Compiled on July 6, 2006

Idempotency of linear combinations of three idempotent matrices, two of which are commuting

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Abstract

Generalizing considerations of Baksalary et al. [J.K. Baksalary, O.M. Baksalary, G.P.H. Styan, Idempotency of linear combinations of an idempotent matrix and a tripotent matrix, Linear Algebra Appl. 354 (2002) 21-34], Baksalary [O.M. Baksalary, Idempotency of linear combinations of three idempotent matrices, two of which are disjoint, Linear Algebra Appl. 388 (2004) 67-78] characterized all situations in which a linear combination $\mathbf{P} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 + c_3\mathbf{P}_3$, with c_i, $i = 1, 2, 3$, being nonzero complex scalars and $\mathbf{P}_i$, $i = 1, 2, 3$, being nonzero complex idempotent matrices such that two of them, $\mathbf{P}_1$ and $\mathbf{P}_2$ say, are disjoint, i.e., satisfy condition $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_1$, is an idempotent matrix. In the present paper, the results given in the aforementioned paper by Baksalary are generalized by weakening the assumption expressing the disjointness of $\mathbf{P}_1$ and $\mathbf{P}_2$ to the commutativity condition $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$.

AMS classification: 15A57, 15A27

Keywords: Oblique projector; Orthogonal projector; Partitioned matrix

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1. Introduction

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices. The symbol $\mathbb{C}_n^{\mathsf{P}}$ will stand for the subset of $\mathbb{C}_{n,n}$ consisting of idempotent matrices (oblique projectors), i.e.,

$$\mathbb{C}_n^{\mathsf{P}} = \{\mathbf{P} \in \mathbb{C}_{n,n} \colon \ \mathbf{P}^2 = \mathbf{P}\},$$

whereas $\mathbb{C}_n^{\mathsf{OP}}$ for the subset of $\mathbb{C}_n^{\mathsf{P}}$ composed of Hermitian idempotent matrices (orthogonal projectors), i.e.,

$$\mathbb{C}_n^{\mathsf{OP}} = \{\mathbf{P} \in \mathbb{C}_{n,n} \colon \ \mathbf{P}^2 = \mathbf{P} = \mathbf{P}^*\},$$

where $\mathbf{P}^*$ is the conjugate transpose of $\mathbf{P}$. Moreover, $\mathbf{I}_n$ will mean the identity matrix of order n and $\mathrm{r}(\mathbf{K})$ will be the rank of $\mathbf{K} \in \mathbb{C}_{m,n}$.

The considerations of this paper were inspired by Baksalary [3] who considered the problem of characterizing all situations in which a linear combination of the form

$$\mathbf{P} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 + c_3\mathbf{P}_3, \tag{1.1}$$

with nonzero $c_i \in \mathbb{C}$, $i = 1, 2, 3$, and nonzero $\mathbf{P}_i \in \mathbb{C}_n^{\mathsf{P}}$, $i = 1, 2, 3$, such that two of them, $\mathbf{P}_1$ and $\mathbf{P}_2$ say, are disjoint, i.e., satisfy condition

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_1, \tag{1.2}$$

is an idempotent matrix. In the present paper, the results given in [3] are generalized by establishing the complete solution to the problem of when a linear combination of the form (1.1) satisfies $\mathbf{P}^2 = \mathbf{P}$ with the assumption (1.2) replaced by an essentially weaker commutativity condition

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1. \tag{1.3}$$

It should be emphasized that an additional motivation to generalize the problem originates from statistics, where considerations concerning the inheritance of the idempotency by linear combinations of idempotent matrices have very useful applications in the theory of distributions of quadratic forms in normal variables; see e.g., Lemma 9.1.2 in [6] or page 68 in [3]. Thus, it is of interest to explore the problem posed in [3] as extensively as possible.

It is noteworthy that the problem of characterizing situations in which a linear combination of the form (1.1) is an idempotent matrix was independently considered by Özdemir and Özban [5], with the use of a different formalism than the one utilized in [3]. However, this formalism has limited applicability, for it can be utilized exclusively to the situations in which matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in (1.1), are different, mutually commuting, i.e., satisfy

$$\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j\mathbf{P}_i,\ i \neq j,\ i, j = 1, 2, 3, \tag{1.4}$$

and such that either

$$\mathbf{P}_i\mathbf{P}_j = \mathbf{0} \quad \text{or} \quad \mathbf{P}_i\mathbf{P}_j = \mathbf{P}_i, \quad i \neq j,\ i, j = 1, 2, 3.$$

Furthermore, due to the intrinsic limitations of the formalism, the authors were able to characterize only some particular sets of sufficient conditions ensuring that $\mathbf{P}^2 = \mathbf{P}$; see Theorem 3.2 in [5].

In the next section we provide three theorems constituting the main result of the paper and show that the extent in which they generalize Theorem 1 in [3] is significant. Section 3 contains some additions results referring to the situations in which matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in (1.1), belong to the set $\mathbb{C}_n^{\mathsf{OP}}$, being of particular interest from the point of view of possible applications in statistics.

2. Main result

The main result of Baksalary [3] is given therein as Theorem 1, which is split into four disjoint parts (a)–(d) referring to situations in which matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in (1.1), in addition to (1.2), satisfy also conditions:

(a) $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_2$,

(b) $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$,

(c) $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_2$,

(d) $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$.

The complete solution to the problem considered in this paper is given in three subsequent theorems, of which Theorem 1 correspond to part (a) of Theorem 1 in [3], Theorem 2 to parts (b) and (c), and Theorem 3 to part (d). As already mentioned, Theorems 1–3 generalize Theorem 1 in [3] and the generalization is included in replacing condition (1.2) by an essentially weaker condition (1.3).

A theorem below generalizes part (a) of Theorem 1 in [3] and Theorem 3.2 in [5].

Theorem 1. *Let $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{P}}$ be nonzero and mutually commuting, i.e., satisfying conditions* (1.4). *Moreover, let $\mathbf{P}$ be a linear combination of the form* (1.1)*, with nonzero $c_1, c_2, c_3 \in \mathbb{C}$. Then the following list comprises characteristics of all cases in which $\mathbf{P}$ is an idempotent matrix:*

(a) $\mathbf{P}_i + \mathbf{P}_j\mathbf{P}_k = \mathbf{P}_i\mathbf{P}_j + \mathbf{P}_i\mathbf{P}_k$ *holds along with* $c_i = -1$, $c_j = 1$, $c_k = 1$,

(b) $\mathbf{P}_i = \mathbf{P}_i\mathbf{P}_j + \mathbf{P}_k$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{0}$, *hold along with* $c_i = -1$, $c_j = 1$, $c_k = 2$,

(c) $\mathbf{P}_i + 2\mathbf{P}_j\mathbf{P}_k = \mathbf{P}_j + \mathbf{P}_k$, $c_j = \frac{1}{2}$, $c_k = \frac{1}{2}$, *hold along with* $c_i = -\frac{1}{2}$ *or* $c_i = \frac{1}{2}$,

(d) $\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j$, $\mathbf{P}_i\mathbf{P}_k = \mathbf{P}_k$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{0}$, *hold along with* $c_i = 1$, $c_j = -1$, $c_k = -1$,

(e) $\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_k$ *holds along with* $c_i = 1$, $c_j = 1$, $c_k = -2$,

(f) $\mathbf{P}_i + \mathbf{P}_j\mathbf{P}_k = \mathbf{P}_j + \mathbf{P}_k$ *holds along with* $c_i = 2$, $c_j = -1$, $c_k = -1$,

(g) $\mathbf{P}_i = \mathbf{P}_j$, $\mathbf{P}_i\mathbf{P}_k = \mathbf{P}_i$, *hold along with* $c_i + c_j = -1$, $c_k = 1$,

(h) $\mathbf{P}_i = \mathbf{P}_j$ *holds along with* $c_i + c_j = 0$, $c_k = 1$,

(i) $\mathbf{P}_i = \mathbf{P}_j$, $\mathbf{P}_i\mathbf{P}_k = \mathbf{P}_k$, *hold along with* $c_i + c_j = 1$, $c_k = -1$,

(j) $\mathbf{P}_i = \mathbf{P}_j$, $\mathbf{P}_i\mathbf{P}_k = \mathbf{0}$, *hold along with* $c_i + c_j = 1$, $c_k = 1$,

(k) $\mathbf{P}_i = \mathbf{P}_j + \mathbf{P}_k$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{0}$, *hold along with* $c_i + c_j = 0$, $c_i + c_k = 0$ *or* $c_i + c_j = 0$, $c_i + c_k = 1$ *or* $c_i + c_j = 1$, $c_i + c_k = 1$,

(l) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0}$, $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$, *hold along with* $c_1 = 1$, $c_2 = 1$, $c_3 = 1$,

(m) $\mathbf{P}_1 = \mathbf{P}_2 = \mathbf{P}_3$ *holds along with* $c_1 + c_2 + c_3 \in \{0, 1\}$,

where in characteristics (a)–(k) $i \neq j,\ i \neq k,\ j \neq k,\ i, j, k = 1, 2, 3$.

