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On linear combinations of two commuting hypergeneralized projectors

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Abstract

The concept of a hypergeneralized projector as a matrix $\mathbf{H}$ satisfying $\mathbf{H}^2 = \mathbf{H}^\dagger$, where $\mathbf{H}^\dagger$ denotes the Moore–Penrose inverse of $\mathbf{H}$, was introduced by Groß and Trenkler in [Generalized and hypergeneralized projectors, Linear Algebra Appl. 264 (1997) 463-474]. Generalizing substantially some preliminary observations given therein, Baksalary et al. in [On some linear combinations of hypergeneralized projectors, Linear Algebra Appl. 413 (2006) 264-273] characterized all situations in which a linear combination $c_1\mathbf{H}_1+c_2\mathbf{H}_2$, where $c_1, c_2 \in \mathbb{C}$ and $\mathbf{H}_1$, $\mathbf{H}_2$ are hypergeneralized projectors such that $\mathbf{H}_1\mathbf{H}_2 = \eta_1\mathbf{H}_1^2 + \eta_2\mathbf{H}_2^2 = \mathbf{H}_2\mathbf{H}_1$ for some $\eta_1, \eta_2 \in \mathbb{C}$, inherits the hypergenerality property. In the present paper, the problem considered in the latter paper is revisited and solved under the essentially weaker assumption that $\mathbf{H}_1\mathbf{H}_2 = \mathbf{H}_2\mathbf{H}_1$.

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1. Introduction

Let $\mathbb{C}_{m,n}$ be the set of $m \times n$ complex matrices. The symbols $\mathbf{K}^*$ and $\text{rk}(\mathbf{K})$ will denote the conjugate transpose and rank of $\mathbf{K} \in \mathbb{C}_{m,n}$, respectively. Further, $\mathbf{K}^\dagger$ will stand for the Moore–Penrose inverse of $\mathbf{K}$, i.e., for the unique matrix satisfying the equations

$$\mathbf{K}\mathbf{K}^\dagger\mathbf{K} = \mathbf{K},\ \mathbf{K}^\dagger\mathbf{K}\mathbf{K}^\dagger = \mathbf{K}^\dagger,\ \mathbf{K}\mathbf{K}^\dagger = (\mathbf{K}\mathbf{K}^\dagger)^*,\ \mathbf{K}^\dagger\mathbf{K} = (\mathbf{K}^\dagger\mathbf{K})^*, \tag{1.1}$$

and $\mathbf{I}_n$ will be the identity matrix of order n. Moreover, $\mathbb{C}_n^{\mathsf{EP}}$, $\mathbb{C}_n^{\mathsf{QP}}$, and $\mathbb{C}_n^{\mathsf{U}}$ will mean the subsets of $\mathbb{C}_{n,n}$ consisting of EP, quadripotent, and unitary matrices, respectively, i.e.,

$$\mathbb{C}_n^{\mathsf{EP}} = \{\mathbf{K} \in \mathbb{C}_{n,n}\colon\ \mathbf{K}\mathbf{K}^\dagger = \mathbf{K}^\dagger\mathbf{K}\}, \tag{1.2}$$

$$\mathbb{C}_n^{\mathsf{QP}} = \{\mathbf{K} \in \mathbb{C}_{n,n}\colon\ \mathbf{K}^4 = \mathbf{K}\}, \tag{1.3}$$

$$\mathbb{C}_n^{\mathsf{U}} = \{\mathbf{K} \in \mathbb{C}_{n,n}\colon\ \mathbf{K}\mathbf{K}^* = \mathbf{I}_n = \mathbf{K}^*\mathbf{K}\}. \tag{1.4}$$

From the point of view of the present paper, the key role is played by matrices belonging to the set of hypergeneralized projectors, defined as

$$\mathbb{C}_n^{\mathsf{HGP}} = \{\mathbf{K} \in \mathbb{C}_{n,n}\colon\ \mathbf{K}^2 = \mathbf{K}^\dagger\}. \tag{1.5}$$

The notion of a hypergeneralized projector was introduced by Groß and Trenkler [6, p. 466], and several characteristics of the set $\mathbb{C}_n^{\mathsf{HGP}}$ are now available in the literature. For instance, according to part (a) $\Leftrightarrow$ (d) of Theorem 2 in [6]

$$\mathbb{C}_n^{\mathsf{HGP}} = \mathbb{C}_n^{\mathsf{EP}} \cap \mathbb{C}_n^{\mathsf{QP}}; \tag{1.6}$$

see also Theorem 3 in [2].

A challenging and relevant question concerning matrices belonging to $\mathbb{C}_n^{\mathsf{HGP}}$ is: when a linear combination of the form

$$\mathbf{H} = c_1\mathbf{H}_1 + c_2\mathbf{H}_2, \tag{1.7}$$

with $c_1, c_2 \in \mathbb{C}$ and $\mathbf{H}_1$, $\mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$, inherits the hypergenerality property? The main difficulty of the problem is included in the fact that the derivation of necessary conditions for $\mathbf{H}^2 = \mathbf{H}^\dagger$ may depend on the formula for the Moore–Penrose inverse of a sum of two matrices, developed in the general case by Hung and Markham [8, Theorem 1], which is not easy to handle. This difficulty was to certain extent avoided by Baksalary et al. [1], who provided the answer to the aforementioned question under the assumption that

$$\mathbf{H}_1\mathbf{H}_2 = \eta_1\mathbf{H}_1^2 + \eta_2\mathbf{H}_2^2 = \mathbf{H}_2\mathbf{H}_1 \tag{1.8}$$

for some $\eta_1, \eta_2 \in \mathbb{C}$. In the present paper, the problem is revisited by utilizing different formalism than the one used in [1]. As a consequence, the complete solution to the problem of when a linear combination of the form (1.7) satisfies $\mathbf{H}^2 = \mathbf{H}^\dagger$, with the assumption (1.8) replaced by an essentially weaker (and more natural) commutativity condition

$$\mathbf{H}_1\mathbf{H}_2 = \mathbf{H}_2\mathbf{H}_1,$$

is established. Interestingly, it turns out that the set of possible situations in which $\mathbf{H} \in \mathbb{C}_n^{\mathsf{HGP}}$ is not influenced by the relevant weakening of the constrains imposed on projectors $\mathbf{H}_1$ and $\mathbf{H}_2$.

In the next section we provide some general results concerning partitioned matrices, which, besides of being useful from the point of view of the present paper, are of independent interest as well. The solution to the problem is given in Section 3.

2. Preliminary results

A crucial role in subsequent considerations is played by the theorem given below, which constitutes part (i) $\Leftrightarrow$ (iv) of Theorem 4.3.1 in [4].

Theorem 1. *Let* $\mathbf{K} \in \mathbb{C}_{n,n}$ *be of rank* r. *Then* $\mathbf{K} \in \mathbb{C}_n^{\mathsf{EP}}$ *if and only if there exists* $\mathbf{U} \in \mathbb{C}_n^{\mathsf{U}}$ *and nonsingular* $\mathbf{K}_1 \in \mathbb{C}_{r,r}$, *such that* $\mathbf{K} = \mathbf{U}\left(\mathbf{K}_1 \oplus \mathbf{0}\right)\mathbf{U}^*$.

