \sum_{j_{n+p}=1}^{2} \mu_1^{j_1} \mu_2^{j_2} \cdots \mu_2^{n+p} = 1$$

(9)

$^2$ Obviously, a trivial change of variable converts any equilibrium point into the origin.
expression (8) is itself a convex combination of \(2^{(n+p)}\) linear models, so after some straightforward manipulations, the system may be expressed as:

\[
\dot{x} = \sum_{i=1}^{r} \mu_i(x,u)(A_i x + B_i u) \tag{10}
\]

where \(r\) is the number of rules, equal to \(2^t\) where \(t = n + p\) is the number of nonlinearities processed\(^3\), and \(\mu_i = \mu_1^j \ldots \mu_t^j\), where \((j_1, \ldots, j_t) \in \{1, 2\}^t\) are the digits of the base-2 representation of number \(i\). As a result, a TS model (1) is obtained that exactly describes the original nonlinear dynamics in a compact neighborhood \(\Omega\) around the origin.

**Remark 1**: the classical 1st-order LTI Jacobian linearisation at the origin can always be obtained as a particular (fixed) convex combination of the vertex models (details omitted for brevity). While the linearization is only accurate in the limit case when the size of the region of interest \(\Omega\) tends to zero, the TS models are exact in the whole \(\Omega\). In this way, TS models may be considered a refinement of the linearization-based control for non-infinitesimal validity regions.

**Example.** As a simple example, consider the system:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\sin(x_1^2)x_1 + u
\end{align*}
\]

and the region of interest \(\Omega = \{(x_1, x_2)| |x_1| \leq 0.5\}\). As, \(0 \leq \sin(x_1^2) \leq \sin(0.5^2) = 0.2474\) for all \(x \in \Omega\), the nonlinearity \(\sin(x_1)\) may then be expressed, by following the above outlined methodology, as a state-dependent interpolation between 0 and 0.2474. The result is a 2-rule model in the form (10) with \(B_1 = B_2 = (0 1)^T\) and

\[
\begin{align*}
A_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & A_2 &= \begin{pmatrix} 0 & 1 \\ -0.2474 & 0 \end{pmatrix}, \\
\mu_1(x) &= \sin(x_1^2)/\sin(0.5^2), & \mu_2(x) &= 1 - \mu_1(x)
\end{align*}
\]

The reader is referred to (Tanaka and Wang, 2001) for further examples. Figure 1 graphically depicts some of the ideas in the transformation (2); this is why the methodology is usually denoted as “sector-nonlinearity modelling”: the basic idea of the approach features maximum and minimum bounds on those expressions that multiply the states.

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\(^3\) In practice, expressions in (2) are sought with a reasonable number of \(\xi_1, \gamma_j\) being constant so that the maximum and minimum are equal. In this case, they do not need the definition of an interpolant membership function, and so the value of \(t\) is drastically reduced.
in the form:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(x, u) \theta_i + \xi^*$$  \hspace{1cm} (12)

where $\xi^*$ is a minimum approximation error, which is progressively reduced as new independent regressors $\mu_i$ are added (universal approximation property (Kosko, 1994)). In particular, fuzzy system identification and adaptive fuzzy control makes use of such a form in order to estimate the parameters $\theta_i$ that fulfill various goals and restrictions; and sometimes even $\mu_i$ have adjustable parameters. The interested reader is referred to (Ge, 2001; Jia et al., 2005; Labiod and Guerra, 2007) and references therein for further information on the main issues which arise when using (12) in adaptive fuzzy control.

3 Fuzzy controllers

The most widely used controller for controlling the systems (10) has membership functions equalising those of the process to be controlled, and so for a state-feedback configuration:

$$u = -\sum_{i=1}^{r} \mu_i K_i x$$  \hspace{1cm} (13)

This is the so-called parallel distributed compensator (PDC). Note that, to compute $u$, all the variables involved in $\mu_i$ must be measurable, and $\mu_i$ should not depend on $u$ (to avoid an algebraic loop). These are common assumptions in the literature. Other non-PDC control structures are mentioned later.

Linear Matrix Inequality (LMI) techniques have become the tool of choice for designing fuzzy controllers (such as (13)) when a fuzzy model of the process is available in the Takagi-Sugeno form. LMIs were introduced in linear and robust control in the late 80’s and 90’s (Boyd et al., 1994), and were then introduced in the late 90's (Tanaka and Wang, 2001) in the fuzzy control community – becoming widespread over the last ten years. Results are available for systems with uncertainty, delay, descriptor forms, etc. Some simple cases are shown below to illustrate the fundamentals of the approach and introduce some notation.

Joining (10) and (13) yields a closed-loop (Tanaka and Wang, 2001) given by:

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (A_i - B_i F_j) x$$  \hspace{1cm} (14)

A simple condition to ensure stability of (14) can be derived from a quadratic Lyapunov function (find $P$ such that $V = x^T P x > 0, -\dot{V} > 0$) as shown in (Wang et al., 1996; Tanaka and Wang, 2001). After a standard change of variable $\psi = P^{-1} x$, stability (actually, decay rate performance $\alpha \geq 0$) is proven (Tanaka and Wang, 2001) if:

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \psi^T Q_{ij} \psi \geq 0$$  \hspace{1cm} (15)

with

$$Q_{ij} = -(A_i X + X A_i^T - B_i M_j - M_j^T B_i^T + 2\alpha X)$$  \hspace{1cm} (16)

for $\psi \neq 0$, where $P^{-1} = X > 0$ and $M_i = F_i X$ are LMI decision variables and $\alpha$ is a user-defined decay-rate parameter ($X > 0$ denotes that $X$ should be a positive-definite matrix).