Proof. Straightforward calculations show that matrix $\mathbf{P}$ of the form (1.1) is idempotent if and only if

$$\begin{aligned} &c_1(c_1 - 1)\mathbf{P}_1 + c_2(c_2 - 1)\mathbf{P}_2 + c_3(c_3 - 1)\mathbf{P}_3 + c_1c_2(\mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_2\mathbf{P}_1) \\ &\qquad + c_1c_3(\mathbf{P}_1\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_1) + c_2c_3(\mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2) = \mathbf{0}. \end{aligned} \tag{2.1}$$

Clearly, taking into account assumptions (1.4), equation (2.1) reduces to

$$\begin{aligned} &c_1(c_1 - 1)\mathbf{P}_1 + c_2(c_2 - 1)\mathbf{P}_2 + c_3(c_3 - 1)\mathbf{P}_3 \\ &\qquad + 2c_1c_2\mathbf{P}_1\mathbf{P}_2 + 2c_1c_3\mathbf{P}_1\mathbf{P}_3 + 2c_2c_3\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}. \end{aligned} \tag{2.2}$$

Sufficiency of the conditions revealed in 13 characteristics provided in the theorem follows by direct verification of criterion (2.2). For the proof of necessity, first observe that in view of (1.4), multiplying (2.2) by the products $\mathbf{P}_1\mathbf{P}_2$ and $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$ leads to

$$(c_1^2 - c_1 + c_2^2 - c_2 + 2c_1c_2)\mathbf{P}_1\mathbf{P}_2 + (c_3^2 - c_3 + 2c_1c_3 + 2c_2c_3)\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 = \mathbf{0} \tag{2.3}$$

and

$$(c_1^2 - c_1 + c_2^2 - c_2 + 2c_1c_2)\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 + (c_3^2 - c_3 + 2c_1c_3 + 2c_2c_3)\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}, \tag{2.4}$$

respectively. From (2.4) it is seen that $\mathbf{P}$ is idempotent only if either

$$\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 = \mathbf{0} \tag{2.5}$$

or

$$(c_1 + c_2 + c_3)(c_1 + c_2 + c_3 - 1) = 0. \tag{2.6}$$

On the other hand, combining (2.3) and (2.4) leads to another necessary condition, namely

$$(c_1 + c_2)(c_1 + c_2 - 1)(\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3) = \mathbf{0},$$

which implies that

$$c_1 + c_2 = 0 \quad \text{or} \quad c_1 + c_2 = 1 \quad \text{or} \quad \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3. \tag{2.7}$$

Observe that (2.2) does not change upon the mutual interchanges of indexes "1", "2", and "3". Therefore, by interchanging, on the one hand, indexes "2" and "3" and, on the other, indexes "1" and "3" from (2.7) we obtain two additional triplets of conditions. Clearly, (2.7) and its two counterparts obtained in such a way can be jointly expressed as

$$c_i + c_j = 0 \quad \text{or} \quad c_i + c_j = 1 \quad \text{or} \quad \mathbf{P}_i\mathbf{P}_j = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3, \quad i \neq j,\ i,j = 1,2,3. \tag{2.8}$$

Triplets of conditions (2.8) are satisfied simultaneously only if at least one condition from each of them is fulfilled. Consequently, we have to consider 27 cases characterized by the following sets of conditions:

(i) $c_i + c_j = 0$, $c_i + c_k = 0$, $c_j + c_k = 0$,

(ii) $c_i + c_j = 0$, $c_i + c_k = 0$, $c_j + c_k = 1$,

(iii) $c_i + c_j = 0$, $c_i + c_k = 1$, $c_j + c_k = 1$,

(iv) $c_i + c_j = 1$, $c_i + c_k = 1$, $c_j + c_k = 1$,

(v) $c_i + c_j = 0$, $c_i + c_k = 0$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$,

(vi) $c_i + c_j = 0$, $c_i + c_k = 1$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$,

(vii) $c_i + c_j = 1$, $c_i + c_k = 1$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$,

(viii) $c_i + c_j = 0$, $\mathbf{P}_i\mathbf{P}_k = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$,

(ix) $c_i + c_j = 1$, $\mathbf{P}_i\mathbf{P}_k = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$,

(x) $\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$, $\mathbf{P}_i\mathbf{P}_k = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$, $\mathbf{P}_j\mathbf{P}_k = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$,

where in each set $i \neq j$, $i \neq k$, $j \neq k$, $i,j,k = 1,2,3$. Notice that: (a) each of sets (i), (iv), and (x) leads to just one triplet of conditions irrespective of what values are attributed to the indexes i,j,k, (b) from set (vi) six different triplets of conditions follow, and (c) from each of the remaining six sets three triplets are obtained. However, to enhance readability of the proof, in its subsequent steps we will assume that $i = 1$, $j = 2$, $k = 3$ and at the end of each step we will expand the conclusions obtained so as to cover all other possible combinations of values of i, j, k.

In the first step observe, that in view of the assumption $c_i \neq 0$, $i = 1, 2, 3$, conditions in sets (i) and (iii) cannot be satisfied. The next observations are that conditions in set (ii) hold merely when

$$c_1 = -\tfrac{1}{2},\ c_2 = \tfrac{1}{2},\ c_3 = \tfrac{1}{2}, \tag{2.9}$$

while conditions in set (iv) when

$$c_1 = \tfrac{1}{2},\ c_2 = \tfrac{1}{2},\ c_3 = \tfrac{1}{2}. \tag{2.10}$$

Since neither (2.9) nor (2.10) satisfies (2.6), it follows that in both these cases $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$. Substituting (2.9) to (2.2) leads to

$$3\mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_3 = 2(\mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3 - \mathbf{P}_2\mathbf{P}_3), \tag{2.11}$$

whereas substituting (2.10) yields

$$\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = 2(\mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3 + \mathbf{P}_2\mathbf{P}_3). \tag{2.12}$$

On account of (2.5), multiplying (2.11) and (2.12) consecutively by $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ reduces each of these relationships to

$$\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3,\ \mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_2\mathbf{P}_3,\ \mathbf{P}_3 = \mathbf{P}_1\mathbf{P}_3 + \mathbf{P}_2\mathbf{P}_3, \tag{2.13}$$

respectively. It is easy to verify that the conjunction of the three conditions in (2.13) is equivalent to

$$\mathbf{P}_1 = \mathbf{P}_2 + \mathbf{P}_3 - 2\mathbf{P}_2\mathbf{P}_3 \tag{2.14}$$

and that (2.14) implies (2.5). Combining (2.9) and (2.10) with (2.14) upon replacement of indexes 1 by i, 2 by j, and 3 by k leads to characteristic (c) of the theorem.

Substituting $c_1 + c_2 = 0$ and $c_1 + c_3 = 0$, being specific versions of the first two conditions in set (v), into (2.6) leads to $c_1(c_1 + 1) = 0$. In view of $c_1 \neq 0$, it is seen that under conditions (v) an alternative

$$c_1 = -1,\ c_2 = 1,\ c_3 = 1,\ \mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 \tag{2.15}$$

or

$$c_1 + c_2 = 0,\ c_1 + c_3 = 0,\ \mathbf{P}_2\mathbf{P}_3 = \mathbf{0} \tag{2.16}$$

is to be considered, where the last condition in (2.16) is obtained by combining (2.5) with matrix condition in set (v). In case (2.15), equation (2.2) reduces to the form

$$\mathbf{P}_1 - \mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_3 + \mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$$

and since this equality implies $\mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$, characteristic (a) of the theorem is established. Under conditions (2.16), equation (2.2) takes the form

$$c_1(\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 - 2\mathbf{P}_1\mathbf{P}_2 - 2\mathbf{P}_1\mathbf{P}_3) = \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_3. \tag{2.17}$$

Multiplying (2.17) by $\mathbf{P}_2$, $\mathbf{P}_3$, and utilizing the fact that $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$, leads to

$$(c_1 + 1)(\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2) = \mathbf{0} \quad \text{and} \quad (c_1 + 1)(\mathbf{P}_3 - \mathbf{P}_1\mathbf{P}_3) = \mathbf{0}, \tag{2.18}$$

respectively. From (2.18) it further follows that either $c_1 = -1$ or $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$. In the former of these situations, (2.17) yields $\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3$, leading to conditions

$$c_1 = -1,\ c_2 = 1,\ c_3 = 1,\ \mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3,\ \mathbf{P}_2\mathbf{P}_3 = \mathbf{0},$$

i.e., to the case, which is already covered by characteristic (a) of the theorem. In the latter situation, multiplying (2.17) by $\mathbf{P}_1$ leads to $(c_1 - 1)(\mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_3) = \mathbf{0}$. In consequence, either $c_1 = 1$ or $\mathbf{P}_1 = \mathbf{P}_2 + \mathbf{P}_3$ and hence the corresponding necessary conditions are

$$c_1 = 1,\ c_2 = -1,\ c_3 = -1,\ \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2,\ \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3,\ \mathbf{P}_2\mathbf{P}_3 = \mathbf{0},$$

leading to characteristic (d), and

$$c_1 + c_2 = 0,\ c_1 + c_3 = 0,\ \mathbf{P}_1 = \mathbf{P}_2 + \mathbf{P}_3,\ \mathbf{P}_2\mathbf{P}_3 = \mathbf{0},$$

leading to the first characteristic in (k).

Next, substituting into (2.6) the conditions

$$c_1 + c_2 = 0,\ c_1 + c_3 = 1, \tag{2.19}$$

being particular versions of the first two equalities in set (vi), leads to $c_1(c_1 - 1) = 0$. Since, in view of the second condition in (2.19), $c_1 = 1$ implies $c_3 = 0$, it is clear that in the considered case the product $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$ must necessarily be equal to zero matrix and, consequently, the last condition in set (vi) takes the form $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$. Substituting (2.19) along with $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$ into (2.2) yields

$$c_1(\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 - 2\mathbf{P}_1\mathbf{P}_2 - 2\mathbf{P}_1\mathbf{P}_3) = \mathbf{P}_1 - \mathbf{P}_2 + \mathbf{P}_3 - 2\mathbf{P}_1\mathbf{P}_3. \tag{2.20}$$

Multiplying (2.20) by $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ leads to

$$(c_1 - 1)(\mathbf{P}_1 - \mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_3) = \mathbf{0}, \tag{2.21}$$

$$(c_1 + 1)(\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2) = \mathbf{0}, \tag{2.22}$$

$$(c_1 - 1)(\mathbf{P}_3 - \mathbf{P}_1\mathbf{P}_3) = \mathbf{0}, \tag{2.23}$$

respectively. In view of (2.19), conditions (2.21)–(2.23) are satisfied simultaneously merely in two situations, namely when either

$$c_1 = -1,\ \mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3,\ \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$$

or

$$\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3,\ \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2,\ \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3.$$

On account of (2.19) and $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$, we conclude that these conditions can be extended to

$$c_1 = -1,\ c_2 = 1,\ c_3 = 2,\ \mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_3,\ \mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$$

and

$$c_1 + c_2 = 0,\ c_1 + c_3 = 1,\ \mathbf{P}_1 = \mathbf{P}_2 + \mathbf{P}_3,\ \mathbf{P}_2\mathbf{P}_3 = \mathbf{0},$$

respectively, leading to characteristic (b) and the middle characteristic in (k) of the theorem.