The following three lemmas will be useful in further derivations.

Lemma 1. *Let* $\mathbf{K} \in \mathbb{C}_{n,n}$ *have a representation* $\mathbf{K} = \mathbf{U}\left(\mathbf{P} \oplus \mathbf{Q}\right)\mathbf{U}^*$, *where* $\mathbf{U} \in \mathbb{C}_n^{\mathsf{U}}$, $\mathbf{P} \in \mathbb{C}_{p,p}$, *and* $\mathbf{Q} \in \mathbb{C}_{n-p,n-p}$. *Then* $\mathbf{K} \in \mathbb{C}_n^{\mathsf{EP}}$ *if and only if* $\mathbf{P} \in \mathbb{C}_p^{\mathsf{EP}}$ *and* $\mathbf{Q} \in \mathbb{C}_{n-p}^{\mathsf{EP}}$.

Proof. The result follows by straightforward verification of definition (1.2). □

In the sequel, the symbol $\|\mathbf{u}\|$ with $\mathbf{u} \in \mathbb{C}_{n,1}$ will mean the euclidean vector norm, whereas $\|\mathbf{K}\|$ with $\mathbf{K} \in \mathbb{C}_{m,n}$ will be the matrix norm induced by the euclidean vector norm (known as the spectral norm); see [9, pp. 270, 281]. The next lemma constitutes a solution to the part of Problem P2.3.2 in [5] referring to the spectral norm.

Lemma 2. *Let* $\mathbf{K} \in \mathbb{C}_{m,n}$ *be partitioned according to*

$$\mathbf{K} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix},$$

where $\mathbf{A} \in \mathbb{C}_{p,q}$, $\mathbf{D} \in \mathbb{C}_{m-p,n-q}$. *Then none of the norms* $\|\mathbf{A}\|$, $\|\mathbf{B}\|$, $\|\mathbf{C}\|$, *and* $\|\mathbf{D}\|$ *is greater than* $\|\mathbf{K}\|$.

Proof. Let $\mathbf{u} \in \mathbb{C}_{q,1}$ be such that $\|\mathbf{u}\| = 1$. It is clear that the following relationships are satisfied

$$\|\mathbf{A}\mathbf{u}\|^2 \leqslant \|\mathbf{A}\mathbf{u}\|^2 + \|\mathbf{C}\mathbf{u}\|^2 = \| \begin{pmatrix} \mathbf{A}\mathbf{u} \\ \mathbf{C}\mathbf{u} \end{pmatrix} \|^2 = \| \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{0} \end{pmatrix} \|^2 \leqslant \|\mathbf{K}\|^2 \|\mathbf{u}\|^2.$$

Thus, $\|\mathbf{A}\mathbf{u}\| \leqslant \|\mathbf{K}\|$ what ensures that $\|\mathbf{A}\| \leqslant \|\mathbf{K}\|$. The proofs concerning the remaining three inequalities are obtained similarly. □

Lemma 3. *Let* $\mathbf{K}_1, \mathbf{K}_2 \in \mathbb{C}_{m,n}$ *be partitioned according to*

$$\mathbf{K}_1 = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \quad \textit{and} \quad \mathbf{K}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{A} \in \mathbb{C}_{p,q}$*,* $\mathbf{B} \in \mathbb{C}_{r,s}$*,* $\mathbf{C} \in \mathbb{C}_{m-r,n-s}$*, and* $\mathbf{D} \in \mathbb{C}_{m-p,n-q}$*. Then*

$$\|\mathbf{K}_1\| = \max\{\|\mathbf{A}\|, \|\mathbf{D}\|\} \quad \textit{and} \quad \|\mathbf{K}_2\| = \max\{\|\mathbf{B}\|, \|\mathbf{C}\|\}. \tag{2.1}$$

Proof. Relationships (2.1) are obtained straightforwardly from the fact that for any $\mathbf{K} \in \mathbb{C}_{m,n}$, the norm $\|\mathbf{K}\|$ is equal to the largest eigenvalue of $\sqrt{\mathbf{K}^*\mathbf{K}}$; see [9, p. 281]. (Parenthetically note that the left-hand side formula in (2.1) constitutes relationship (5.2.12) in [9].) $\square$

The theorem below concerns relationships between spectral norms of submatrices involved in two partitioned matrices, of which at least one is EP. A particular case of the theorem, covered by the corollary following it, will be of key importance in establishing the main result of the paper.

Theorem 2. *Let* $\mathbf{K}_1 \in \mathbb{C}_n^{\mathsf{EP}}$*,* $\mathbf{K}_2 \in \mathbb{C}_{n,n}$*, and nonnegative* $\epsilon \in \mathbb{R}$ *satisfy* $\|\mathbf{K}_1\mathbf{K}_2 - \mathbf{K}_2\mathbf{K}_1\| \leqslant \epsilon$*. Moreover, let* $\mathrm{rk}(\mathbf{K}_1) = r$*. Then there exists* $\mathbf{U} \in \mathbb{C}_n^{\mathsf{U}}$ *such that*

$$\mathbf{K}_1 = \mathbf{U}\begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^* \quad \textit{and} \quad \mathbf{K}_2 = \mathbf{U}\begin{pmatrix} \mathbf{A}_2 & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}\mathbf{U}^*, \tag{2.2}$$

where $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{C}_{r,r}$*,* $\mathbf{D} \in \mathbb{C}_{n-r,n-r}$*,* $\mathrm{rk}(\mathbf{A}_1) = r$*, and* $\|\mathbf{B}\|, \|\mathbf{C}\| \leqslant \epsilon\|\mathbf{K}_1^\dagger\|$*.*

Proof. The existence of the representation of $\mathbf{K}_1$ given in (2.2), with nonsingular $\mathbf{A}_1$, is ensured by Theorem 1. Since,

$$\mathbf{K}_1\mathbf{K}_2 - \mathbf{K}_2\mathbf{K}_1 = \mathbf{U}\begin{pmatrix} \mathbf{A}_1\mathbf{A}_2 - \mathbf{A}_2\mathbf{A}_1 & \mathbf{A}_1\mathbf{B} \\ -\mathbf{C}\mathbf{A}_1 & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

on account of Lemma 2, assumption $\|\mathbf{K}_1\mathbf{K}_2 - \mathbf{K}_2\mathbf{K}_1\| \leqslant \epsilon$ implies $\|\mathbf{A}_1\mathbf{B}\|, \|\mathbf{C}\mathbf{A}_1\| \leqslant \epsilon$. Hence,

$$\|\mathbf{B}\| = \|\mathbf{A}_1^{-1}\mathbf{A}_1\mathbf{B}\| \leqslant \|\mathbf{A}_1^{-1}\|\|\mathbf{A}_1\mathbf{B}\| \leqslant \epsilon\|\mathbf{A}_1^{-1}\|.$$

In view of $\mathbf{K}_1^{-1} = \mathbf{U}\left(\mathbf{A}_1^{-1} \oplus \mathbf{0}\right)\mathbf{U}^*$, from Lemma 3 it follows that $\|\mathbf{A}_1^{-1}\| = \|\mathbf{K}_1^\dagger\|$, and, thus, inequality $\|\mathbf{B}\| \leqslant \epsilon\|\mathbf{K}_1^\dagger\|$ is established. The proof of $\|\mathbf{C}\| \leqslant \epsilon\|\mathbf{K}_1^\dagger\|$ is obtained analogously. □