This is the simplest example of a class of widely-used conditions for stability or performance of a closed-loop fuzzy control system. These conditions may be expressed, for some matrices $Q_{ij}$, in the form (15). The left-hand term of expression (15) will be denoted as double fuzzy summation. Note, importantly, that if $Q_{ij}$ are linear in some matrix unknowns, then the above referenced LMI techniques (Tanaka and Wang, 2001) may be used to check condition (15) by restating it as the requirement of positive-definiteness in the matrix

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j Q_{ij}$$

Another example of performance-related condition uses

$$Q_{ij} = \begin{pmatrix} P A_i^T + R_i^T B_{i1}^T + A_i P + B_{2i} R_j & B_{i1} & P C_i^T + R_i^T D_{12i}^T \\ B_{i1}^T & -\gamma I & D_{11i}^T \\ C_i P + D_{12i} R_j & D_{11i} & -\gamma I \end{pmatrix}$$  \hspace{1cm} (17)

to prove that the $H_{\infty}$ norm (i.e., $\mathcal{L}_2$ to $\mathcal{L}_2$ induced norm) of a TS fuzzy system given by:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z) (A_i x + B_{i1} v + B_{2i} u)$$  \hspace{1cm} (18)

$$y = \sum_{i=1}^{r} \mu_i(z) (C_i x + D_{11i} v + D_{12i} u)$$  \hspace{1cm} (19)

is lower than $\gamma$. The reader is referred to (Tuan et al., 2001) for details on how (17) is obtained.

Other well-known performance and robustness requirements for fuzzy systems can also be cast as (15), as well as conditions for discrete-time TS systems $x_{n+1} = \sum_{i=1}^{r} \mu_i(x_n + B_i u_n)$. The reader is referred to (Tanaka et al., 1998; Oliveira et al., 1999; Tanaka and Wang, 2001), etc. for details.

As a generalisation of (15), other fuzzy control results require positiveness of a $p$-dimensional fuzzy summation,
i.e., checking $\forall x \neq 0$

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \cdots \sum_{i_p=1}^{r} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_p}x^T Q_{i_1i_2\ldots i_p}x > 0$$

(20)

The case $p = 2$ reduces to (15). Conditions requiring $p = 3$ are, for instance, the fuzzy dynamic controllers in (Li et al., 1999; Tanaka and Wang, 2001), using $Q_{ijk} = E_{ijk} + E_{ijk}^T$, with

$$E_{ijk} = \begin{pmatrix} A_{i}Q_{11} + B_{i}C_{jk} & A_{i} + B_{i}D_{i}C_{k} \\ A_{ijk} & A_{i}P_{11} + B_{ij}C_{k} \end{pmatrix} < 0$$

(21)

or the output-feedback conditions in (Fang et al., 2006; Chen et al., 2005). There are other non-PDC control laws, for instance (Guerra and Vermeiren, 2004; Kruszewski et al., 2007), that also yield fuzzy summation conditions. Also, when the memberships of the fuzzy controller, say $\eta_i$, are not coincident with those of the process, the results are equations in the form

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \eta_j \psi^T_i Q_{ij} \psi \geq 0$$

(22)

### 3.1 Positivity conditions for fuzzy summations

In the fuzzy literature, once the control requirements have been expressed as (15) or (20), some conditions to prove such positivity must be stated. As $\mu_i$ (shorthand for $\mu_i(x)$) depends on the state, the summation is a complex expression which, in a general case, is fairly difficult to handle. To avoid such issues, the dependence on $x$ of $\mu_i$ is usually disregarded (see Section 4.4) and conditions are proven for all possible $\mu_i > 0$ so that $\sum_i \mu_i = 1$. The set to which $\mu_i$ belongs is denoted the $r - 1$-dimensional standard simplex.

In this way, only sufficient conditions are obtained, and these are routinely used to set up a set of LMIs (see below) which, if feasible, provide a solution for the original fuzzy control problem.

Basic sufficient conditions for (15) are $Q_{ii} > 0$, $Q_{ij} + Q_{ji} > 0$ (Tanaka and Wang, 2001). Perhaps the most widely used improved conditions are those in (Liu and Zhang, 2003), and shown below:

(Liu and Zhang, 2003, Theorem 2). Expression (15), under fuzzy partition conditions $\sum_i \mu_i = 1$, $\mu_i \geq 0$, holds if there are matrices $X_{ij} = X_{ji}^T$ so that:

$$X_{ii} \leq Q_{ii}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} \text{ } i \neq j$$

(23)

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \cdots & X_{rr} \end{pmatrix} > 0$$

(24)

(25)

Less conservative examples appear in (Fang et al., 2006). Conditions not involving additional slack variables appear in (Tuan et al., 2001). Other recently published examples (Sala and Ariño, 2007a) will be later discussed.

With regard to (22) in shape-independent setups (i.e., requiring (22) to hold for all values of $\mu_i$, $\eta_i$ positive adding), it holds if and only if $Q_{ij} > 0$ (usually resulting in much more conservative performance bounds than the PDC case $\mu \equiv \eta$). If some relationships between $\mu$ and $\eta$ are known, then relaxations in (Ariño and Sala, 2008) may be used to overcome such conservatism.

### 4 Sources of conservatism in the fuzzy approach

Basically, there are quite a few sources of conservatism in the above reviewed mainstream fuzzy control results. This section will discuss a selection of these in some detail.

#### 4.1 Conservatism of the Lyapunov approach

In fuzzy control, the Lyapunov function search is only made over a particular family of candidate functions; hence, a stable system may not have a Lyapunov function in the particular family (parametrization) being sought. In fact, this problem is common to nonlinear control theory: the Lyapunov approach is not constructive.