Substituting now

$$c_1 + c_2 = 1,\ c_1 + c_3 = 1, \tag{2.24}$$

i.e., particular versions of the first two conditions in (vii), into (2.6) leads to $(c_1 - 1)(c_1 - 2) = 0$. Since on account of (2.24), $c_1 = 1$ implies $c_2 = 0$, $c_3 = 0$, in the present case we have to consider an alternative $c_1 = 2$ or $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$. In the former situation, (2.24) implies $c_2 = -1$, $c_3 = -1$, and (2.2) reduces to the form

$$\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 - 2\mathbf{P}_1\mathbf{P}_2 - 2\mathbf{P}_1\mathbf{P}_3 + \mathbf{P}_2\mathbf{P}_3 = \mathbf{0}. \tag{2.25}$$

By multiplying (2.25) by $\mathbf{P}_2$, $\mathbf{P}_3$, and utilizing $\mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$, we obtain respectively $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$ and substituting these conditions back into (2.25) leads to

$$\mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_3 + \mathbf{P}_2\mathbf{P}_3 = \mathbf{0}. \tag{2.26}$$

Since (2.26) implies $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$, and $\mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$, in view of (2.24), it follows that the next set of necessary conditions is

$$c_1 = 2,\ c_2 = -1,\ c_3 = -1,\ \mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_2 + \mathbf{P}_3,$$

leading to characteristic (f). On the other hand, substituting (2.24) and $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$

(which is a consequence of $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$), and taking into account that $c_1 \neq 1$, into (2.2) reduces it to the form

$$\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 - 2\mathbf{P}_1\mathbf{P}_2 - 2\mathbf{P}_1\mathbf{P}_3 = \mathbf{0}. \tag{2.27}$$

Multiplying the equality above by $\mathbf{P}_2$ and $\mathbf{P}_3$, leads to $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$ and $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$, respectively, and substituting these conditions to (2.27) gives

$$\mathbf{P}_1 = \mathbf{P}_2 + \mathbf{P}_3. \tag{2.28}$$

In view of the fact that the conjunction of (2.28) and $\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$ implies both $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$ and $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$, we conclude this step of the proof by a quadruple of necessary conditions

$$c_1 + c_2 = 1,\ c_1 + c_3 = 1,\ \mathbf{P}_1 = \mathbf{P}_2 + \mathbf{P}_3,\ \mathbf{P}_2\mathbf{P}_3 = \mathbf{0},$$

which can be expanded to the last part of characteristic (k) of the theorem.

Let's now consider set (viii). Substituting

$$c_1 + c_2 = 0, \tag{2.29}$$

obtained from the first condition listed therein, to (2.6) leads to $c_3(c_3 - 1) = 0$. In view of $c_3 \neq 0$, it is clear that this time we have to consider an alternative $c_3 = 1$ or $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$. If $c_3 = 1$, then on account of (2.29) and $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2\mathbf{P}_3$ (obtained from the remaining two conditions in (viii)) equation (2.2) reduces to the form

$$c_1(\mathbf{P}_1 + \mathbf{P}_2 - 2\mathbf{P}_1\mathbf{P}_2) = \mathbf{P}_1 - \mathbf{P}_2. \tag{2.30}$$

Multiplying (2.30) by $\mathbf{P}_2$ yields $(c_1+1)(\mathbf{P}_2-\mathbf{P}_1\mathbf{P}_2) = \mathbf{0}$, from where we conclude that either $c_1 = -1$ or $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$. In the former case, from (2.30) it follows that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$. Since $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2\mathbf{P}_3$ clearly implies $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$ and $\mathbf{P}_2\mathbf{P}_3 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$, we obtain the next characteristic, namely

$$c_1 = -1,\ c_2 = 1,\ c_3 = 1,\ \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1,\ \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2\mathbf{P}_3,$$

which is a stronger version of characteristic (a) and is also covered by characteristic (h) of the theorem. On the other hand, if $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$, then from (2.30) it follows that $(c_1 - 1)(\mathbf{P}_1 - \mathbf{P}_2) = \mathbf{0}$, which further leads to an alternative $c_1 = 1$ or $\mathbf{P}_1 = \mathbf{P}_2$. In the former case, the necessary conditions form set

$$c_1 = 1,\ c_2 = -1,\ c_3 = 1,\ \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2,\ \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2\mathbf{P}_3,$$

which is again covered by characteristic (a), and in the latter one the necessary conditions are

$$c_1 + c_2 = 0,\ c_3 = 1,\ \mathbf{P}_1 = \mathbf{P}_2,$$

leading to characteristic (h).

To complete the considerations concerning conditions (viii) we still have to consider the case corresponding to $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 = \mathbf{0}$. In view of the last two conditions in (viii) it is seen that, in addition to (2.29), in this situation also $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_3$ hold, in which case (2.2) reduces to

$$c_1(c_1 - 1)\mathbf{P}_1 + c_1(c_1 + 1)\mathbf{P}_2 + c_3(c_3 - 1)\mathbf{P}_3 - 2c_1^2\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}. \tag{2.31}$$

Multiplying this condition by $\mathbf{P}_1$ and $\mathbf{P}_2$ leads to

$$(c_1 - 1)(\mathbf{P}_1 - \mathbf{P}_1\mathbf{P}_2) = \mathbf{0} \quad \text{and} \quad (c_1 + 1)(\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2) = \mathbf{0}, \tag{2.32}$$

respectively. From (2.32) it follows that three situations may occur, namely: $c_1 = 1$, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$ or $c_1 = -1$, $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$ or $\mathbf{P}_2 = \mathbf{P}_1$. Since $\mathbf{P}_3 \neq \mathbf{0}$, in each of them (2.31) implies $c_3 = 1$ and, correspondingly, three sets of necessary conditions are obtained

$$c_1 = 1,\ c_2 = -1,\ c_3 = 1,\ \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2,\ \mathbf{P}_1\mathbf{P}_3 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_3,$$

$$c_1 = -1,\ c_2 = 1,\ c_3 = 1,\ \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1,\ \mathbf{P}_1\mathbf{P}_3 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_3,$$

$$c_1 + c_2 = 0,\ c_3 = 1,\ \mathbf{P}_1 = \mathbf{P}_2,\ \mathbf{P}_1\mathbf{P}_3 = \mathbf{0}.$$

As easy to observe, the first two of them are covered by characteristic (a), and the third one by characteristic (h).

Two sets left to be considered, i.e., (ix) and (x). Substituting

$$c_1 + c_2 = 1, \tag{2.33}$$

obtained from the first condition in (ix), into (2.6) yields $c_3(c_3 + 1) = 0$. Thus, we obtain an alternative $c_3 = -1$ or $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_3$, where the matrix conditions are consequences of combining (2.5) with the last two conditions in (ix). If $c_3 = -1$, then on account of (2.33) and $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2\mathbf{P}_3$, equation (2.2) reduces to

$$c_1(c_1 - 1)(\mathbf{P}_1 + \mathbf{P}_2 - 2\mathbf{P}_1\mathbf{P}_2) + 2(\mathbf{P}_3 - \mathbf{P}_1\mathbf{P}_3) = \mathbf{0}, \tag{2.34}$$

and multiplying (2.34) by $\mathbf{P}_1$ further gives $c_1(c_1 - 1)(\mathbf{P}_1 - \mathbf{P}_1\mathbf{P}_2) = \mathbf{0}$. Consequently, $c_1 = 1$ or $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$. However, in view of (2.33), $c_1 = 1$ leads to $c_2 = 0$, which contradicts the assumptions. Hence, only condition $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$ is to be considered and substituting it into (2.34) yields

$$c_1(c_1 - 1)(\mathbf{P}_2 - \mathbf{P}_1) + 2(\mathbf{P}_3 - \mathbf{P}_1\mathbf{P}_3) = \mathbf{0}. \tag{2.35}$$

Equation (2.35), multiplied by $\mathbf{P}_2$, implies $\mathbf{P}_1 = \mathbf{P}_2$, which substituted back into (2.35) gives $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$. Thus, it follows that another set of necessary conditions is

$$c_1 + c_2 = 1,\ c_3 = -1,\ \mathbf{P}_1 = \mathbf{P}_2,\ \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3,$$

and constitute characteristic (i) of the theorem. On the other hand, if $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_3$, then, having in mind that (2.33) holds, (2.2) yields

$$c_1(c_1 - 1)(\mathbf{P}_1 + \mathbf{P}_2 - 2\mathbf{P}_1\mathbf{P}_2) + c_3(c_3 - 1)\mathbf{P}_3 = \mathbf{0}. \tag{2.36}$$

Multiplying (2.36) by $\mathbf{P}_1$ and $\mathbf{P}_2$ gives

$$c_1(c_1-1)(\mathbf{P}_1-\mathbf{P}_1\mathbf{P}_2)=\mathbf{0} \quad \text{and} \quad c_1(c_1-1)(\mathbf{P}_2-\mathbf{P}_1\mathbf{P}_2)=\mathbf{0},$$

respectively, and, in consequence, leads to an alternative $c_1 = 1$ or $\mathbf{P}_1 = \mathbf{P}_2$. Since, as already mentioned, $c_1 = 1$ implies $c_2 = 0$, we need to consider only the latter of these conditions. Substituting $\mathbf{P}_1 = \mathbf{P}_2$ into (2.36), in view of $\mathbf{P}_3 \neq \mathbf{0}$, gives $c_3 = 1$, and thus the next set of necessary conditions is

$$c_1+c_2=1,\ c_3=1,\ \mathbf{P}_1=\mathbf{P}_2,\ \mathbf{P}_1\mathbf{P}_3=\mathbf{0}.$$

In consequence, characteristic (j) is established.