Corollary 1. *Let* $\mathbf{K}_1 \in \mathbb{C}_n^{\mathsf{EP}}$, $\mathbf{K}_2 \in \mathbb{C}_{n,n}$ *satisfy* $\mathbf{K}_1\mathbf{K}_2 = \mathbf{K}_2\mathbf{K}_1$. *Moreover, let* $\mathrm{rk}(\mathbf{K}_1) = r$. *Then there exists* $\mathbf{U} \in \mathbb{C}_n^{\mathsf{U}}$ *such that*

$$\mathbf{K}_1 = \mathbf{U}\left(\mathbf{A}_1 \oplus \mathbf{0}\right)\mathbf{U}^* \quad \textit{and} \quad \mathbf{K}_2 = \mathbf{U}\left(\mathbf{A}_2 \oplus \mathbf{D}\right)\mathbf{U}^*, \tag{2.3}$$

where $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{C}_{r,r}$, $\mathbf{D} \in \mathbb{C}_{n-r,n-r}$, $\mathrm{rk}(\mathbf{A}_1) = r$, *and* $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$.

Proof. The result follows from Theorem 2 by taking $\epsilon = 0$. □

The next two results are obtained directly from Corollary 1. The first of them provides a solution to Exercise 14 in Chapter 4 in [4]; see also Theorem 6 in [3].

Corollary 2. *Let* $\mathbf{K}_1, \mathbf{K}_2 \in \mathbb{C}_n^{\mathsf{EP}}$ *satisfy* $\mathbf{K}_1\mathbf{K}_2 = \mathbf{K}_2\mathbf{K}_1$. *Then* $(\mathbf{K}_1\mathbf{K}_2)^\dagger = \mathbf{K}_1^\dagger\mathbf{K}_2^\dagger$.

Proof. Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be of the forms given in (2.3). Then, clearly, $(\mathbf{K}_1\mathbf{K}_2)^\dagger = \mathbf{K}_1^\dagger\mathbf{K}_2^\dagger$ is equivalent to $(\mathbf{A}_1\mathbf{A}_2)^\dagger = \mathbf{A}_1^{-1}\mathbf{A}_2^\dagger$, whereas $\mathbf{K}_1\mathbf{K}_2 = \mathbf{K}_2\mathbf{K}_1$ is satisfied if and only if $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$. Furthermore, from Lemma 1 it follows that $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{C}_r^{\mathsf{EP}}$ and thus $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1 \Leftrightarrow \mathbf{A}_1^{-1}\mathbf{A}_2^\dagger = \mathbf{A}_2^\dagger\mathbf{A}_1^{-1}$; see Solutions 26-4.1–26-4.3 [IMAGE – The Bulletin of the International Linear Algebra Society 27 (2001) pp. 30-32] to the problem posed by Tian [11]. In view of the equalities constituting the last equivalence, it is seen that $(\mathbf{A}_1^{-1}\mathbf{A}_2^\dagger)(\mathbf{A}_1\mathbf{A}_2) = \mathbf{A}_2^\dagger\mathbf{A}_2$ and $(\mathbf{A}_1\mathbf{A}_2)(\mathbf{A}_1^{-1}\mathbf{A}_2^\dagger) = \mathbf{A}_2\mathbf{A}_2^\dagger$. Hence, direct verifications of definition (1.1) show that $\mathbf{A}_1^{-1}\mathbf{A}_2^\dagger$ is indeed the Moore–Penrose inverse of $\mathbf{A}_1\mathbf{A}_2$ and, thus, the assertion is established. □

In general, the Moore–Penrose inverse of a sum of two matrices is not equal to a sum of the Moore–Penrose inverses of the matrices. Nevertheless, one of the situations in which this is the case was pointed out by Groß and Trenkler [6, p. 471], who observed that for $\mathbf{H}_1, \mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$ such that $\mathbf{H}_1\mathbf{H}_2 = \mathbf{0} = \mathbf{H}_2\mathbf{H}_1$, equality $(\mathbf{H}_1 + \mathbf{H}_2)^\dagger = \mathbf{H}_1^\dagger + \mathbf{H}_2^\dagger$ is necessarily satisfied. In what follows, this result is extended and generalized. According to the convention utilized, for $c_i \in \mathbb{C}$, $i = 1, 2$, $c_i^\dagger = 0$ when $c_i = 0$ and $c_i^\dagger = c_i^{-1}$ when $c_i \neq 0$.

Corollary 3. *Let $c_1, c_2 \in \mathbb{C}$ and let $\mathbf{K}_1 \in \mathbb{C}_n^{\mathsf{EP}}$, $\mathbf{K}_2 \in \mathbb{C}_{n,n}$ satisfy $\mathbf{K}_1\mathbf{K}_2 = \mathbf{0} = \mathbf{K}_2\mathbf{K}_1$. Then $(c_1\mathbf{K}_1 + c_2\mathbf{K}_2)^\dagger = c_1^\dagger\mathbf{K}_1^\dagger + c_2^\dagger\mathbf{K}_2^\dagger$. Moreover, if $\mathbf{K}_2 \in \mathbb{C}_n^{\mathsf{EP}}$, then $c_1\mathbf{K}_1 + c_2\mathbf{K}_2 \in \mathbb{C}_n^{\mathsf{EP}}$.*

Proof. Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be of the forms given in (2.3). Then, clearly, $\mathbf{K}_1\mathbf{K}_2 = \mathbf{0}$ if and only if $\mathbf{A}_1\mathbf{A}_2 = \mathbf{0}$. Since $\mathbf{A}_1$ is nonsingular, it is further seen that $\mathbf{K}_1\mathbf{K}_2 = \mathbf{0} \Leftrightarrow \mathbf{A}_2 = \mathbf{0}$. Hence, representations (2.3) lead to

$$c_1\mathbf{K}_1 + c_2\mathbf{K}_2 = \mathbf{U}\left(c_1\mathbf{A}_1 \oplus c_2\mathbf{D}\right)\mathbf{U}^*, \tag{2.4}$$

from where we obtain

$$(c_1\mathbf{K}_1 + c_2\mathbf{K}_2)^\dagger = \mathbf{U}\left(c_1^\dagger\mathbf{A}_1^{-1} \oplus c_2^\dagger\mathbf{D}^\dagger\right)\mathbf{U}^* = c_1^\dagger\mathbf{U}\left(\mathbf{A}_1^{-1} \oplus \mathbf{0}\right)\mathbf{U}^* + c_2^\dagger\mathbf{U}\left(\mathbf{0} \oplus \mathbf{D}^\dagger\right)\mathbf{U}^*.$$

In consequence, $(c_1\mathbf{K}_1 + c_2\mathbf{K}_2)^\dagger = c_1^\dagger\mathbf{K}_1^\dagger + c_2^\dagger\mathbf{K}_2^\dagger$. Moreover, if $\mathbf{K}_2 \in \mathbb{C}_n^{\mathsf{EP}}$, then, on account of Lemma 1, we have $\mathbf{D} \in \mathbb{C}_{n-r}^{\mathsf{EP}}$. Combining this fact with $\mathbf{A}_1 \in \mathbb{C}_r^{\mathsf{EP}}$, being ensured by the nonsingularity of $\mathbf{A}_1$, and referring again to Lemma 1, from (2.4) it is seen that $c_1\mathbf{K}_1 + c_2\mathbf{K}_2 \in \mathbb{C}_n^{\mathsf{EP}}$. The proof is complete. □

The last result of the section refers to the notion of diagonalizability, which will play an important role in establishing the main result of the paper.