In the fuzzy control literature, the approach using the quadratic Lyapunov function $V(x) = x^TPx$ has been deeply explored. Improvements are available: a piecewise quadratic function (Feng and Harris, 2001; Johansson, 1999), and fuzzy Lyapunov functions $V(x) = \sum_i \mu_i x^TP_i x$ for continuous (Tanaka et al., 2003; Tanaka et al., 2007a) or $V(x) = x^T(\sum_i \mu_i G_i)^{-1}x$ for discrete (Guerra and Vermeiren, 2004) systems.

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4 The issues in this section have been put forward based on the author’s experience. Other issues, such as delay, uncertainty structures, adaptive approaches, hybrid and non-smooth fuzzy systems, etc., are also important. However, despite deserving further attention, they are not discussed here.
Then, proving a “global stability” result in the TS system, say, $\Sigma_{TS}$, obtained from a nonlinear system, $\Sigma_{NL}$, so that the TS system exactly models $\Sigma_{NL}$ in a compact region $\Omega$, by using the approach in Section 2.

4.2 Locality of TS models.

Another source of conservativeness arises from the usually local nature of TS models. Indeed, consider a TS system $\Sigma_{TS}$ not exactly local models $\Sigma_{NL}$ in a compact region $\Omega$, by using the approach in Section 2.

Then, proving a “global stability” result in the TS system $\Sigma_{TS}$, i.e., finding a decreasing Lyapunov function $V(x)$ for any state $x$ and memberships $\mu(x)$ even out of $\Omega$! Justifying “local stability” for $\Sigma_{NL}$, for instance, by the fuzzy methodology to result in stable behaviour for $\Sigma_{NL}$.

Similar issues would arise in other input-output results - such as proving bounds for disturbance-rejection $H_\infty$ (i.e., $L_2$ induced norm bounds) performance: too big a disturbance would force the state out of $\Omega_{\nu,m}$ during the transient and, hence, all guarantees for $\Sigma_{NL}$ would be lost even if “global” $H_\infty$ bounds exist for $\Sigma_{TS}$.

4.3 Conservatism of the positivity conditions for fuzzy summations.

In Section 3.1 the positivity conditions in the current literature have been recalled.

As there are only sufficient conditions for (15) to hold, a Lyapunov function fulfilling (15) may exist but the conditions above may fail to find it.

In fact, apart for being conservative for disregarding the dependence of $\mu_i$ on $x$ (see Section 4.4 below), the conditions are conservative, even when considering all $\mu_i$ in the standard simplex.

For instance, consider the double fuzzy summation

$$\phi = \sum_i \sum_j \mu_i \mu_j q_{ij} = \mu_1^2 - 1.8 \mu_1 \mu_2 + \mu_2^2$$

for $q_{11} = 1$, $q_{12} = q_{21} = -0.9$, $q_{22} = 1$. By replacing $\mu_2 = 1 - \mu_1$, as $\phi = 3.8(\mu_1^2 - \mu_1) + 1$, it is easy to check positivity for all $\mu_1 \in [0,1]$ but, for instance, the simple condition $q_{12} + q_{21} < 0$ from (Tanaka and Wang, 2001) does not hold. Other (more sophisticated) counterexamples can be found for the conditions in (Liu and Zhang, 2003; Fang et al., 2006). Due to this fact, LMI conditions in the literature may fail to obtain valid solutions. There is a quest in the fuzzy community to find necessary and sufficient conditions for the positivity of (15): unfortunately, the quest is basically futile as the expression (15) is a case of the co-NP-complete matrix copositivity problem in (Murty and Kabadi, 1987). Conditions which reduce the discussed conservativeness (at the cost of ever-increasing computational cost) are outlined in Section 5.2.

4.4 Intrinsic conservativeness of the fuzzy approach versus a nonlinear one: shape-independence.

A key source of conservativeness of fuzzy approaches is the partial use of the knowledge of the membership function shape. Indeed, to implement a fuzzy PDC controller (13), the values of $\mu_i$ should be explicitly known function of some measurable variables. There is a quest in the fuzzy community to find necessary and sufficient conditions for the positivity of (15) - as the expression (15) is a case of the co-NP-complete matrix copositivity problem in (Murty and Kabadi, 1987). Conditions which reduce the discussed conservativeness (at the cost of ever-increasing computational cost) are outlined in Section 5.2.

Indeed, even if a nonlinear system, $\Sigma_{NL}$, is exactly modeled as a TS system, $\Sigma_{TS}$, in the form (10), for some explicitly known $\mu_i(x)$, the conditions from references in Section 3.1 prove stability or performance for the whole family of systems $\Sigma_{TS}^E$ as defined by

$$\dot{x} = \sum_{i=1}^r \mu_i(t) (A_i x + B_i u)$$

for any $\mu_i(t)$ as long as $0 \leq \mu_i \leq 1$, $\sum_i \mu_i = 1$. Evidently, $\Sigma_{TS} \subseteq \Sigma_{TS}^E$ as $\mu_i(x(t))$ are a particular case, but $\Sigma_{TS}^E$ is now a family of linear-time-varying systems.

Shape-independence may be good for two reasons: (a) it enables the use of generalisations of LMI techniques for linear systems; and (b) the results (if feasible) apply to a large set of nonlinear time-varying systems (as long as they share the same “vertices” $A_i, B_i$) so there is a sort of inherent robustness. However, the approach is,
evidently, conservative for a particular system (i.e., a Lyapunov function in a particular family may exist for \( \Sigma_{TS} \), but not for all members of \( \Sigma_{TS}^E \)).

On the contrary, Lyapunov-based nonlinear control usually explicitly takes into account the exact shape of the nonlinearity in order to derive results, being less conservative than the fuzzy approach when used for a particular nonlinear system.