Finally, substituting $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2\mathbf{P}_3$, obtained from set (x), into (2.2) leads to

$$c_1(c_1-1)\mathbf{P}_1+c_2(c_2-1)\mathbf{P}_2+c_3(c_3-1)\mathbf{P}_3+2(c_1c_2+c_1c_3+c_2c_3)\mathbf{P}_1\mathbf{P}_2=\mathbf{0}. \tag{2.37}$$

Assume first that (2.5) holds. Then (2.37) reduces to

$$c_1(c_1-1)\mathbf{P}_1+c_2(c_2-1)\mathbf{P}_2+c_3(c_3-1)\mathbf{P}_3=\mathbf{0},$$

and multiplying this condition by $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, in view of $c_i \neq 0$, $\mathbf{P}_i \neq \mathbf{0}$, $i = 1,2,3$, it respectively follows that $c_1 = 1$, $c_2 = 1$, $c_3 = 1$. Thus, another set of necessary conditions constitute characteristic (l) of the theorem. On the other hand, if (2.6) holds, then multiplying (2.37) by $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, gives

$$c_1(c_1-1)(\mathbf{P}_1-\mathbf{P}_1\mathbf{P}_2)=\mathbf{0},\ c_2(c_2-1)(\mathbf{P}_2-\mathbf{P}_1\mathbf{P}_2)=\mathbf{0},\ c_3(c_3-1)(\mathbf{P}_3-\mathbf{P}_1\mathbf{P}_2)=\mathbf{0},$$

respectively. These conditions are fulfilled in eight cases, namely when:

$$c_1=1,\ c_2=1,\ c_3=1,$$

$$c_i=1,\ c_j=1,\ \mathbf{P}_1\mathbf{P}_2=\mathbf{P}_k, \quad i\neq j,\ i\neq k,\ j\neq k,\ i,j,k=1,2,3,$$

$$c_i=1,\ \mathbf{P}_j=\mathbf{P}_k, \quad i\neq j,\ i\neq k,\ j\neq k,\ i,j,k=1,2,3,$$

$$\mathbf{P}_1=\mathbf{P}_2=\mathbf{P}_3.$$

If $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, then (2.6) does not hold, so this case is excluded from further considerations. Next, if two of c_is, c_1 and c_2 say, are equal to one, then (2.6) reduces to $(c_3+2)(c_3+1) = 0$, showing that either $c_3 = -2$ or $c_3 = -1$. Since $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_3$, being the third condition characterizing the present case, implies $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2\mathbf{P}_3$, we arrive at the following set of necessary conditions

$$c_1 = 1,\ c_2 = 1,\ c_3 = -2 \text{ or } c_3 = -1,\ \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_3.$$

However, situation corresponding to $c_3 = -1$ is already covered by characteristic (a) of the theorem, and thus only situation corresponding to $c_3 = -2$ is introduced to the theorem as characteristic (e). Further, if only one of c_is, c_1 say, is equal to one, then (2.6) reduces to the form $(c_2 + c_3)(c_2 + c_3 + 1) = 0$, from which it follows that either $c_2 + c_3 = 0$ or $c_2 + c_3 = -1$. Since $\mathbf{P}_2 = \mathbf{P}_3$ clearly implies $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_3$, but not necessarily $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_3$ (or $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2\mathbf{P}_3$), conditions on c_2 and c_3 must be supplemented by $\mathbf{P}_2 = \mathbf{P}_3 = \mathbf{P}_1\mathbf{P}_2$. The case corresponding to $c_2 + c_3 = 0$ is a stronger version of characteristic (h), while the case corresponding to $c_2 + c_3 = -1$ leads to characteristic (g) of the theorem. In the last step of the proof observe that if $\mathbf{P}_1 = \mathbf{P}_2 = \mathbf{P}_3$ then, in view of $\mathbf{P}_i \neq \mathbf{0}$, $i = 1, 2, 3$, also (2.6) necessarily holds. In such situation, condition (2.37) is clearly fulfilled and thus characteristic (m) of the theorem is established. □

Theorem 1 is supplemented by an analysis showing that the extent in which it generalizes part (a) of Theorem 1 in [3] is indeed essential. The following list indicates that only seven out of 13 characteristics listed in Theorem 1 above have their counterparts in Theorem 1 in [3]. Moreover, two from among the seven characteristics provide generalizations of their counterparts.

Theorem 1	Theorem 1 in [3]
characteristic (a) generalizes:	characteristics (a_2), (a_3), (a_5), and first cases in characteristics (a_6), (a_7)

characteristic (b) corresponds to	second cases in characteristics (a_6), (a_7)
characteristic (d) corresponds to	characteristic (a_4)
characteristic (h) generalizes	first cases in characteristics (a_8), (a_9)
characteristic (j) corresponds to	second cases in characteristics (a_8), (a_9)
characteristic (k) corresponds to	characteristic (a_{10})
characteristic (l) corresponds to	characteristic (a_1)

The next two theorems provide characteristics of situations in which matrices $\mathbf{P}_i$, $i = 1, 2, 3$, are such that, in addition to (1.3), they satisfy conditions $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$ and $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1$, $\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$, respectively. The proofs of these theorems are based on a formalism, referring to partitioned matrices, which turns out to be a very powerful tool in dealing with the situations of interest. In what follows – whenever the size of a given square zero matrix will not be definite – we will use a subscripted symbol $\mathbf{0}$ to indicate its order.

A theorem below generalizes parts (b) and (c) of Theorem 1 in [3].

Theorem 2. *Let* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{P}}$ *be nonzero and such that*

$$\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2, \quad \mathbf{P}_1\mathbf{P}_i = \mathbf{P}_i\mathbf{P}_1,\ i = 2, 3. \tag{2.38}$$

Moreover, let $\mathbf{P}$ *be a linear combination of the form* (1.1), *with nonzero* $c_1, c_2, c_3 \in \mathbb{C}$ *constituting* $\alpha = c_1(c_1 - 1)/c_2c_3$. *Then the following list comprises characteristics of all cases in which* $\mathbf{P}$ *is an idempotent matrix:*

(a) $\mathbf{P}_1\mathbf{P}_j = \mathbf{P}_1$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1 - \mathbf{P}_1\mathbf{P}_k$, *hold along with* $c_1 = -1$, $c_j = 2$, $c_k = -1$,

(b) $\frac{1}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = 2\mathbf{P}_k + \mathbf{P}_1\mathbf{P}_j - \mathbf{P}_1\mathbf{P}_k$ *holds along with* $c_1 = \frac{1}{2}$, $c_j = 1$, $c_k = -1$,

(c) $\mathbf{P}_1\mathbf{P}_j = \mathbf{0}$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1\mathbf{P}_k$, *hold along with* $c_1 = 1$, $c_j = 2$, $c_k = -1$,

(d) $\mathbf{P}_1\mathbf{P}_j = \mathbf{0}$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, $c_2 + c_3 = 1$, *hold along with* $c_1 + c_k = 0$ *or* $c_1 + c_k = 1$,

(e) $\mathbf{P}_1\mathbf{P}_j = \mathbf{P}_j$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1 + \mathbf{P}_k - \mathbf{P}_1\mathbf{P}_k$, *hold along with* $2c_1 + c_j = 0$, $c_k = 1$,

(f) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1$, *hold along with* $2c_1 + c_2 + c_3 = 1$,

(g) $\mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{2}$, *hold along with* $c_1 = -\frac{1}{2}$ *or* $c_1 = \frac{1}{2}$,

(h) $\frac{3}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3$ *holds along with* $c_1 = -\frac{1}{2}$, $c_2 = 1$, $c_3 = 1$,

(i) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$, *hold along with* $c_1 = -1$, $c_2 + c_3 = 1$,

(j) $\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2 - \mathbf{P}_3$, $4c_2^2(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, *hold along with* $c_1 = \frac{1}{2}$, $c_2 + c_3 = 0$,

(k) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$, *hold along with* $c_1 = 1$, $c_2 + c_3 = 1$,

where in characteristics (a)–(e) $j \neq k$, $j, k = 2, 3$.

Proof. Under assumptions (2.38), equation (2.1) reduces to

$$c_1(c_1 - 1)\mathbf{P}_1 + c_2(c_2 - 1)\mathbf{P}_2 + c_3(c_3 - 1)\mathbf{P}_3 \\ +2c_1c_2\mathbf{P}_1\mathbf{P}_2 + 2c_1c_3\mathbf{P}_1\mathbf{P}_3 + c_2c_3(\mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2) = \mathbf{0}. \tag{2.39}$$

On account of an obvious relationship $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_2 + \mathbf{P}_3 - \mathbf{P}_2\mathbf{P}_3 - \mathbf{P}_3\mathbf{P}_2$, condition (2.39) can be rewritten in the form

$$c_1(c_1 - 1)\mathbf{P}_1 + c_2(c_2 + c_3 - 1)\mathbf{P}_2 + c_3(c_2 + c_3 - 1)\mathbf{P}_3 \\ +2c_1c_2\mathbf{P}_1\mathbf{P}_2 + 2c_1c_3\mathbf{P}_1\mathbf{P}_3 = c_2c_3(\mathbf{P}_2 - \mathbf{P}_3)^2. \tag{2.40}$$

Sufficiency of the conditions revealed in 11 characteristics provided in the theorem follows by direct verification of criterion (2.40). For the proof of necessity, first observe that (2.40) as well as (2.38) are invariant with respect to interchanging indexes "2" and "3". Thus, similarly as in the proof of Theorem 1, it is reasonable to introduce two indexes, "j" and "k" say, such that $j, k \in \{2, 3\}$, and to use them, under the assumption that $j \neq k$, to express the necessary conditions in a possibly compact way.

It is known that every idempotent matrix is diagonalizable (see e.g., [7, Theorem 4.1]), and thus there exists a nonsingular matrix $\mathbf{W} \in \mathbb{C}_{n,n}$ such that

$$\mathbf{P}_1 = \mathbf{W}(\mathbf{I}_r \oplus \mathbf{0}_{n-r})\mathbf{W}^{-1}, \tag{2.41}$$

where $r = \mathrm{r}(\mathbf{P}_1)$ and "$\oplus$" denotes a direct sum. Clearly, $0 < r \leqslant n$ and if $r = n$, then the latter of the summands in representation (2.41) vanishes. Since $\mathbf{P}_1$, $\mathbf{P}_2$ as well as $\mathbf{P}_1$, $\mathbf{P}_3$ commute, we can represent $\mathbf{P}_2$ and $\mathbf{P}_3$ as

$$\mathbf{P}_2 = \mathbf{W}(\mathbf{X} \oplus \mathbf{Y})\mathbf{W}^{-1} \quad \text{and} \quad \mathbf{P}_3 = \mathbf{W}(\mathbf{S} \oplus \mathbf{T})\mathbf{W}^{-1}, \tag{2.42}$$

with $\mathbf{X}, \mathbf{S} \in \mathbb{C}_{r,r}$, $\mathbf{Y}, \mathbf{T} \in \mathbb{C}_{n-r,n-r}$, where $\mathbf{Y}$ and $\mathbf{T}$ vanish when $\mathbf{P}_1$ in (2.41) is nonsingular. From the idempotency of matrices $\mathbf{P}_2$ and $\mathbf{P}_3$ it follows that $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{S}$, and $\mathbf{T}$ are all idempotent. A consequence of this fact is that there exists a nonsingular matrix $\mathbf{U} \in \mathbb{C}_{r,r}$ such that

$$\mathbf{X} = \mathbf{U}(\mathbf{I}_x \oplus \mathbf{0}_{r-x})\mathbf{U}^{-1}, \tag{2.43}$$

where $x = \mathrm{r}(\mathbf{X})$. Clearly, $0 \leqslant x \leqslant r$, and if $x = 0$ then the former, whereas if $x = r$ then the latter, of the summands in representation (2.43) vanishes. Additional useful observations concerning matrices given in (2.41) and (2.42) are, on the one hand, that the idempotency of a linear combination (1.1) implies idempotency of both $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ and $c_2\mathbf{Y} + c_3\mathbf{T}$, and, on the other hand, that the first condition in (2.38) ensures that either $\mathbf{XS} \neq \mathbf{SX}$ or $\mathbf{YT} \neq \mathbf{TY}$.