Lemma 4. *Let $\mathbf{K} \in \mathbb{C}_{n,n}$ be diagonalizable and have two distinct eigenvalues, say, $\alpha, \mu \in \mathbb{C}$. Then $\mathbf{K}^2 + \alpha\mu\mathbf{I}_n = (\alpha + \mu)\mathbf{K}$.*

Proof. Since $\mathbf{K}$ is diagonalizable, there exists nonsingular $\mathbf{S} \in \mathbb{C}_{n,n}$ such that $\mathbf{K} = \mathbf{S}(\lambda\mathbf{I}_r \oplus \mu\mathbf{I}_{n-r})\mathbf{S}^{-1}$. Hence, clearly $\mathbf{K} - \lambda\mathbf{I}_n = \mathbf{S}(\mathbf{0} \oplus -(\lambda - \mu)\mathbf{I}_{n-r})\mathbf{S}^{-1}$ and $\mathbf{K} - \mu\mathbf{I}_n = \mathbf{S}((\lambda - \mu)\mathbf{I}_r \oplus \mathbf{0})\mathbf{S}^{-1}$. In consequence, we get $(\mathbf{K} - \lambda\mathbf{I}_n)(\mathbf{K} - \mu\mathbf{I}_n) = \mathbf{0}$, whence the assertion follows. $\square$

3. The main result

In what follows, we assume that $c_1, c_2 \in \mathbb{C}$ and $\mathbf{H}_1, \mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$ involved in linear combination (1.7) are nonzero. Furthermore, we exclude from the considerations situations in which one of the projectors occurring in (1.7) is a scalar multiple of the other, e.g., $\mathbf{H}_1 = c\mathbf{H}_2$ for some nonzero $c \in \mathbb{C}$. As pointed out in [1, p. 266], such situations lead to trivial characterizations only.

Theorem 3. *Let nonzero $\mathbf{H}_1, \mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$ be such that they are not scalar multiples of each other and satisfy $\mathbf{H}_1\mathbf{H}_2 = \mathbf{H}_2\mathbf{H}_1$. Then for nonzero $c_1, c_2 \in \mathbb{C}$, a linear combination $\mathbf{H} = c_1\mathbf{H}_1 + c_2\mathbf{H}_2$ belongs to $\mathbb{C}_n^{\mathsf{HGP}}$ if and only if any of the following disjoint sets of conditions holds:*

(i) $c_1 \in \sqrt[3]{1}$, $c_2 \in \sqrt[3]{1}$, $\mathbf{H}_1\mathbf{H}_2 = \mathbf{0}$,

(ii) $c_1 \in \sqrt[3]{1}$, $c_2 \in \sqrt[3]{-1}$, $(c_1\mathbf{H}_1 + c_2\mathbf{H}_2)\mathbf{H}_2 = \mathbf{0}$,

(iii) $c_1 \in \sqrt[3]{-1}$, $c_2 \in \sqrt[3]{1}$, $(c_1\mathbf{H}_1 + c_2\mathbf{H}_2)\mathbf{H}_1 = \mathbf{0}$,

(iv) $c_1 \in \sqrt[3]{1}$, $c_2 \in \sqrt[6]{-27}$, *and there exists* $\lambda \in \sqrt[3]{1}$ *such that* $c_1 + \lambda c_2 \in \sqrt[3]{1}$ *holds along with* $\mathbf{H}_2^2 = \lambda\mathbf{H}_1\mathbf{H}_2$,

(v) $c_1 \in \sqrt[6]{-27}$, $c_2 \in \sqrt[3]{1}$, *and there exists* $\mu \in \sqrt[3]{1}$ *such that* $\mu c_1 + c_2 \in \sqrt[3]{1}$ *holds along with* $\mathbf{H}_1^2 = \mu\mathbf{H}_1\mathbf{H}_2$,

(vi) *there exist* $\lambda \in \sqrt[3]{-1}$ and $\mu \in \sqrt[3]{1}$ *such that* $\lambda \neq \mu$, $c_1 + \lambda c_2 \in \sqrt[3]{1}$, and $c_1 + \mu c_2 \in \{0\} \cup \sqrt[3]{1}$ hold along with $\lambda\mu\mathbf{H}_1^2 + \mathbf{H}_2 = (\lambda + \mu)\mathbf{H}_1\mathbf{H}_2$ *and* $\mathbf{H}_1^2\mathbf{H}_2$ is a normal matrix.

Proof. Let $\mathrm{rk}(\mathbf{H}_1) = r_1$. Then, on account of Corollary 1, we can represent $\mathbf{H}_1$ and $\mathbf{H}_2$ as

$$\mathbf{H}_1 = \mathbf{U}\left(\mathbf{A}_1 \oplus \mathbf{0}\right)\mathbf{U}^* \quad \text{and} \quad \mathbf{H}_2 = \mathbf{U}\left(\mathbf{A}_2 \oplus \mathbf{D}_2\right)\mathbf{U}^*, \tag{3.1}$$

where $\mathbf{U} \in \mathbb{C}_n^{\mathsf{U}}$, $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{C}_{r_1}^{\mathsf{HGP}}$, $\mathbf{D}_2 \in \mathbb{C}_{n-r_1}^{\mathsf{HGP}}$, $\mathrm{rk}(\mathbf{A}_1) = r_1$, and $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$. Moreover, denoting $\mathrm{rk}(\mathbf{A}_2) = x$, again by Corollary 1, we can represent $\mathbf{A}_1$ and $\mathbf{A}_2$ as

$$\mathbf{A}_1 = \mathbf{V}\left(\mathbf{B}_1 \oplus \mathbf{C}_1\right)\mathbf{V}^* \quad \text{and} \quad \mathbf{A}_2 = \mathbf{V}\left(\mathbf{B}_2 \oplus \mathbf{0}\right)\mathbf{V}^*, \tag{3.2}$$

where $\mathbf{V} \in \mathbb{C}_{r_1}^{\mathsf{U}}$, $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{C}_x^{\mathsf{HGP}}$, $\mathbf{C}_1 \in \mathbb{C}_{r_1-x}^{\mathsf{HGP}}$, $\mathrm{rk}(\mathbf{B}_2) = x$, and $\mathbf{B}_1\mathbf{B}_2 = \mathbf{B}_2\mathbf{B}_1$. Furthermore, with $\mathrm{rk}(\mathbf{D}_2) = y$, in view of Theorem 1, we can represent $\mathbf{D}_2$ as

$$\mathbf{D}_2 = \mathbf{W}\left(\mathbf{C}_2 \oplus \mathbf{0}\right)\mathbf{W}^*, \tag{3.3}$$

where $\mathbf{W} \in \mathbb{C}_{n-r_1}^{\mathsf{U}}$, $\mathbf{C}_2 \in \mathbb{C}_y^{\mathsf{HGP}}$, and $\mathrm{rk}(\mathbf{C}_2) = y$. Concluding, from (3.1)–(3.3) we obtain