As a trivial example, consider \( \dot{x} = \mu_1(z)x + (1 - \mu_1(z))(-x) \). It cannot be proven stable for an arbitrary \( \mu_1 \), \( 0 \leq \mu_1(z) \leq 1 \) (it is unstable for \( \mu_1(z) = 1 \)). However, it is stable for, say, \( \mu_1 = 0.2 + 0.2 \sin(x) \) because \( \dot{x} = (-1 + 2\mu_1)x \) is, trivially, an exponentially stable first-order nonlinear system when \( \mu_1 \leq b < 0.5 \), \( b \in \mathbb{R} \).

This example is a clear case indicating that this situation may happen even in first-order systems. By using nonlinear control ideas, once the explicit formula of the membership functions and the consequent vertex models are known. However, the number of nonlinearities considered, and so grouping required may be zero as possible (as previously commented, the TS representation is non-unique). Alternatively, some nonlinearities may be conservatively modelled as “uncertainty” in the vertex models, as in (Bouarrar et al., 2007). Obviously, considering nonlinearities as uncertainties increases conservativeness. In general, as discussed in next section, even if (theoretically) the conservativeness of the fuzzy approach can be

4.6 Unmeasured scheduling variables.

Most fuzzy control results (both in state- and output-feedback form) assume that the membership function arguments (scheduling variables) are measurable. This is not the case if membership functions depend on some unmeasured state variables that are computed in the controller via estimated outputs from an observer. Results on this topic in the TS field are preliminary and very conservative; for instance, conditions in (Guerra et al., 2006) are fulfilled by a robust linear (non-fuzzy) regulator.

4.7 Computational power - the curse of dimensionality.

The number of decision variables in some of the latest LMI results is huge. Even if LMI solvers use polynomial-time algorithms, the exponent of the system order is large, and many results can only be implemented for simple systems. Also, the number of rules using the sector-nonlinearity methodology explodes exponentially as the number of nonlinearities grow. To keep problems tractable, an expression (2) may be sought with as many \( \xi_j \) identically zero as possible (as previously commented, the TS representation is non-unique). Alternatively, some nonlinearities may be conservatively modelled as “uncertainty” in the vertex models, as in (Bouarrar et al., 2007). Obviously, considering nonlinearities as uncertainties increases conservativeness.
greatly reduced, it is often at the cost of a basically unsurmountable increase in computational requirements.

5 Some (partial) ideas

There are some ideas for reducing conservativeness in fuzzy control designs (in some of the first cases considered in the previous section), but these ideas greatly increase computational power, and so there is a tradeoff. The results, however, diminish the gap between fuzzy and nonlinear control, at least in theory. This section outlines some of the possibilities (the reader is referred to the provided literature for details).

5.1 Arbitrarily complex Lyapunov functions.

In Section 4.1, some issues regarding limited choices of Lyapunov functions in fuzzy control were discussed. It is worthwhile mentioning three possibilities for obtaining arbitrarily complex Lyapunov functions:

(1) higher-degree homogeneous Lyapunov functions (Chesi et al., 2003) or, even, non-homogeneous polynomial Lyapunov functions (see Section 6);
(2) exacerbate the piecewise idea (the point-wise approach in (Johansen, 2000));
(3) use “resampling”, checking for $V(t + k) - V(t) \leq 0$, $k > 1$ (Kruszewski and Guerra, 2005). Indeed, if and only if a system is stable, there will be a $k$ such that even $V(x) = x^T x$ will fulfill $V(t + k) - V(t) \leq 0$. Complexity lies in the predictors needed in the approach, and in an underlying non-delayed Lyapunov function expression.

Importantly, the proposed Lyapunov function structures are asymptotically exact when locally modeling a nonlinear system: if there exists a sufficiently smooth Lyapunov function for the system, it will be approximated to an arbitrary small error with a polynomial function, or an interpolation look-up table on a progressively denser grid; in the third case, there is no need to even search for it (a large value of $k$ will solve the problem).

Note, however, that the locality-related issues discussed in Section 4.2 still remain unsolved: fuzzy TS models do not extrapolate, and so the found Lyapunov functions may, if ill-conditioned, be insufficiently useful.

5.2 Asymptotically necessary and sufficient conditions for fuzzy summations.

Consider a multi-dimensional index variable $i \in \{1, \ldots, r\}^n$ where $r$ is the number of rules and $n$ is an arbitrary complexity parameter. Denote the permutations of $i$ by $P(i)$. Then, results in (Liu and Zhang, 2003; Fang et al., 2006) are particular cases of finding a multi-dimensional arrangement of matrices (tensor) fulfilling, for all $i$:

$$\sum_{j \in P(i)} Q_{j,1,2} > \sum_{j \in P(i)} \frac{1}{2}(X_j + X_j^T)$$  \hspace{1cm} (27)

and the inequality (with complexity $n - 2$):

$$\sum_{k \in B_{n-2}} \mu_k \xi^T \left[ \begin{array}{ccc} X_{(k,1,1)} & \cdots & X_{(k,1,r)} \\ \vdots & \ddots & \vdots \\ X_{(k,r,1)} & \cdots & X_{(k,r,r)} \end{array} \right] \xi > 0$$  \hspace{1cm} (28)

In a suitable recursive framework, it can be proven that the above conditions become necessary and sufficient with $n \to \infty$, and establish some “tolerance” parameter for finite $n$. The basic idea stems from Polya’s theorem on positiveness of homogeneous forms (Polya, 1928; de Llora and Santos, 2001; Scherer, 2006). The reader is referred to (Sala and Arino, 2007a) for details of the above idea in a fuzzy control framework.