Assume first that $\mathbf{XS} \neq \mathbf{SX}$, which means that $\mathbf{X}$ and $\mathbf{S}$ are nonzero and singular. Utilizing matrix $\mathbf{U}$ used in (2.43), we represent $\mathbf{S}$ as

$$\mathbf{S} = \mathbf{U}\begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{pmatrix}\mathbf{U}^{-1},$$

where $\mathbf{S}_1 \in \mathbb{C}_{x,x}$, $\mathbf{S}_2 \in \mathbb{C}_{x,r-x}$, $\mathbf{S}_3 \in \mathbb{C}_{r-x,x}$, and $\mathbf{S}_4 \in \mathbb{C}_{r-x,r-x}$. From $\mathbf{XS} \neq \mathbf{SX}$ it follows that either $\mathbf{S}_2 \neq \mathbf{0}$ or $\mathbf{S}_3 \neq \mathbf{0}$. Furthermore, the idempotency of $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ entails

$$c_1(c_1 - 1)\mathbf{I}_r + c_2(2c_1 + c_2 - 1)\mathbf{X} + c_3(2c_1 + c_3 - 1)\mathbf{S} + c_2c_3(\mathbf{XS} + \mathbf{SX}) = \mathbf{0}. \tag{2.44}$$

Premultiplying (2.44) by $\mathbf{U}^{-1}$ and postmultiplying it by $\mathbf{U}$ leads to

$$c_1(c_1-1)\begin{pmatrix}\mathbf{I}_x & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{r-x}\end{pmatrix}+c_2(2c_1+c_2-1)\begin{pmatrix}\mathbf{I}_x & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}+c_3(2c_1+c_3-1)\begin{pmatrix}\mathbf{S}_1 & \mathbf{S}_2\\ \mathbf{S}_3 & \mathbf{S}_4\end{pmatrix}$$
$$+c_2c_3\begin{pmatrix}2\mathbf{S}_1 & \mathbf{S}_2\\ \mathbf{S}_3 & \mathbf{0}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}. \tag{2.45}$$

It follows from the equation attributed to the upper-right submatrix in (2.45) when $\mathbf{S}_2$ is nonzero, and lower-left submatrix when $\mathbf{S}_3$ is nonzero, that

$$2c_1+c_2+c_3=1. \tag{2.46}$$

Substituting (2.46) into (2.44) yields

$$\alpha\mathbf{I}_r=(\mathbf{X}-\mathbf{S})^2, \tag{2.47}$$

where $\alpha=c_1(c_1-1)/c_2c_3$. Clearly, if $\mathbf{P}_1$ is nonsingular, i.e., $\mathbf{P}_1=\mathbf{I}_n$ or, equivalently, $r=n$, then $\mathbf{P}_2=\mathbf{WXW}^{-1}$ and $\mathbf{P}_3=\mathbf{WSW}^{-1}$. Hence, by premultiplying and postmultiplying (2.47) by $\mathbf{W}$ and $\mathbf{W}^{-1}$, respectively, we obtain a stronger version of characteristic (f) of the theorem.

On the other hand, if $\mathbf{P}_1$ is singular, then the latter of the summands in representation (2.41) is present and two situations can occur, namely $\mathbf{YT}\neq\mathbf{TY}$ and $\mathbf{YT}=\mathbf{TY}$. In the former of them, from Theorem in [1] it follows that the idempotency of a linear combination $c_2\mathbf{Y}+c_3\mathbf{T}$ entails $c_2+c_3=1$. Substituting this condition into (2.46) yields $c_1=0$, which is in a contradiction with the assumptions. If $\mathbf{YT}=\mathbf{TY}$, then $c_2\mathbf{Y}+c_3\mathbf{T}$ can be idempotent in seven situations characterized by the following sets of conditions:

(i) $\mathbf{Y}=\mathbf{0}$, $\mathbf{T}=\mathbf{0}$, $c_2,c_3\in\mathbb{C}$,

(ii) $\mathbf{Y}=\mathbf{T}$, $\mathbf{Y},\mathbf{T}\neq\mathbf{0}$, $c_2+c_3\in\{0,1\}$,

(iii) $\mathbf{Y}=\mathbf{0}$, $\mathbf{T}\neq\mathbf{0}$, $c_2\in\mathbb{C}$, $c_3=1$,

(iv) $\mathbf{Y}\neq\mathbf{0}$, $\mathbf{T}=\mathbf{0}$, $c_2=1$, $c_3\in\mathbb{C}$,

(v) $\mathbf{YT} = \mathbf{0}$, $c_2 = 1$, $c_3 = 1$,

(vi) $\mathbf{YT} = \mathbf{T}$, $c_2 = 1$, $c_3 = -1$,

(vii) $\mathbf{YT} = \mathbf{Y}$, $c_2 = -1$, $c_3 = 1$.

Sets (i)–(iv) constitute characterizations of situations in which a scalar multiple of an idempotent matrix is also idempotent, whereas sets (v)–(vii) follow straightforwardly from Theorem in [1]. Clearly, conditions (iii), (iv) and (vi), (vii) are analogues of each other obtained by interchanging matrices $\mathbf{Y}$ and $\mathbf{T}$ as well as scalars c_2 and c_3. In consequence, only five sets from the list above are to be considered.

If both $\mathbf{Y}$ and $\mathbf{T}$ are equal zero matrices, then from (2.41) and (2.42) it follows that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$ and $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$. Substituting these conditions along with (2.46) into (2.40) leads to $\alpha\mathbf{P}_1 = (\mathbf{P}_2 - \mathbf{P}_3)^2$. In consequence, characteristic (f) is obtained.

From (2.46) it follows that $c_2 + c_3 \neq 1$, for otherwise, as already mentioned, $c_1 = 0$. Thus, the last condition in set (ii) reduces to $c_2 + c_3 = 0$ and substituting it into (2.46) leads to $c_1 = \frac{1}{2}$. Another observation is that with $\mathbf{Y} = \mathbf{T}$, (2.41) and (2.42) entail $\mathbf{P}_1(\mathbf{P}_2 - \mathbf{P}_3) = \mathbf{P}_2 - \mathbf{P}_3$. Substituting this condition along with $c_1 = \frac{1}{2}$ and $c_2 + c_3 = 0$ into (2.40) gives $\mathbf{P}_1 = 4c_2^2(\mathbf{P}_2 - \mathbf{P}_3)^2$ and thus characteristic (j) of the theorem is established.

Substituting $c_3 = 1$, i.e., the last condition in set (iii), into (2.46) leads to $2c_1 + c_2 = 0$. Furthermore, on account of $\mathbf{Y} = \mathbf{0}$, it follows from (2.41) and (2.42) that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$. Hence, equation (2.40) reduces to

$$(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1 + \mathbf{P}_3 - \mathbf{P}_1\mathbf{P}_3,$$

and, in consequence, characteristic (e) follows.

Two sets remain to be considered, namely (v) and (vi). Combining $c_2 = 1$, $c_3 = 1$ with (2.46) gives $c_1 = -\frac{1}{2}$ and substituting these values of scalars c_i, $i = 1, 2, 3$, into (2.40) entails

$$\tfrac{3}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = \mathbf{P}_1(\mathbf{P}_2 + \mathbf{P}_3).$$

Consequently, we arrive at characteristic (h).

Similarly, combining $c_2 = 1$, $c_3 = -1$ with (2.46) gives $c_1 = \frac{1}{2}$ and hence (2.40) reduces to

$$\tfrac{1}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = 2\mathbf{P}_3 + \mathbf{P}_1(\mathbf{P}_2 - \mathbf{P}_3),$$

leading to characteristic (b).

Let's now consider situation in which $\mathbf{XS} = \mathbf{SX}$. Then, in view of $\mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2$, the latter of the summands in representation (2.41) is present and $\mathbf{YT} \neq \mathbf{TY}$. Hence, since a linear combination $c_2\mathbf{Y} + c_3\mathbf{T}$ is idempotent, from Theorem in [1] it follows that $(\mathbf{Y} - \mathbf{T})^2 = \mathbf{0}$ and

$$c_2 + c_3 = 1. \tag{2.48}$$

The remaining part of the proof will be based on an observation that $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ is a linear combination of three mutually commuting idempotent matrices $\mathbf{I}_r$, $\mathbf{X}$, $\mathbf{S}$ and thus – assuming that matrices $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ in a linear combination (1.1) are represented by $\mathbf{I}_r$, $\mathbf{X}$, and $\mathbf{S}$, respectively – we can utilize Theorem 1 of the present paper to characterize its idempotency.

First notice, that characteristics (a), (d), (e), (g), (h), and (l) of Theorem 1 cannot be reconciled with (2.48).

Combining conditions in characteristic (b) of Theorem 1 with (2.48) it follows that either $i = 2$, $k = 3$ or $i = 3$, $k = 2$ holds along with $j = 1$. In the former of these cases, the second matrix condition in (b) leads to $\mathbf{S} = \mathbf{0}$, whereas in the latter case to $\mathbf{X} = \mathbf{0}$. Each of these conditions is in a contradiction with the assumptions of Theorem 1.

Next, we consider characteristic (c) of Theorem 1. Combining first $c_i = -\frac{1}{2}$, $c_j = \frac{1}{2}$, $c_k = \frac{1}{2}$ with (2.48) leads to a conclusion that, in addition to $i = 1$, either $j = 2$, $k = 3$ or $j = 3$, $k = 2$. In both these cases, matrix condition in (c) yields $(\mathbf{X} - \mathbf{S})^2 = \mathbf{I}_r$. Hence, in view of $(\mathbf{Y} - \mathbf{T})^2 = \mathbf{0}$, it follows from (2.41) and (2.42) that

$$(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1. \tag{2.49}$$

This equation holds also when $c_i = -\frac{1}{2}$ is replaced by $c_i = \frac{1}{2}$ (which is an alternate case in (c)), for in this situation matrix condition in characteristic (c) entails $(\mathbf{X}-\mathbf{S})^2 = \mathbf{I}_r$ when $i = 1$ and $\mathbf{X} + \mathbf{S} = \mathbf{I}_r$ when $i = 2$ or $i = 3$. Since these conditions are equivalent (each of them implies $\mathbf{XS} = \mathbf{0}$), this step is concluded by an observation that substituting (2.49) along with possible triplets c_i, $i = 1, 2, 3$, into (2.40) gives $\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3$. Thus, characteristic (g) of the theorem is established.