$$\mathbf{H}_1 = \mathbf{X}\left(\mathbf{B}_1 \oplus \mathbf{C}_1 \oplus \mathbf{0} \oplus \mathbf{0}\right)\mathbf{X}^* \quad \text{and} \quad \mathbf{H}_2 = \mathbf{X}\left(\mathbf{B}_2 \oplus \mathbf{0} \oplus \mathbf{C}_2 \oplus \mathbf{0}\right)\mathbf{X}^*, \tag{3.4}$$

where $\mathbf{X} \in \mathbb{C}_n^{\mathsf{U}}$ is of the form $\mathbf{X} = \mathbf{U}(\mathbf{V} \oplus \mathbf{W})$. Since the nonsingularity of $\mathbf{A}_1$ implies the nonsingularity of $\mathbf{B}_1$ and $\mathbf{C}_1$, it is seen that the nonzero summands in (3.4) satisfy $\mathbf{B}_i^3 = \mathbf{I}_x$, $i = 1, 2$, $\mathbf{C}_1^3 = \mathbf{I}_{r_1-x}$, and $\mathbf{C}_2^3 = \mathbf{I}_y$.

First, we shall prove the necessity of the six sets of conditions listed in the theorem. In this part of the proof, it is assumed (without loss of generality) that the fourth summands in representations of $\mathbf{H}_1$ and $\mathbf{H}_2$ given in (3.4) are not present, i.e.,

$$\mathbf{H}_1 = \mathbf{X}\left(\mathbf{B}_1 \oplus \mathbf{C}_1 \oplus \mathbf{0}\right)\mathbf{X}^* \quad \text{and} \quad \mathbf{H}_2 = \mathbf{X}\left(\mathbf{B}_2 \oplus \mathbf{0} \oplus \mathbf{C}_2\right)\mathbf{X}^*. \tag{3.5}$$

From (3.5) it follows that $c_1\mathbf{H}_1 + c_2\mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$ can be equivalently expressed as the conjunction

$$c_1\mathbf{B}_1 + c_2\mathbf{B}_2 \in \mathbb{C}_x^{\mathsf{HGP}},\ c_1\mathbf{C}_1 \in \mathbb{C}_{r_1-x}^{\mathsf{HGP}},\ c_2\mathbf{C}_2 \in \mathbb{C}_y^{\mathsf{HGP}}. \tag{3.6}$$

As easy to verify, for nonzero scalar c and any nonzero hypergeneralized projector, say $\mathbf{P}$, the product $c\mathbf{P}$ is a hypergeneralized projector if and only if $c \in \sqrt[3]{1}$, or, equivalently, $c \in \{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\}$. Thus, if $r_1 - x > 0$ (i.e., if the second summands in representations (3.5) are present), then the middle condition in (3.6) is satisfied if and only if $c_1 \in \sqrt[3]{1}$, and, analogously, if $y > 0$ (i.e., if the third summands in representations (3.5) are present), then the right-hand side condition in (3.6) is satisfied if and only if $c_2 \in \sqrt[3]{1}$.

Assume first that $x = 0$, i.e., summands $\mathbf{B}_1$ and $\mathbf{B}_2$ in (3.5) are not present. One clear consequence of such an assumption is that $\mathbf{H}_1\mathbf{H}_2 = \mathbf{0}$. Another one is that, in view of $\mathbf{H}_1 \neq \mathbf{0}$ and $\mathbf{H}_2 \neq \mathbf{0}$, we have $r_1 - x > 0$ and $y > 0$. Hence, conjunction (3.6) yields $c_1, c_2 \in \sqrt[3]{1}$, and we arrive at nine cases covered by the set (i) of the theorem.

Suppose now that $x > 0$, and let us focus on the left-hand side condition in (3.6). In view of $\mathbf{B}_1\mathbf{B}_2 = \mathbf{B}_2\mathbf{B}_1$, postmultiplying $(c_1\mathbf{B}_1 + c_2\mathbf{B}_2)^2 = (c_1\mathbf{B}_1 + c_2\mathbf{B}_2)^\dagger$ by $\mathbf{B}_1$ $(= \mathbf{B}_1^{-2})$, and using Corollary 2, leads to

$$c_1\mathbf{B}_1 + c_2\mathbf{B}_2 \in \mathbb{C}_x^{\mathsf{HGP}} \;\Leftrightarrow\; [(c_1\mathbf{B}_1 + c_2\mathbf{B}_2)\mathbf{B}_1^{-1}]^2 = [(c_1\mathbf{B}_1 + c_2\mathbf{B}_2)\mathbf{B}_1^{-1}]^\dagger.$$

Hence, $c_1\mathbf{B}_1 + c_2\mathbf{B}_2 \in \mathbb{C}_x^{\mathsf{HGP}}$ if and only if

$$c_1\mathbf{I}_x + c_2\mathbf{G} \in \mathbb{C}_x^{\mathsf{HGP}}, \tag{3.7}$$

where

$$\mathbf{G} = \mathbf{B}_2\mathbf{B}_1^{-1}. \tag{3.8}$$

In view of $\mathbf{B}_i^3 = \mathbf{I}_x$, $i = 1, 2$, and $\mathbf{B}_1\mathbf{B}_2 = \mathbf{B}_2\mathbf{B}_1$, it is clear that $\mathbf{G}^3 = \mathbf{I}_x$.

Condition (3.7), although being much simpler than the original one $c_1\mathbf{H}_1 + c_2\mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$, is still difficult to handle. Fortunately, as is shown in what follows, it can be further simplified. Let $\mathrm{rk}(c_1\mathbf{I}_x + c_2\mathbf{G}) = s$. Since (3.7) ensures that $c_1\mathbf{I}_x + c_2\mathbf{G} \in \mathbb{C}_x^{\mathsf{EP}}$, from Theorem 1 it follows that there exists $\mathbf{Y} \in \mathbb{C}_x^{\mathsf{U}}$ and nonsingular $\mathbf{F} \in \mathbb{C}_{s,s}$ such that

$$c_1\mathbf{I}_x + c_2\mathbf{G} = \mathbf{Y}\left(\mathbf{F} \oplus \mathbf{0}\right)\mathbf{Y}^*. \tag{3.9}$$

Combining the nonsingularity of $\mathbf{F}$ with $\mathbf{F} \in \mathbb{C}_s^{\mathsf{HGP}}$, being an obvious consequence of

(3.7) and (3.9), gives $\mathbf{F}^3 = \mathbf{I}_s$. Premultiplying and postmultiplying (3.9) by $\mathbf{Y}^*$ and $\mathbf{Y}$, respectively, leads to

$$c_1 \begin{pmatrix} \mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{x-s} \end{pmatrix} + c_2 \begin{pmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{M} & \mathbf{N} \end{pmatrix} = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{3.10}$$

where the latter matrix on the left-hand side, with $\mathbf{K} \in \mathbb{C}_{s,s}$ and $\mathbf{N} \in \mathbb{C}_{x-s,x-s}$, represents the product $\mathbf{Y}^*\mathbf{G}\mathbf{Y}$. From (3.10) if follows that $c_1\mathbf{I}_s + c_2\mathbf{K} = \mathbf{F}$ holds along with

$$\mathbf{L} = \mathbf{0},\ \mathbf{M} = \mathbf{0},\ c_1\mathbf{I}_{x-s} + c_2\mathbf{N} = \mathbf{0}. \tag{3.11}$$

Taking (3.11) into account, matrix $\mathbf{G}$ can be expressed as

$$\mathbf{G} = \mathbf{Y}\left(\mathbf{K} \oplus -(c_1/c_2)\mathbf{I}_{x-s}\right)\mathbf{Y}^*, \tag{3.12}$$

and, thus, $\mathbf{G}^3 = \mathbf{I}_x$ entails $\mathbf{K}^3 = \mathbf{I}_s$ (provided that $s > 0$) and $c_1/c_2 \in \sqrt[3]{-1}$, or, equivalently, $c_1/c_2 \in \{-1, \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\}$, (provided that $x - s > 0$).