However, the main issue with these conditions is that they are only necessary asymptotically; hence, infeasibility does not imply that the original fuzzy control problem is unsolvable (as the used conditions are only sufficient); it only implies that more computing resources may be needed to find a feasible solution. The issue, then, is determining when to stop increasing the computing complexity. This issue is addressed by using a triangulation approach in (Kruszewski et al., 2008): a complementary set of necessary (also asymptotically exact) conditions is considered there so conclusive infeasibility answers can be obtained for some cases (the available computer resources determine the size of the gap between the provably feasible and provably infeasible fuzzy control problems). The latter infeasible problems must then be approached with a different choice for either Lyapunov function parametrization, or the controller structure, or by incorporating membership-shape information (see below).

5.3 Look at the membership shape.

Incorporating membership-shape information may relax conservativeness. Indeed, as the membership functions for fuzzy controllers are known, some of their bounds can be computed. There are two types of such information:

- inequalities in the memberships themselves, say $\mu_1(x)\mu_2(x) < 0.1 \forall x \in \Omega$, obtained from the explicit expressions (4) for $\mu_1$ and $\mu_2$ in a particular system.
- Restrictions on the membership shape in particular regions of the state space (such as $\{x_1 \geq 0 \Rightarrow \mu_2 < 0.4\}$, or $\{\mu_3 < 0.2 \Rightarrow x_1^2 + x_2^2 < 0.7\}$). The basic idea is: if
in a certain zone of the state space a particular model is not totally active, the Lyapunov function existence conditions may take this into account (instead of just assuming that any value of the memberships is possible for any value of $x$, as most current literature does).

Ideas on incorporating both of those sources of information are outlined below. Another promising possibility is purposely designing the memberships of the fuzzy controller (not necessarily the same, nor the same number, as those of the plant) to achieve some performance objectives. A preliminary approach in that direction appears in (Lam and Leung, 2007).

5.3.1 Membership-only restrictions (LTV).

(a) Overlap bounds. As the membership functions for the fuzzy controllers are known, the following set of bounds can be easily computed:

$$0 \leq \mu_i(z)\mu_j(z) \leq \beta_{ij} \forall z$$  \hspace{1cm} (29)

The bounds $\beta_{ij}$ may be used to establish some relaxed LMIs. Early literature (Tanaka and Wang, 2001) already realized that $\beta_{ij} = 0$ enabled the corresponding term $Q_{ij}$ in (15) to be disregarded. However, even when the overlap is non-zero, some relaxations can be found: from (Sala and Ariño, 2007b), expression (15) holds that if there are matrices $X_{ij} = X^T_{ji}$ and symmetric $R_{ij} \geq 0$, $i \leq j$, such that, defining $\Lambda = \sum_{k=1}^{n} \sum_{i,j} \beta_{ki} R_{kl}$, then equations (23) and (24) are replaced, respectively by:

$$X_{ii} \leq Q_{ii} + R_{ii} - \Lambda$$  
$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} + R_{ij} - 2\Lambda$$  \hspace{1cm} (30)\hspace{1cm} (31)

The idea can be extended to any polynomial restriction in the memberships such as $\mu_i^2 - \mu_j^2 + 0.05 < 0$, as discussed in (Sala and Ariño, 2008) and Section 6. In this way, adding information about the “overlap” between membership functions enables additional performance to be extracted from a PDC controller for a particular system.

(b) Tensor-product fuzzy systems. As discussed in Section 2, fuzzy systems arising from sector-nonlinearity modelling actually appear in the tensor-product form (8) before “flattening” to (10). Such tensor structure can be exploited to obtain less conservative conditions: (Ariño and Sala, 2007) adapts (Liu and Zhang, 2003), and (Kruszewski, 2006) adapts (Tuan et al., 2001) to the tensor-product case.

Fig. 3. Membership function $\mu_i(x_1, x_2)$ for the example in the main text (contour plot).

(c) Uncertain memberships. If there is uncertainty in the values of the membership functions, the controller implements a set of memberships, $\eta_i$, different to those in the process $\mu_i$. As a result, expressions (22) appear. If some inequalities can be asserted relating $\eta_i$ and $\mu_i$, say $|\mu_i - \eta_i| \leq \delta_i$ or $\mu_i/\eta_i \leq \rho_i$, then conditions that are much less conservative than the shape-independent conditions $Q_{ij} > 0$ may be stated, as discussed in (Lam and Leung, 2005; Ariño and Sala, 2008).

5.3.2 Nonlinear restrictions (membership-state inequalities).

As previously discussed, one of the main sources of conservativeness in the fuzzy approach is considering memberships $\mu_i$ to be independent of the process state $x$. Approaches that consider such information (in the form of constraints which involve both $x$ and $\mu$) have appeared in recent literature.

As an example, consider the system $\dot{x} = \sum_{i=1}^{2} \mu_i(x)A_i x$ with

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & -1 \end{pmatrix}$$

with membership functions given by $\mu_1(x) = x_1^2/(x_1^2 + x_2^2) + (1 - \exp(-x_1^2 - x_2^2)) + 0.5 \cdot \exp(-x_1^2 - x_2^2)$, $\mu_2(x) = 1 - \mu_1(x)$ (a contour plot of $\mu_1$ is depicted in Figure 3).

Note that both $A_i$ are not Hurwitz, hence the usual fuzzy control approach of testing feasibility of $A_i^T P + PA_i < 0$ fails as no such $P > 0$ exists.