Characteristic (f) of Theorem 1 is to be considered next. Combining $c_i = 2$, $c_j = -1$, $c_k = -1$ with (2.48) shows that four situations are possible, namely: (i) $i = 2$, $j = 3$, $k = 1$, (ii) $i = 2$, $j = 1$, $k = 3$, (iii) $i = 3$, $j = 2$, $k = 1$, and (iv) $i = 3$, $j = 1$, $k = 2$. In the first of them, $c_1 = -1$, $c_2 = 2$, $c_3 = -1$, and matrix condition in (f) gives $\mathbf{X} = \mathbf{I}_r$. In consequence, (2.41) and (2.42) entail $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$ and substituting this condition along with the values of c_1, c_2, c_3 into (2.40) leads to $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1 - \mathbf{P}_1\mathbf{P}_3$. The considerations concerning the remaining three situations are limited to two observations. The first of them is that characteristic (f) of Theorem 1 is invariant with respect to an interchange of indexes "j" and "k" and thus cases (i) and (ii) as well as (iii) and (iv) correspond to the same situations. The second observation is that cases (i) and (iii) are analogues of each other obtained by interchanging "2" and "3", and thus conditions obtained above can be expanded to characteristic (a).

The next characteristic of Theorem 1 to be considered is (i). Comparison of conditions on scalars c_1, c_2, c_3 provided therein with (2.48) shows that six cases are to be analyzed, namely: (i) $i = 1$, $j = 2$, $k = 3$, (ii) $i = 1$, $j = 3$, $k = 2$, (iii) $i = 2$, $j = 1$, $k = 3$, (iv) $i = 2$, $j = 3$, $k = 1$, (v) $i = 3$, $j = 1$, $k = 2$, (vi) $i = 3$, $j = 2$, $k = 1$. However, cases (i), (iii) as well as (ii), (v) and (iv), (vi) lead to the same situations. Moreover, cases (i) and (ii) are counterparts of each other obtained by interchanging "2" and "3". Consequently, it suffices to consider two cases only, say,

(i) in which $c_1 = -1$, $c_2 = 2$, $c_3 = -1$ and (iv) in which $c_2 + c_3 = 1$, $c_1 = -1$. In the former of them, the first matrix condition in (i) leads to $\mathbf{X} = \mathbf{I}_r$. From the discussion above, it follows that this case is already covered by characteristic (a) of the theorem. On the other hand, in case (iv), by utilizing matrix conditions in characteristic (i) of Theorem 1, we arrive at $\mathbf{X} = \mathbf{I}_r$, $\mathbf{S} = \mathbf{I}_r$. Hence, from (2.41) and (2.42) it follows that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$ and substituting these conditions along with $c_2 + c_3 = 1$, $c_1 = -1$ into (2.40) gives $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$, establishing characteristic (i) of the theorem.

Similarly as with respect to characteristic (i) of Theorem 1, also in the case of its characteristic (j) six situations are to be considered. However, as easy to verify by comparing corresponding conditions on scalars c_1, c_2, c_3 with (2.48), in two of them it follows that $c_2 = 0$ and in other two that $c_3 = 0$. Furthermore, in the remaining two situations, from matrix conditions in (j) we obtain $\mathbf{X} = \mathbf{0}$, $\mathbf{S} = \mathbf{0}$. Thus, each of these six situations is irreconcilable with the assumptions of Theorem 1.

The last two characteristics of Theorem 1 are to be considered. From the second matrix condition in (k) it is seen that $i = 1$, for otherwise $\mathbf{X} = \mathbf{0}$ or $\mathbf{S} = \mathbf{0}$. Consequently, there are four situations to be analyzed, namely when in addition to (2.48) the following conditions are fulfilled: (i) $c_1 + c_2 = 0$, $c_1 + c_3 = 0$, (ii) $c_1 + c_2 = 0$, $c_1 + c_3 = 1$, (iii) $c_1 + c_2 = 1$, $c_1 + c_3 = 0$, (iv) $c_1 + c_2 = 1$, $c_1 + c_3 = 1$. However, each of situations (ii) and (iii) entails $c_1 = 0$, which contradicts the assumptions, so they are excluded from further considerations. In the remaining two situations, i.e., (i) and (iv), $c_1 = -\frac{1}{2}$ and $c_1 = \frac{1}{2}$, respectively, hold along with $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{2}$. Since matrix conditions corresponding to those situations are equivalent to $(\mathbf{X} - \mathbf{S})^2 = \mathbf{I}_r$, utilizing the same arguments as those used in the proof corresponding to characteristic (c) of Theorem 1, we arrive at the characteristic already listed in the theorem as (g).

Finally, we consider characteristic (m) of Theorem 1. Comparing the last condition given therein with (2.48) shows that $c_1 = -1$. Furthermore, matrix condition in (m) entails $\mathbf{X} = \mathbf{I}_r$, $\mathbf{S} = \mathbf{I}_r$. Hence, from (2.41) and (2.42) it follows that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$, and substituting these conditions into (2.40) gives $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$. Thus, we again arrived at the characteristic already listed in the theorem, this time as (i).

In the last step of the proof, we consider three particular cases not covered by Theorem 1, in which either $\mathbf{X} = \mathbf{0}$ or $\mathbf{S} = \mathbf{0}$. Let us first assume that both these conditions are satisfied. Hence, from (2.41) and (2.42) it follows that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0}$. Furthermore, the idempotency of a linear combination $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ (now equal to $c_1\mathbf{I}_r$) entails $c_1 = 1$ and substituting $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0}$, $c_1 = 1$, $c_2 + c_3 = 1$ into (2.40) gives $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$. Since this condition combined with either $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$ or $\mathbf{P}_1\mathbf{P}_3 = \mathbf{0}$ implies the other of these relationships, we obtained characteristic (k) of the theorem. On the other hand, if $\mathbf{X} = \mathbf{0}$ and $\mathbf{S} \neq \mathbf{0}$, then the idempotency of $c_1\mathbf{I}_r + c_2\mathbf{X} + c_3\mathbf{S}$ means that $c_1\mathbf{I}_r + c_3\mathbf{S}$ is idempotent. We will separately consider the cases corresponding to $\mathbf{S} = \mathbf{I}_r$ and $\mathbf{S} \neq \mathbf{I}_r$. In the former of them, clearly, either $c_1 + c_3 = 0$ or $c_1 + c_3 = 1$, and, in view of $\mathbf{X} = \mathbf{0}$, $\mathbf{S} = \mathbf{I}_r$, (2.41) and (2.42) give $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$, $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$. Hence, on account of (2.48), equation (2.40) entails $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, and, since combining this condition with either $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$ or $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_1$ implies the other of these relationships, characteristic (d) follows. If now $\mathbf{S} \neq \mathbf{I}_r$, then, in view of (2.48), from Theorem in [1] it follows that $c_1 = 1$, $c_2 = 2$, $c_3 = -1$. Moreover, with $\mathbf{X} = \mathbf{0}$, (2.41) and the left identity in (2.42) yield $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$. With these conditions taken into account, (2.40) entails $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1\mathbf{P}_3$, leading to characteristic (c). We conclude this step with an observation that the case corresponding to $\mathbf{X} \neq \mathbf{0}$ and $\mathbf{S} = \mathbf{0}$ is a counterpart of the previous one obtained by interchanging in the resultant conditions indexes "2" and "3" and was taken into account by introducing indexes "j" and "k" in characteristics (d) and (c) of the theorem. The proof is complete. □

It is noteworthy that only three out of 11 characteristics listed in Theorem 2 have their counterparts in Theorem 1 in [3]. Namely, its characteristic (c) corresponds to characteristics (b_1), (c_1), its characteristic (d) to characteristics (b_3), (c_3), and its characteristic (k) to characteristics (b_2), (c_2). Clearly, the remaining eight characteristics in Theorem 2 originate from replacing the assumption (1.2) by (1.3).

The following result corresponds to the situation in which from among three possible pairs of matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in a linear combination (1.1), only $\mathbf{P}_1$ and $\mathbf{P}_2$ commute. It generalizes part (d) of Theorem 1 in [3].

Theorem 3. *Let $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{P}}$ be nonzero and such that*

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1, \quad \mathbf{P}_i\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_i, \ i = 1, 2. \tag{2.50}$$

Moreover, let $\mathbf{P}$ be a linear combination of the form (1.1)*, with nonzero $c_1, c_2, c_3 \in \mathbb{C}$. Then the following list comprises characteristics of all cases in which $\mathbf{P}$ is an idempotent matrix:*

(a) $(2c_j/c_3)(\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_j) = \mathbf{P}_3 - (\mathbf{P}_j - \mathbf{P}_3)^2 + (\mathbf{P}_k - \mathbf{P}_3)^2$ *holds along with* $c_1 + c_2 = 0$, $c_k + c_3 = 1$,

(b) $(2c_1/c_3)(\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1 - \mathbf{P}_2) = \mathbf{P}_3 + (\mathbf{P}_1 - \mathbf{P}_3)^2 + (\mathbf{P}_2 - \mathbf{P}_3)^2$ *holds along with* $c_1 = c_2$, $3c_1 + c_3 = 1$,

(c) $(2c_1/c_3)\mathbf{P}_1\mathbf{P}_2 = (\mathbf{P}_1 - \mathbf{P}_3)^2 + (\mathbf{P}_2 - \mathbf{P}_3)^2 - \mathbf{P}_3$ *holds along with* $c_1 = c_2$, $c_1 + c_3 = 1$,

(d) $c_1c_2(\mathbf{P}_1 - \mathbf{P}_2)^2 + c_1c_3(\mathbf{P}_1 - \mathbf{P}_3)^2 + c_2c_3(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{0}$ *holds along with* $c_1 + c_2 + c_3 = 1$,

where in characteristic (a) $j \neq k$, $j, k = 1, 2$.

Proof. Under assumptions (2.50), equation (2.1) reduces to

$$c_1(c_1 - 1)\mathbf{P}_1 + c_2(c_2 - 1)\mathbf{P}_2 + c_3(c_3 - 1)\mathbf{P}_3 + 2c_1c_2\mathbf{P}_1\mathbf{P}_2 \\ + c_1c_3(\mathbf{P}_1\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_1) + c_2c_3(\mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2) = \mathbf{0}. \tag{2.51}$$

Sufficiency of the conditions revealed in four characteristics provided in the theorem follows by direct verification of criterion (2.51). For the proof of necessity, observe that assumptions (2.50) and equation (2.51) are invariant with respect to an interchange of indexes “1” and “2”. Thus, also in the present proof it is reasonable to use indexes

$j,k \in \{1,2\}$, $j \neq k$, in order to shorten the derivations (and the resultant list) of necessary conditions.