In what follows, we consider separately the situations characterized by $s = 0$ and $0 < s \leqslant x$. In the former of them, (3.9) takes the form $c_1\mathbf{I}_x + c_2\mathbf{G} = \mathbf{0}$, what means that the conjunction

$$c_1/c_2 \in \sqrt[3]{-1},\ c_1\mathbf{B}_1 + c_2\mathbf{B}_2 = \mathbf{0} \tag{3.13}$$

necessarily holds. Four disjoint cases are possible regarding the presence of the summands in representations (3.5), namely:

(i) only the first summands are present, i.e., $\mathbf{H}_1 = \mathbf{X}\mathbf{B}_1\mathbf{X}^*$ and $\mathbf{H}_2 = \mathbf{X}\mathbf{B}_2\mathbf{X}^*$. The right-hand side condition in (3.13) ensures then that $c_1\mathbf{H}_1 + c_2\mathbf{H}_2 = \mathbf{0}$, what is irreconcilable with the assumption that $\mathbf{H}_1$ and $\mathbf{H}_2$ are not scalar multiples of each other;

(ii) only the first and second summands are present, i.e., $\mathbf{H}_1 = \mathbf{X}(\mathbf{B}_1 \oplus \mathbf{C}_1)\mathbf{X}^*$ and $\mathbf{H}_2 = \mathbf{X}(\mathbf{B}_2 \oplus \mathbf{0})\mathbf{X}^*$. Then, the middle condition in (3.6) yields $c_1 \in \sqrt[3]{1}$, what, in view of the left-hand side condition in (3.13), gives $c_2 \in \sqrt[3]{-1}$. Moreover, in view of the right-hand side condition in (3.13), we have $c_1\mathbf{H}_1 + c_2\mathbf{H}_2 = \mathbf{X}(\mathbf{0} \oplus c_1\mathbf{C}_1)\mathbf{X}^*$. Hence, $(c_1\mathbf{H}_1 + c_2\mathbf{H}_2)\mathbf{H}_2 = \mathbf{0}$, and, thus, the set (ii) of the theorem is established.

(iii) only the first and third summands are present, i.e., $\mathbf{H}_1 = \mathbf{X}(\mathbf{B}_1 \oplus \mathbf{0})\mathbf{X}^*$ and $\mathbf{H}_2 = \mathbf{X}(\mathbf{B}_2 \oplus \mathbf{C}_2)\mathbf{X}^*$. This case is a counterpart of case (ii) and leads to the set (iii) of the theorem.

(iv) all summands are present. Then, from (3.6) we obtain $c_1, c_2 \in \sqrt[3]{1}$. However, these inclusions are irreconcilable with the left-hand side condition in (3.13).

Let us now assume that $0 < s \leqslant x$. By combining (3.9) with (3.12), and utilizing $\mathbf{F}^3 = \mathbf{I}_s$, it is clear that in such a situation

$$(c_1\mathbf{I}_s + c_2\mathbf{K})^3 = \mathbf{I}_s, \tag{3.14}$$

where $\mathbf{K}^3 = \mathbf{I}_s$. On account of Corollary 3.3.8 in [7], equality $\mathbf{K}^3 = \mathbf{I}_s$ ensures that $\mathbf{K}$ is diagonalizable; see also [10, p. 410]. Thus, there exists nonsingular $\mathbf{S} \in \mathbb{C}_{s,s}$ such that $\mathbf{K} = \mathbf{S}\,\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_s)\,\mathbf{S}^{-1}$, with $\lambda_1, \lambda_2, ..., \lambda_s \in \sqrt[3]{1}$, and, in consequence, (3.14) can be rewritten as

$$(c_1 + c_2\lambda_i)^3 = 1, \;\; i = 1, 2, ..., s. \tag{3.15}$$

In what follows we consider separately three disjoint cases of (3.15) in which eigenvalues λ_i, $i = 1, 2, ..., s$,: (i) are all equal, (ii) take exactly two different values, and (iii) take exactly three different values.

Assume first that all eigenvalues of $\mathbf{K}$ are equal to, say, λ, i.e.,

$$\mathbf{K} = \lambda\mathbf{I}_s. \tag{3.16}$$

In such a case, equations (3.15) reduce to

$$c_1 + c_2\lambda \in \sqrt[3]{1}. \tag{3.17}$$

Having in mind that $\lambda \in \sqrt[3]{1}$, direct calculations show that if $c_1 \in \sqrt[3]{1}$, then (3.17) entails $c_2 \in \sqrt[6]{-27}$, or, equivalently, $c_2 \in \{\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{i}, \frac{3}{2}-\frac{\sqrt{3}}{2}\mathrm{i}, -\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{i}, -\frac{3}{2}-\frac{\sqrt{3}}{2}\mathrm{i}, \sqrt{3}\mathrm{i}, -\sqrt{3}\mathrm{i}\}$. Hence, it is clear that each of the pairs $c_1 \in \sqrt[3]{1}$, $c_2 \in \sqrt[3]{1}$ and $c_1 \in \sqrt[3]{1}$, $c_2 \in \sqrt[3]{-1}$ is irreconcilable with (3.17). Furthermore, also the pair $c_1 \in \sqrt[3]{-1}$, $c_2 \in \sqrt[3]{1}$ is in contradiction with (3.17). This means, on the one hand, that the second and third summands in representations (3.5) cannot be present simultaneously, and, on the other hand, that the presence of any of these summands must be accompanied with $x = s$, for otherwise $c_1 \in \sqrt[3]{1}$ or $c_2 \in \sqrt[3]{1}$ would be irreconcilable with the left-hand side condition in (3.13). In consequence, the following three situations regarding the presence of the summands in representations (3.5) are to be considered: (i) only the first summands are present, (ii) only the first and second summands are present with $x = s$, and (iii) only the first and third summands are present with $x = s$. Before analyzing these situations, observe that when $x = s$, then combining (3.8), (3.12), and (3.16), entails $\mathbf{B}_2 = \lambda\mathbf{B}_1$.