However, simulations show that the nonlinear system is stable. The trick is realizing that in the sector given by $x_1^2 - x_2^2 \geq 0$, there is $\mu_1 \geq \mu_2$ (so, when $x_1 > x_2$, $x_1$ decreases), whereas $\mu_1 \leq \mu_2$ is in the complementary sector $-x_1^2 + x_2^2 \geq 0$ (when $x_1 < x_2$, $x_2$ decreases).
When \( x_1 = x_2, \mu_1 = \mu_2 = 0.5 \) and the stable dynamics \( 0.5(A_1 + A_2) \) keeps \( x_1 = x_2 \) invariant.

The knowledge that there are regions of the state (described by quadratic inequalities) where memberships fulfill some other linear, or quadratic inequalities, may be incorporated in the LMI framework using the results in (Sala, 2008b). In the considered example, it is proven that a quadratic Lyapunov function exists for the above system. If the quadratic case fails, a different Lyapunov function may be searched for each region, leading to a shape-dependent piecewise approach (Bernal et al., 2008).

In a general case, partitioning the state space in arbitrarily small balls, or hypercubes, would enable a detailed description of the relationship between memberships and states to be made. This, in theory, could remove the shape-independence conservatism (with an exponentially growing computational cost, however).

As an alternative to the above piecewise approach, the above regions, inequalities or Lyapunov functions may be chosen to be not quadratic: for instance, polynomial expressions via algebraic transformations, as:

\[
\dot{x}_i = p_i(z, x, u)
\]

where \( p_i(z, x, u) \) are polynomials depending on input \( u \), state \( x \) and a vector \( z \) of continuous functions \( z = z(x, u, m) \) of any relevant state, input, or exogenous variable \( m \). Additionally, in order to fulfill later equilibrium conditions, condition \( p_i(z(x, u, m), 0, 0) = 0 \) for any \( x, u, m \) must hold.

Indeed, (2) is a particular case of the above, with \( p_i \) bilinear (linear in \( (x, u) \) and linear in \( z \)), being \( z \) composed of functions \( \xi \) and \( \gamma \) in (2).

In such a case, by finding upper and lower bounds for each component of \( z(x, u, m) \) in a particular operation region \( \Omega \), such a component can be expressed in a similar way to (3). Subsequently, the original nonlinear system may be expressed in the polynomial form:

\[
\dot{x}_1 = \pi_i(\mu, x, u) \quad j = 1, \ldots, n \forall x \in \Omega
\]  

where \( \pi_i \) is a polynomial on the membership functions, the state and the input, whose equilibrium point is zero for any value of \( \mu \).

For instance, considering a nonlinear system

\[
\dot{x} = \begin{pmatrix}
-x_1^3 - (0.05 + 0.95\sin x_1^2) x_1 x_3^2 \\
-(1 + \sin x_1^2) \ast x_2 - x_1 x_2 \\
-x_3 + 3 \ast x_1^2 \ast x_3 - 3 x_3
\end{pmatrix}
\]

and writing the nonlinearity \( z(x, u, m) = \sin x_1 \) as a fuzzy expression, the result from the methodology in Section 2, is:

\[
\sin x_1 = \mu_1 \ast (+1) + \mu_2 \ast (-1), \mu_1 + \mu_2 = 1
\]

Then, the system (34) may be expressed as a polynomial fuzzy system, by replacing the sinusoidal nonlinearity with \( \mu_1 - \mu_2 \), so the squared sinusoid is written as \( (\mu_1 - \mu_2)^2 = \mu_1^2 + \mu_2^2 + 2\mu_1 \mu_2 \).

**Remark:** In fact, a more precise description of the sinusoid could make use of its Taylor expansion around the origin (Sala and Ariño, 2009):

\[
\sin x_1 = x_1 - \frac{x_1^3}{3!} + T(x_1) \frac{x_1^5}{5!}
\]

where it is well-known that \( T(x_1) \) is equal to the fifth derivative of \( \sin x_1 \) at an intermediate point between zero and \( x_1 \). As such a derivative is continuous, the maximum \( (T) \) and minimum \( (T) \) of \( T(x_1) \) in any compact region of interest \( \Omega \) can be obtained. Therefore, \( T(x_1) = \mu_1(x_1) T + \mu_2(x_1) \) can be replaced in the above Taylor series and, hence, the sinusoidal can be expressed as the interpolation between two polynomials of degree
5. In this way, the sector-nonlinearity approach in Section 2 is the particular case of the 1st-order Taylor expansion. In the same way that sector-nonlinearity models converge to the linearized model when the size of the region of interest \( \Omega \) tends to zero, so the “polynomial fuzzification” converges to the Taylor approximation of order greater than 1 (for instance, the sinusoidal would converge to \( x_1 + x_2^2/3! + x_3^2/5! \) for small \( \Omega \)).

Hence, the sector-nonlinearity methodology can be extended so that: (1) non-polynomial nonlinearities, such as \( \sin x_1 \), are “fuzzified” as interpolation between two polynomials (usually locally, as in the local sector-nonlinearity technique), and; (2) polynomial nonlinearities are kept as such (including polynomial expressions of simpler nonlinearities, as \( (\sin x_1)^2 \) in the above example).

In this way, a continuous nonlinear system may be locally expressed via this expanded fuzzy modeling as a general polynomial fuzzy TS model: in which derivatives of state variables are expressed as polynomials in two sets of variables:

- state and input variables
- membership functions that are a nonlinear function of state variables and other exogenous inputs.

The membership functions are positive and, as usual, 

\[
\sum_i \mu_i \leq 1
\]

Note that the polynomial-TS-modeling methodology outlined above does not involve any “approximation error”. Note also that the representation is also non-unique, sharing with ordinary TS modeling techniques the same issues discussed in Section 4.5.