Let $\mathbf{P}_1$ have a representation (2.41) and note that, since $\mathbf{P}_1$ and $\mathbf{P}_2$ commute, the representation of $\mathbf{P}_2$ given in (2.42) can also be utilized in the present proof. From $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1$ it follows, on the one hand, that $\mathbf{P}_1$ is singular (i.e., $0 < r < n$), and, in consequence, the latter of the summands in (2.41) is present, and, on the other hand, that representation of $\mathbf{P}_3$ in (2.42) does not hold. Thus, we represent $\mathbf{P}_3$ as

$$\mathbf{P}_3 = \mathbf{W}\begin{pmatrix}\mathbf{K} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{pmatrix}\mathbf{W}^{-1},$$

with $\mathbf{K} \in \mathbb{C}_{r,r}$, $\mathbf{L} \in \mathbb{C}_{r,n-r}$, $\mathbf{M} \in \mathbb{C}_{n-r,r}$, and $\mathbf{N} \in \mathbb{C}_{n-r,n-r}$, where, on account of $\mathbf{P}_1\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_1$, it is seen that either $\mathbf{L} \neq \mathbf{0}$ or $\mathbf{M} \neq \mathbf{0}$. Consequently, premultiplying and postmultiplying (2.51) by $\mathbf{W}^{-1}$ and $\mathbf{W}$, respectively, yields

$$c_1(c_1-1)\begin{pmatrix}\mathbf{I}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} + c_2(c_2-1)\begin{pmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}\end{pmatrix} + c_3(c_3-1)\begin{pmatrix}\mathbf{K} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{pmatrix} + 2c_1c_2\begin{pmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$$
$$+c_1c_3\begin{pmatrix}2\mathbf{K} & \mathbf{L}\\ \mathbf{M} & \mathbf{0}\end{pmatrix} + c_2c_3\begin{pmatrix}\mathbf{XK}+\mathbf{KX} & \mathbf{XL}+\mathbf{LY}\\ \mathbf{YM}+\mathbf{MX} & \mathbf{YN}+\mathbf{NY}\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}. \tag{2.52}$$

Assume first that $\mathbf{L}$ is nonzero and observe that from the equation attributed to the upper-right submatrix in (2.52) it follows that

$$(c_1 + c_3 - 1)\mathbf{L} + c_2(\mathbf{XL} + \mathbf{LY}) = \mathbf{0}. \tag{2.53}$$

Recall that matrices $\mathbf{X}$ and $\mathbf{Y}$ are idempotent (parenthetically notice that since $\mathbf{P}_1$ is nonzero and nonsingular, both of them are necessarily present in (2.42)), and thus there exist nonsingular matrices $\mathbf{U} \in \mathbb{C}_{r,r}$ and $\mathbf{V} \in \mathbb{C}_{n-r,n-r}$ such that

$$\mathbf{X} = \mathbf{U}(\mathbf{I}_x \oplus \mathbf{0}_{r-x})\mathbf{U}^{-1} \quad \text{and} \quad \mathbf{Y} = \mathbf{V}(\mathbf{I}_y \oplus \mathbf{0}_{n-r-y})\mathbf{V}^{-1}, \tag{2.54}$$

where $x = \mathrm{r}(\mathbf{X})$ and $y = \mathrm{r}(\mathbf{Y})$. Clearly, $0 \leqslant x \leqslant r$ and $0 \leqslant y \leqslant n-r$, and if $x = 0$ and/or $y = 0$ then the former, whereas if $x = r$ and/or $y = n-r$ then the latter, of the summands in the left and/or right identities in (2.54) vanish. Denoting matrices $\mathbf{I}_x \oplus \mathbf{0}_{r-x}$ and $\mathbf{I}_y \oplus \mathbf{0}_{n-r-y}$ occurring in (2.54) by $\mathbf{D_X}$ and $\mathbf{D_Y}$, respectively, gives

$$\mathbf{XL} + \mathbf{LY} = \mathbf{U}(\mathbf{D_X}\mathbf{U}^{-1}\mathbf{LV} + \mathbf{U}^{-1}\mathbf{LVD_Y})\mathbf{V}^{-1},$$

and, consequently, from (2.53) we obtain

$$(c_1 + c_3 - 1)\mathbf{U}^{-1}\mathbf{LV} + c_2(\mathbf{D_X}\mathbf{U}^{-1}\mathbf{LV} + \mathbf{U}^{-1}\mathbf{LVD_Y}) = \mathbf{0}. \tag{2.55}$$

Notice that from $\mathbf{L} \neq \mathbf{0}$ and the nonsingularity of $\mathbf{U}$ and $\mathbf{V}$ it follows that $\mathbf{U}^{-1}\mathbf{LV} \neq \mathbf{0}$. Let us represent matrix $\mathbf{L}$ as

$$\mathbf{L} = \mathbf{U}\begin{pmatrix} \mathbf{L}_1 & \mathbf{L}_2 \\ \mathbf{L}_3 & \mathbf{L}_4 \end{pmatrix}\mathbf{V}^{-1}, \tag{2.56}$$

with $\mathbf{L}_1 \in \mathbb{C}_{x,y}$, $\mathbf{L}_2 \in \mathbb{C}_{x,n-r-y}$, $\mathbf{L}_3 \in \mathbb{C}_{r-x,y}$, and $\mathbf{L}_4 \in \mathbb{C}_{r-x,n-r-y}$, where: $\mathbf{L}_1$, $\mathbf{L}_2$ vanish if $x = 0$; $\mathbf{L}_3$, $\mathbf{L}_4$ vanish if $x = r$; $\mathbf{L}_1$, $\mathbf{L}_3$ vanish if $y = 0$; and $\mathbf{L}_2$, $\mathbf{L}_4$ vanish if $y = n - r$. Substituting (2.56) into (2.55) yields

$$(c_1 + c_3 - 1)\begin{pmatrix} \mathbf{L}_1 & \mathbf{L}_2 \\ \mathbf{L}_3 & \mathbf{L}_4 \end{pmatrix} + c_2\begin{pmatrix} 2\mathbf{L}_1 & \mathbf{L}_2 \\ \mathbf{L}_3 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{2.57}$$

and taking into account that at least one of $\mathbf{L}_i$s, $i = 1, ..., 4$, is nonzero, it follows from (2.57) that one of the conditions

$$\text{(i) } c_1 + 2c_2 + c_3 = 1, \quad \text{(ii) } c_1 + c_2 + c_3 = 1, \quad \text{(iii) } c_1 + c_3 = 1, \tag{2.58}$$

must be satisfied. As easy to observe, if $x = 0$ or $y = 0$, then condition (i) does not hold, whereas if $x = r$ or $y = n-r$, then condition (iii) does not hold. (Parenthetically notice that conditions (2.58) are mutually excluding.) Combining (2.58) with the fact that assumptions (2.50) and equation (2.51) do not change upon an interchange of indexes "1" and "2", results in another set of necessary conditions, namely

$$\text{(iv) } 2c_1 + c_2 + c_3 = 1, \quad \text{(v) } c_1 + c_2 + c_3 = 1, \quad \text{(vi) } c_2 + c_3 = 1, \tag{2.59}$$

of which at least one must be satisfied.

To conclude this step of the proof we need to consider nine situations corresponding to the possible conjunctions of one condition from (2.58) and one condition from (2.59). However, two of these conjunctions, i.e., (i), (v) and (iii), (v), lead to $c_2 = 0$, and the other two, i.e., (ii), (iv) and (ii), (vi), lead to $c_1 = 0$. Moreover, situations corresponding to conjunctions (i), (vi) and (iii), (iv) are counterparts of each other obtained by interchanging indexes "1" and "2". Consequently, four situations are to be considered.

Combining (i) with (iv) shows that $c_1 = c_2$, $3c_1 + c_3 = 1$. With these relationships taken into account, (2.51) yields matrix condition in characteristic (b) of the theorem. Next, from conjunction (i) and (vi) we obtain $c_1 + c_2 = 0$, $c_2 + c_3 = 1$. Substituting these conditions into (2.51), and including also the case obtained by interchanging indexes "1" and "2" leads to matrix condition in (a). Further, since conditions characterizing cases (ii) and (v) are the same, we cannot obtain more information about scalars c_1, c_2, c_3 than just $c_1 + c_2 + c_3 = 1$. In this situation, however, equation (2.51) can be rewritten as in characteristic (d). Finally, combining (iii) with (vi) entails $c_1 = c_2$, $c_1 + c_3 = 1$ and these conditions are in characteristic (c) of the theorem accompanied by an equality obtained from (2.51).

The proof is concluded with an observation that if $\mathbf{M} \neq \mathbf{0}$, then from the equation attributed to the lower-left submatrix in (2.52), an analogues condition to (2.53) is obtained, simply with $\mathbf{L}$ replaced by $\mathbf{M}$ and, additionally, $\mathbf{X}$ and $\mathbf{Y}$ interchanged. Thus, following the steps of the proof corresponding to the situation in which $\mathbf{L} \neq \mathbf{0}$, also when $\mathbf{M} \neq \mathbf{0}$ we would obtain characteristics already listed in the theorem. The proof is complete. $\square$

In a comment to Theorem 3 we emphasize that the extent in which it generalizes part (d) of Theorem 1 in [3] is essential, for only two out of four characteristics listed therein have their counterparts in Theorem 1 in [3]. Namely, characteristic (c) generalizes case (d_1), whereas characteristic (d) generalizes case (d_2).

3. Additional results

In this section we consider situations in which idempotent matrices $\mathbf{P}_i$, $i = 1, 2, 3$, occurring in a linear combination (1.1), are Hermitian. As already mentioned, such situations are of particular interest from the point of view of possible applications in statistics.