In situation (i), it is necessary that $0 < s < x$, for if $x = s$, then $\mathbf{H}_2 = \lambda\mathbf{H}_1$, contradicting the assumptions. In view of (3.16), representation (3.12) reduces to

$$\mathbf{G} = \mathbf{Y}(\lambda\mathbf{I}_s \oplus \mu\mathbf{I}_{x-s})\mathbf{Y}^*, \tag{3.18}$$

with $\mu = -c_1/c_2$ satisfying $\mu \in \sqrt[3]{1}$. From Lemma 4 it follows that (3.18) implies

$$\mathbf{G}^2 + \lambda\mu\mathbf{I}_x = (\lambda + \mu)\mathbf{G}. \tag{3.19}$$

Combining this equality with (3.8) leads to $\mathbf{B}_2^2\mathbf{B}_1^{-2} + \lambda\mu\mathbf{I}_x = (\lambda + \mu)\mathbf{B}_2\mathbf{B}_1^{-1}$, what multiplied by $\mathbf{B}_1^2$ entails

$$\mathbf{B}_2^2 + \lambda\mu\mathbf{B}_1^2 = (\lambda + \mu)\mathbf{B}_2\mathbf{B}_1. \tag{3.20}$$

Taking into account that only the first summands in (3.5) are present, equality (3.20) implies

$$\mathbf{H}_2^2 + \lambda\mu\mathbf{H}_1^2 = (\lambda + \mu)\mathbf{H}_1\mathbf{H}_2.$$

Moreover, (3.18) implies that $\mathbf{G} = \mathbf{B}_1^2\mathbf{B}_2$ is normal, hence $\mathbf{H}_1^2\mathbf{H}_2$ is also normal. Thus, set (vi) of the theorem has been obtained.

In the situation (ii), when only the first and second summands in representations (3.5) are present with $x = s$, on account of $\mathbf{B}_2 = \lambda\mathbf{B}_1$, we have

$$\lambda\mathbf{H}_1\mathbf{H}_2 = \lambda^2\mathbf{X}(\mathbf{B}_1^2 \oplus \mathbf{0})\mathbf{X}^* = \mathbf{H}_2^2.$$

Furthermore, the presence of the second summands in (3.5) ensures that $c_1 \in \sqrt[3]{1}$. In consequence, we arrive at the set (iv) of the theorem.

In the next situation, corresponding to the presence of only the first and third summands in representations (3.5) with $x = s$, similar arguments to the ones utilized in the proof leading to the set (iv) of the theorem, lead to its set (v). Thus, the part of the proof under the assumption that all eigenvalues of $\mathbf{K}$ are equal is completed.

Let us now assume that $\mathbf{K}$ has two distinct eigenvalues say, λ and μ. Then, from (3.15) we obtain

$$c_1 + c_2\lambda \in \sqrt[3]{1} \quad \text{and} \quad c_1 + c_2\mu \in \sqrt[3]{1},$$

whence it is clear that also here we have to consider three situations regarding the presence of the summands in representations (3.5), namely: (i) only the first summands are present, (ii) only the first and second summands are present with $x = s$, and (iii) only the first and third summands are present with $x = s$.

In the first of them, we shall still distinguish two cases, namely $x = s$ and $s < x$. If $x = s$, then (3.12) ensures that the eigenvalues of $\mathbf{G}$ are equal to the eigenvalues of $\mathbf{K}$, i.e., λ and μ. Combining the fact that $\mathbf{G}$ is diagonalizable with Lemma 4, leads to (3.19), from where the set (vi) of the theorem is (again) derived. ???If, on the other hand, $s < x$, then utilizing the fact that $c_1\mathbf{I}_s + c_2\mathbf{K}$ is nonsingular, being a trivial consequence of (3.14), Theorem 3 in [1] ensures that $c_1, c_2 \in \sqrt[3]{-1}$, what further leads to $c_1/c_2 \in \sqrt[3]{1}$. However, this inclusion is irreconcilable with $c_1/c_2 \in \sqrt[3]{-1}$.???

In the situation (ii), characterized by the presence of only the first and second summands in representations (3.5) with $x = s$, we again make use of the fact that λ and μ, being the two distinct eigenvalues of $\mathbf{K}$, are simultaneously the eigenvalues of diagonalizable $\mathbf{G}$. Hence, employing once more Lemma 4, we arrive again at (3.19). Clearly, account of the implication $\lambda, \mu \in \sqrt[3]{1} \Rightarrow \lambda + \mu \neq 0$, (3.19) shows that $\mathbf{G} \cdot \mathbf{I}_x$ can be expressed as a linear combination of $\mathbf{G}^2$ and $\mathbf{I}_x^2$. ???In consequence, since $c_1\mathbf{I}_x + c_2\mathbf{G}$ has maximal possible rank (from (3.9) it is seen that $x = s$ ensures $c_1\mathbf{I}_x + c_2\mathbf{G}$ is nonsingular), by utilizing Theorem 3 in [1] we get $c_1 \in \sqrt[3]{-1}$.??? But this inclusion is in contradiction with $c_1 \in \sqrt[3]{1}$, being a consequence of the presence of the second summands of representations (3.5).

The present step of the proof is concluded by the observation that an analogous contradiction is obtained in situation (iii), characterized by the presence of only the first and third summands in (3.5) with $x = s$.

Finally, let us assume that $\mathbf{K}$ has three different eigenvalues: 1, θ, and $\overline{\theta}$, where $\theta = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$. From (3.15) we get solvable system

$$c_1 + c_2 = u, \; c_1 + \theta c_2 = v, \; c_1 + \overline{\theta} c_2 = w, \tag{3.21}$$

where $u, v, w \in \sqrt[3]{1}$. Combining the first with the second equality in (3.21) yields $(1 - \theta)c_2 = u - v$, what implies $u \neq v$. In a similar way, we obtain $u \neq w$ and $v \neq w$. Thus, $\{u, v, w\} = \{1, \theta, \overline{\theta}\}$. Since the system (3.21) is solvable,

$$\det(\begin{pmatrix} 1 & 1 & u \\ 1 & \theta & v \\ 1 & \overline{\theta} & w \end{pmatrix}) = 0, \tag{3.22}$$

and detailed analysis shows that only three out of six possible combinations of the values of scalars u, v, and w satisfy (3.22), namely:

$$(u, v, w) = (1, \theta, \overline{\theta}), \; (u, v, w) = (\theta, \overline{\theta}, 1), \; (u, v, w) = (\overline{\theta}, 1, \theta).$$

In each of them we have $v = u\theta$. Hence, combining the first two equalities in (3.21), yields $u = c_2$, what substituted to the first equation in (3.21) leads to $c_1 = 0$, being in a contradiction with the assumptions. Thus, the necessity part of the proof is completed.

Let us now show that the sets listed in the theorem are also sufficient for $c_1\mathbf{H}_1 + c_2\mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$. Furthermore, the sets (ii) and (iii) as well as (vi) and (v) are symmetrical in the sense that one of them is obtained from the other by interchanging indexes "1" and "2". In consequence, only ???three??? sets are to be considered.

If conditions given in the set (i) hold, in view of the nonsingularity of $\mathbf{B}_1$ and $\mathbf{B}_2$, the first summands in (3.4) are not present. Hence,

$$c_1\mathbf{H}_1 + c_2\mathbf{H}_2 = \mathbf{X}(c_1\mathbf{C}_1 \oplus c_2\mathbf{C}_2 \oplus \mathbf{0})\mathbf{X}^*. \tag{3.23}$$

Since $c_1, c_2 \in \sqrt[3]{1}$, $\mathbf{C}_1^3 = \mathbf{I}_{r_1-x}$, and $\mathbf{C}_2^3 = \mathbf{I}_y$, it follows from (3.23) that $c_1\mathbf{H}_1 + c_2\mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$.