6.2 Stability analysis

Once a nonlinear system is equivalently expressed as a fuzzy-polynomial one, tools for polynomial systems may be applied to the resulting model. In a sense, polynomial TS systems are a “bridge” between the conservative linear-in-memberships, the linear-in-state, the linear-in-input plain TS model (10), and a general (Lipschitz continuous) nonlinear system \( \hat{x} = f(x, u, m) \). SOSTOOLS may be used with the polynomial model, whereas the original one may be difficult to deal with unless written in a particular form amenable to some nonlinear control techniques. By increasing the degree of the Taylor-series approximation, increasingly accurate bounding polynomials may be obtained (if the Taylor series uniformly converges to \( f \) in the region of interest \( \Omega \)).

For brevity, further details are omitted and outlined using an example on how the above referenced SOSTOOLS package may be easily used (at least in stability analysis) to obtain polynomial-in-state Lyapunov functions for the above class of polynomial TS systems.

For instance, consider the nonlinear system (34) and model \((\sin x_1)^2 = \mu_1 + \mu_2 \ast 1\). Then, force the non-negativeness of \( \mu_1 \) and \( \mu_2 \) by stating \( \mu_1 = \psi_1^2, \mu_2 = \psi_2^2 \), and force homogeneous by multiplying by \( \psi_1^2 + \psi_2^2 \) so that all the monomials are quadratic in \( \psi \). The result is a system given by:

\[
\dot{x} = \left( -\psi_1^2 x_2 - (\psi_1^2 + \psi_2^2)x_1 x_2 + \psi_2^2 (-2x_2) \right) \\
\left( \psi_1^2 + \psi_2^2 \right) \left( -x_3 + 3 \ast x_1^2 \ast x_3 - 3 \ast x_3 \right)
\]

Now consider a Lyapunov function \( V(x) \) in the form of a degree 4 polynomial in the state variables \( x_1, x_2, x_3 \).

Then, in a sum of squares programming package, the positiveness of \( V(x) \) is set as requiring \( V(x) \) to be SOS. The negativeness of \( \dot{V}(x) \) is set as requiring \( -\dot{V}(x) \) to be SOS, obtaining \( \frac{dV}{dt} \) with a symbolic algebra toolbox 6.

Using SOSTOOLS and ScDuMi, the solver finds a Lyapunov function given by:

\[
V(x) = 8.3931 \ast x_1^2 + 6.1584 \ast x_2^2 + 2.0284 \ast x_3^2 + 12.539 \ast x_1^4 + 0.24121 \ast 10^{-2} \ast x_1^3 + 12.074 \ast x_1^2 \ast x_2^2 + 1.5372 \ast x_1^3 \ast x_3^2 + 0.15023 \ast x_3^2 \ast x_3^3
\]

which is, evidently, SOS and whose time derivative with a changed sign is also SOS. Note that the Lyapunov function is not homogeneous (it is the sum of second- and fourth-order polynomials); this allows greater flexibility with respect to other approaches in the literature.

In the considered example, the proposed system is fortunately stable in all state space. If the search for a Lyapunov function had been unsuccessful, there would be some options to explore:

- Searching for a higher-degree global Lyapunov function (this approach may quickly exhaust computational resources).
- Pursuing a local approach to prove Lyapunov conditions in a region defined by \( g(x) \geq 0 \), being \( g(x) \) a known polynomial vector 7. This would be carried out

---

6 Note that the derivative of the Lyapunov function will then be non-positive for any value of \( \psi_i \), i.e., for any non-negative \( \mu_i \). As the fuzzy system is an homogeneous polynomial in \( \mu_i \), being non-positive in the standard simplex \( \sum \mu_i = 1 \) is trivially equivalent to being non-positive in the first quadrant.

7 Clearly, if the polynomial fuzzy system had been obtained from local modelling of a nonlinear system, then this local approach should have been pursued in the first place, by expressing the region of interest \( \Omega \) in the form \( g(x) \geq 0 \).
by searching for vectors of SOS polynomials η_1, η_2 so that V(z) = η_1 g(x) and -dV/dz = η_2 g(x) are SOS. η_1, η_2 are the analogue in SOS programming of the Lagrange multipliers in constrained optimization.

Note that, in fact, η_1 and η_2 need not be SOS, and just need to be positive in g(x) ≥ 0, and so such conditions can be nested (Sala, 2008b), following the idea of the so-called Positivstellensatz argumentation which generalizes the S-procedure and Karush-Kuhn-Tucker conditions to SOS problems (Prajna et al., 2004b).

- Adding shape-dependent information, if available, in the form of polynomial constraints: partitioning the state space in regions Ω_k where a set of polynomial constraints \( R_k(µ, x) \) are known to hold (for instance, for a region Ω_1 contained in a ball centered at the origin where the known memberships fulfill \{ (µ_2 - 1)^2 + µ_1^2 < 0.5, x_1^2 + x_2^2 ≤ 2 \}, ∀x \in Ω_1). They can be incorporated in the Lyapunov conditions, as discussed in (Sala, 2008b).

### 6.3 Controller design for fuzzy polynomial systems

If the fuzzy polynomial system is affine in control, the procedures in (Tanaka et al., 2007b) (an adaptation to the fuzzy case of those in (Prajna et al., 2004a)) may be readily applied. Indeed, consider an affine in control \( u \) variables and a control law:

\[
\dot{z} = \sum_{j=1}^{r} \mu_j(x, u, m) (A_j(x)z + B_j(x)u)
\]  

(36)

where \( z \) is a vector of known polynomials in the state variables and a control law: \( u = \sum_{j=1}^{r} \mu_j K(x)Q(x)z \).