We begin with arguments showing that all 13 characteristics listed in Theorem 1 remain valid when $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$. The reasoning is based on the fact that, since $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ are mutually commuting idempotent matrices, they are simultaneously diagonalizable, i.e., there exists a nonsingular matrix $\mathbf{U} \in \mathbb{C}_{n,n}$, say, such that $\mathbf{D}_i = \mathbf{U}\mathbf{P}_i\mathbf{U}^{-1}$, $i = 1, 2, 3$, are diagonal matrices with diagonal entries being equal to either zero or one; see e.g., Theorem 1.3.19 in [4]. Thus, $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3 \in \mathbb{C}_n^{\mathsf{OP}}$. In consequence, it is obvious that if given mutually commuting $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$ satisfy any matrix condition (for instance one of those occurring in characteristics (i)–(m) of Theorem 1), then $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3 \in \mathbb{C}_n^{\mathsf{OP}}$ satisfy this condition as well.

The following theorem shows that when $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$, then six of 11 characteristics listed in Theorem 2 are no longer valid, and two from among five characteristics which are valid have more restrictive forms.

Theorem 4. *Let* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero and such that conditions* (2.38) *are satisfied. Moreover, let* $\mathbf{P}$ *be a linear combination of the form* (1.1)*, with nonzero* $c_1, c_2, c_3 \in \mathbb{C}$ *constituting* $\alpha = c_1(c_1 - 1)/c_2c_3$*. Then* $\alpha > 0$ *and the following list comprises characteristics of all cases in which* $\mathbf{P}$ *is an idempotent* (*and Hermitian*) *matrix:*

(a) $\frac{1}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = 2\mathbf{P}_k + \mathbf{P}_1\mathbf{P}_j - \mathbf{P}_1\mathbf{P}_k$ *holds along with* $c_1 = \frac{1}{2}$*,* $c_j = 1$*,* $c_k = -1$*,*

(b) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$*,* $\mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_3$*,* $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1$*, hold along with* $2c_1 + c_2 + c_3 = 1$*,*

(c) $\frac{3}{4}\mathbf{P}_1 + \mathbf{P}_2\mathbf{P}_3 + \mathbf{P}_3\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_3$ *holds along with* $c_1 = -\frac{1}{2}$, $c_2 = 1$, $c_3 = 1$,

(d) $\mathbf{P}_1\mathbf{P}_j = \mathbf{P}_j$, $(\mathbf{P}_2 - \mathbf{P}_3)^2 = \alpha\mathbf{P}_1 + \mathbf{P}_k - \mathbf{P}_1\mathbf{P}_k$, *hold along with* $2c_1 + c_j = 0$, $c_k = 1$, $c_1 < 1$,

(e) $\mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_3 = \mathbf{P}_2 - \mathbf{P}_3$, $4c_2^2(\mathbf{P}_2 - \mathbf{P}_3)^2 = \mathbf{P}_1$, *hold along with* $c_1 = \frac{1}{2}$, $c_2 + c_3 = 0$, $c_2 \in \mathbb{R}$,

where in characteristics (a) *and* (d) $j \neq k$, $j, k = 2, 3$.

Proof. We shall use the notation utilized in the proof of Theorem 2. It is known that for every $\mathbf{K} \in \mathbb{C}_n^{\mathsf{OP}}$, there exists a unitary matrix $\mathbf{U} \in \mathbb{C}_{n,n}$ such that $\mathbf{K} = \mathbf{U}(\mathbf{I}_k \oplus \mathbf{0}_{n-k})\mathbf{U}^*$, where $k = \mathrm{r}(\mathbf{K})$. Thus, since $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{C}_n^{\mathsf{OP}}$, we can assume that matrix $\mathbf{W}$ occurring in (2.41) and (2.42) satisfies $\mathbf{W}^{-1} = \mathbf{W}^*$. In consequence, it is seen from (2.42) that $\mathbf{X}, \mathbf{S} \in \mathbb{C}_r^{\mathsf{OP}}$ and $\mathbf{Y}, \mathbf{T} \in \mathbb{C}_{n-r}^{\mathsf{OP}}$.

Similarly as in the proof of Theorem 2, assume first that $\mathbf{XS} \neq \mathbf{SX}$. Then, from (2.47) it follows that

$$\alpha\mathbf{I}_r = (\mathbf{X} - \mathbf{S})^*(\mathbf{X} - \mathbf{S}). \tag{2.60}$$

Combining (2.60) with a known fact that for every $\mathbf{K} \in \mathbb{C}_{m,n}$, the product $\mathbf{K}^*\mathbf{K}$ is nonnegative definite, and, moreover, $\mathbf{K}^*\mathbf{K} = \mathbf{0}$ if and only if $\mathbf{K} = \mathbf{0}$, entails $\alpha > 0$.

Recall that under the assumption $\mathbf{XS} \neq \mathbf{SX}$, five characteristics of Theorem 2 were obtained, namely (b), (e), (f), (h), and (j). Three of them, i.e., (b), (f), and (h) correspond to characteristics (a), (b), and (c) of the present theorem, respectively. Now, from conditions $2c_1 + c_j = 0$ and $c_k = 1$, being a part of characteristic (e) of Theorem 2, we get $\alpha = (1 - c_1)/2$, regardless of whether $j = 2$, $k = 3$ or $j = 3$, $k = 2$. Hence, $c_1 < 1$, leading to characteristic (d) of the theorem. Analogously, from the scalar conditions in characteristic (j) of Theorem 2 we obtain $\alpha = 1/4c_2^2$. Thus, $c_2^2 > 0$, which means that $c_2 \in \mathbb{R}$, and we arrive at characteristic (e) of the theorem.

Consider now situation in which $\mathbf{XS} = \mathbf{SX}$. Recall that from the proof of Theorem 2 it follows that then $\mathbf{YT} \neq \mathbf{TY}$ and $(\mathbf{Y} - \mathbf{T})^2 = \mathbf{0}$. However, from the latter of these conditions we have $(\mathbf{Y}-\mathbf{T})^*(\mathbf{Y}-\mathbf{T})=\mathbf{0}$, which yields $\mathbf{Y}=\mathbf{T}$, being in a contradiction with the former condition. The proof is complete. $\square$

The last result refers to Theorem 3 and shows that two out of four characteristics listed therein are no longer valid when $\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3\in\mathbb{C}_n^{\mathsf{OP}}$.

Theorem 5. *Let $\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3\in\mathbb{C}_n^{\mathsf{OP}}$ be nonzero and such that conditions* (2.50) *are satisfied. Moreover, let $\mathbf{P}$ be a linear combination of the form* (1.1)*, with nonzero $c_1,c_2,c_3\in\mathbb{C}$. Then the following list comprises characteristics of all cases in which $\mathbf{P}$ is an idempotent* (*and Hermitian*) *matrix:*

(a) $(2c_1/c_3)(\mathbf{P}_1\mathbf{P}_2-\mathbf{P}_1-\mathbf{P}_2)=\mathbf{P}_3+(\mathbf{P}_1-\mathbf{P}_3)^2+(\mathbf{P}_2-\mathbf{P}_3)^2$ *holds along with* $c_1=c_2$, $3c_1+c_3=1$,

(b) $c_1c_2(\mathbf{P}_1-\mathbf{P}_2)^2+c_1c_3(\mathbf{P}_1-\mathbf{P}_3)^2+c_2c_3(\mathbf{P}_2-\mathbf{P}_3)^2=\mathbf{0}$ *holds along with* $c_1+c_2+c_3=1$.

Proof. Referring to representations of $\mathbf{P}_2,\mathbf{P}_3\in\mathbb{C}_n^{\mathsf{P}}$ utilized in the proof of Theorem 3, and taking into account that these matrices are now Hermitian, we can represent them as

$$\mathbf{P}_2=\mathbf{W}\begin{pmatrix}\mathbf{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}\end{pmatrix}\mathbf{W}^* \quad\text{and}\quad \mathbf{P}_3=\mathbf{W}\begin{pmatrix}\mathbf{K} & \mathbf{L}\\ \mathbf{M} & \mathbf{N}\end{pmatrix}\mathbf{W}^*, \tag{2.61}$$

where $\mathbf{X},\mathbf{K}\in\mathbb{C}_{r,r}$ and $\mathbf{Y},\mathbf{N}\in\mathbb{C}_{n-r,n-r}$. From the left identity in (2.61) it is seen that $\mathbf{X}\in\mathbb{C}_r^{\mathsf{OP}}$, $\mathbf{Y}\in\mathbb{C}_{n-r}^{\mathsf{OP}}$, whereas from the right identity we get $\mathbf{K}^*=\mathbf{K}$, $\mathbf{N}^*=\mathbf{N}$, and $\mathbf{L}^*=\mathbf{M}$. Moreover, the idempotency (along with Hermitancy) of $\mathbf{P}_3$ entails

$$\mathbf{N}=\mathbf{L}^*\mathbf{L}+\mathbf{N}^2. \tag{2.62}$$

In the remaining part of the proof we will show that characteristics (a) and (c) of Theorem 3 do not hold when $\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3$ are Hermitian. We will refer to condition

$$c_2(c_2-1)\mathbf{Y}+c_3(c_3-1)\mathbf{N}+c_2c_3(\mathbf{YN}+\mathbf{NY})=\mathbf{0}, \tag{2.63}$$

obtained from the equation attributed to the lower-right submatrix in (2.52). Substituting $c_1 + c_2 = 0$ and $c_2 + c_3 = 1$, being a part of characteristic (a) of Theorem 3 with $k = 2$, into (2.63) gives

$$\mathbf{N} = \mathbf{YN} + \mathbf{NY} - \mathbf{Y}. \tag{2.64}$$

Combining (2.62) with (2.64) leads to $\mathbf{L}^*\mathbf{L} + \mathbf{N}^2 = \mathbf{YN} + \mathbf{NY} - \mathbf{Y}$. Since $\mathbf{Y}$ and $\mathbf{N}$ are Hermitian, this condition can be equivalently expressed as

$$\mathbf{L}^*\mathbf{L} + (\mathbf{Y} - \mathbf{N})^*(\mathbf{Y} - \mathbf{N}) = \mathbf{0},$$

i.e., as a sum of two nonnegative definite matrices. Hence, $\mathbf{L} = \mathbf{0}$, and from (2.41) and (2.61) it follows that $\mathbf{P}_1$ and $\mathbf{P}_3$ commute, what is in a contradiction with assumptions (2.50). The same contradiction is obtained if in characteristic (a) of Theorem 3, $k = 1$. This fact is seen by noticing that the roles of $\mathbf{P}_1$ and $\mathbf{P}_2$ are symmetrical in both Theorem 3 and the present theorem.

The proof is concluded by an observation that substituting $c_1 = c_2$ and $c_1 + c_3 = 1$, i.e., conditions occurring in characteristic (c) of Theorem 3, into (2.63) leads to (2.64) as well. $\square$

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