Next, if $(c_1\mathbf{H}_1 + c_2\mathbf{H}_2)\mathbf{H}_2 = \mathbf{0}$, being a condition given in the set (ii), then representations (3.4) entail

$$(c_1\mathbf{H}_1 + c_2\mathbf{H}_2)\mathbf{H}_2 = \mathbf{X}((c_1\mathbf{B}_1 + c_2\mathbf{B}_2)\mathbf{B}_2 \oplus \mathbf{0} \oplus c_2\mathbf{C}_2^2 \oplus \mathbf{0})\mathbf{X}^* = \mathbf{0}.$$

It is thus seen that the nonsingularity of $\mathbf{C}_2$ implies that the third summands in (3.4) are not present. On the other hand, since $\mathbf{H}_2 \neq \mathbf{0}$, the first summands in (3.4) are necessarily present and the nonsingularity of $\mathbf{B}_2$ entails $c_1\mathbf{B}_1 + c_2\mathbf{B}_2 = \mathbf{0}$. In consequence,

$$c_1\mathbf{H}_1 + c_2\mathbf{H}_2 = \mathbf{X}(\mathbf{0} \oplus c_1\mathbf{C}_1 \oplus \mathbf{0})\mathbf{X}^*,$$

and, in view of $c_1 \in \sqrt[3]{1}$, $\mathbf{C}_1^3 = \mathbf{I}_{r_1-x}$, the validity of inclusion $c_1\mathbf{H}_1 + c_2\mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$ is clear.

To show the sufficiency of the set (iv), first observe that from (3.4) we have

$$\mathbf{H}_2^2 = \mathbf{X}(\mathbf{B}_2^2 \oplus \mathbf{0} \oplus \mathbf{C}_2^2 \oplus \mathbf{0})\mathbf{X}^* \quad \text{and} \quad \mathbf{H}_1\mathbf{H}_2 = \mathbf{X}(\mathbf{B}_1\mathbf{B}_2 \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0})\mathbf{X}^*.$$

Since $\mathbf{H}_2^2 = \lambda\mathbf{H}_1\mathbf{H}_2$, the nonsingularity of $\mathbf{C}_2$ entails that the third summands in

(3.4) are not present. Moreover, the nonsingularity of $\mathbf{B}_2$ ensures that $\mathbf{B}_2^2 = \lambda \mathbf{B}_1 \mathbf{B}_2$ can be reduced to $\mathbf{B}_2 = \lambda \mathbf{B}_1$. Hence,

$$c_1 \mathbf{H}_1 + c_2 \mathbf{H}_2 = \mathbf{X}((c_1 + \lambda c_2)\mathbf{B}_1 \oplus c_1 \mathbf{C}_1 \oplus \mathbf{0})\mathbf{X}^*,$$

from where, on account of $\mathbf{B}_1^3 = \mathbf{I}_x$, $\mathbf{C}_1^3 = \mathbf{I}_{r_1 - x}$, and $c_1 + \lambda c_2, c_1 \in \sqrt[3]{1}$, we arrive at $c_1 \mathbf{H}_1 + c_2 \mathbf{H}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$.

Finally, we shall prove the sufficiency of the set (vi). From (3.4) we have

$$\mathbf{H}_2^2 + \lambda\mu \mathbf{H}_1^2 = \mathbf{X}(\mathbf{B}_2^2 + \lambda\mu \mathbf{B}_1^2 \oplus \lambda\mu \mathbf{C}_1^2 \oplus \mathbf{C}_2^2 \oplus \mathbf{0})\mathbf{X}^*,$$

and

$$(\lambda + \mu)\mathbf{H}_1 \mathbf{H}_2 = \mathbf{X}((\lambda + \mu)\mathbf{B}_1 \mathbf{B}_2 \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0})\mathbf{X}^*.$$

The nonsingularity of $\mathbf{C}_1$ and $\mathbf{C}_2$ imply that the second and the third summands in (3.4) are not present. Moreover, we can deduce $\mathbf{B}_2^2 + \lambda\mu \mathbf{B}_1^2 = (\lambda + \mu)\mathbf{B}_1 \mathbf{B}_2$. If we define define $\mathbf{G} = \mathbf{B}_1^{-1} \mathbf{B}_2$, we get

$$\mathbf{G}^2 + \lambda\mu \mathbf{I}_x = (\lambda + \mu)\mathbf{G}. \tag{3.24}$$

On the other hand, the normality of $\mathbf{H}_1^2 \mathbf{H}_2$ imply the normality of $\mathbf{G}$. In view of (3.24), there exists $\mathbf{Y} \in \mathbb{C}_x^{\mathsf{U}}$ such that $\mathbf{G} = \mathbf{Y}(\lambda \mathbf{I}_s \oplus \mu \mathbf{I}_{x-s})\mathbf{Y}^*$ for some natural s with $0 < s < x$. Now,

$$c_1 \mathbf{I}_x + c_2 \mathbf{G} = \mathbf{Y}((c_1 + \lambda c_2)\mathbf{I}_s \oplus (c_1 + \mu c_2)\mathbf{I}_{x-s})\mathbf{Y}^*,$$

which in view of $c_1 + \lambda c_2 \in \sqrt[3]{1}$ and $c_1 + \mu c_2 \in \{0\} \cup \sqrt[3]{1}$, implies $(c_1 \mathbf{I}_x + c_2 \mathbf{G})^3 = \mathbf{Y}(\mathbf{I}_s \oplus \delta \mathbf{I}_{x-s})\mathbf{Y}^*$, where $\delta \in \{0, 1\}$. Hence $(c_1 \mathbf{I}_x + c_2 \mathbf{G})^3$ is Hermitian and $(c_1 \mathbf{I}_x + c_2 \mathbf{G})^4 = c_1 \mathbf{I}_x + c_2 \mathbf{G}$. These two latter equalities are equivalent to

$$(c_1 \mathbf{I}_x + c_2 \mathbf{G})^\dagger = (c_1 \mathbf{I}_x + c_2 \mathbf{G})^2. \tag{3.25}$$

In particular $c_1 \mathbf{I}_x + c_2 \mathbf{G} \in \mathbb{C}_x^{\mathsf{EP}}$. By corollary 2,

$$((c_1\mathbf{I}_x + c_2\mathbf{G})\mathbf{B}_1)^\dagger = (c_1\mathbf{I}_x + c_2\mathbf{G})^\dagger\mathbf{B}_1^\dagger.$$

Now apply (3.25) in order to get $(c_1\mathbf{B}_1 + c_2\mathbf{B}_2)^\dagger = (c_1\mathbf{I}_x + c_2\mathbf{G})^2\mathbf{B}_1^2 = (c_1\mathbf{B}_1 + c_2\mathbf{B}_2)^2$, which yields to $c_1\mathbf{B}_1 + c_2\mathbf{B}_2 \in \mathbb{C}_n^{\mathsf{HGP}}$.

The proof is complete. $\square$

In a comment to Theorem 3 observe that its set (i) corresponds to Corollary 2 in [1], ...

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