Define a \( t \times n \) Jacobian matrix \( M(x) \) with elements:

\[
M_{ij} = \frac{∂z_i}{∂x_j} \quad \dot{z} = M \dot{x}
\]  

(37)

and, also, define a candidate Lyapunov function:

\[
V = z^T \sum_{k=1}^{r} \mu_k (Az + Bu)
\]

(39)

So, by replacing the control action, and denoting \( Qz = v \) we obtain:

\[
\frac{dV}{dt} = \sum_{k=1}^{r} \sum_{l=1}^{r} \mu_k \mu_l v^T (\sum_{j \in J} \frac{dP}{dx_i} A_k^T z) + 2M(\sum_{k=1}^{r} \mu_k (Az + Bu))
\]

(38)

The reader is referred to (Tanaka et al., 2007b; Prajna et al., 2004a) for details. After the change of variable \( \mu = \sigma_2 \), if the above expression is an SOS polynomial in \( (v, x, σ) \), a stabilizing controller has been found. Other state-feedback design criteria (such as \( H_∞ \), etc...) in (Prajna et al., 2004a) may also be adapted to the fuzzy polynomial case (details omitted for brevity).

**Discussion.** In the author’s opinion, the potential for using polynomial TS systems (obtained via the fuzzy Taylor series approach) in generalizing TS control techniques (yielding less conservative results) is interesting. However, many of the sources of conservativeness in Section 4 still apply in the fuzzy polynomial case. Also, some specific remarks are pertinent:

- The sum-of-squares approach for non-input affine polynomial TS systems is not so straightforward when feedback control laws are being designed. This is because, except in particular cases, products of decision variables appear. The issue requires further research.
- Shape-dependent conditions (constraints between \( µ \) and \( x \)) cannot be used after changes of variable - such as \( Qz = v \) before (39).
- Also, even if using polynomial-complexity LMI and semi-definite programming solvers underneath, the number of decision variables heavily increases with the system order, the number of rules, the degrees of the system polynomial bounds, and the degree of the polynomial Lyapunov function. Currently, the approach, albeit theoretically elegant, is only applicable in practice to simple problems (say, second- or third-order nonlinear systems where all the involved polynomials are of degree 6 or lower).
- Positivstellensatz conditions are used in order to incorporate shape constraints, local stability results, etc. Under certain conditions, these are asymptotically exact when the number of Lagrange multipliers tends to infinity (i.e., all the powers and cross-products of the restrictions are considered). However, the conditions are only sufficient when a finite number of Lagrange multipliers are used, which is the case in practical implementations.

### 7 Conclusions

This paper has reviewed the conservativeness of fuzzy control approaches when used to control a nonlinear system whose (non-fuzzy) model is originally available. Widely-used modeling and fuzzy control results have
been discussed. The most important sources of conservativeness were: (1) conservativeness in the choice of Lyapunov functions; (2) locality of the results (even if global results are proven for the TS model) and conditioning issues; (3) conservativeness of some fuzzy-summation positiveness conditions; (4) conservativeness due to shape-independence; (5) problems in identifying accurate TS models from data; (6) issues arising with non-measured scheduling variables.

Issues (2), (5), and (6) are basically unsolved. However, they may also be considered as open issues in mainstream non-fuzzy nonlinear control. The paper reviews paths for possible solutions of the other issues. In particular, asymptotically exact results are available for issues (1) and (3) above, as a complexity parameter \( n \) increases (computational requirements increase heavily with \( n \)). Some shape-dependence ideas for (4) are reviewed, both in an LTV setting (restrictions on \( \mu_i \)), and in a true nonlinear sense (combined restrictions between memberships and states).

Finally, the Taylor-series extension of sector-nonlinearity modeling ideas to polynomial TS systems is outlined for systems where non-polynomial nonlinearities appear in polynomials that themselves multiply polynomials in states and inputs. Such a class of systems is very general. In this way, tools for polynomial systems may be applied to polynomial TS systems: polynomial TS systems are a “bridge” between the conservative TS models and a “general” nonlinear system \( \dot{x} = f(x, u, m) \). The latter may be hard to deal with unless written in a particular form that is amenable to some nonlinear control techniques. Stability analysis and control design (for affine-in-control fuzzy polynomial systems) have been outlined.

In summary, the conservativeness of LMI fuzzy control as a tool for general nonlinear control can be significantly reduced providing that the actual shape of the membership functions is used in the LMs (exploiting some nonlinearity knowledge) and some “fuzzy” sufficient conditions are also made (asymptotically) necessary. This fact, jointly with the use of more general Lyapunov functions (in particular polynomial functions and those stemming from multiple-step-ahead prediction) look promising. Polynomial fuzzy systems also offer the potential to decrease part of the conservativeness, especially when high-order Taylor-series and shape-dependent conditions are considered.

The main drawback of the results is that many of them are only asymptotically exact, meaning that, in practice, most require plenty of computing power to achieve significant conservativeness reduction. Research is needed on the balance between the computational complexity and (non-asymptotic) accuracy of the results, as well as on the unsolved issues of identification and non-measurable scheduling variables.

References


\dot{x}_i = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \cdots \sum_{j_{n+p}=1}^{2} \mu_{j_1}^1 \mu_{j_2}^2 \cdots \mu_{j_{n+p}}^{n+p} \left( \sum_{k=1}^{n} \xi_{k,j_k} x_k + \sum_{k=1}^{p} \gamma_{k,j_k+n} u_k \right) \tag{7}

\dot{x} = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \cdots \sum_{j_{n+p}=1}^{2} \mu_{j_1}^1 \mu_{j_2}^2 \cdots \mu_{j_{n+p}}^{n+p} \left( A_{j_1,j_2,\ldots,j_{n+p}} x + B_{j_{n+1},\ldots,j_{n+p}} u \right) \tag{8}