Applications of CS decomposition in linear combinations of two orthogonal projectors

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Abstract
We study the spectrum and the rank of a linear combination of two orthogonal projectors. We characterize when this linear combination is EP, diagonalizable, idempotent, tripotent, involutive, nilpotent, generalized projector, and hypergeneralized projector. Also we derive the Moore-Penrose inverse of a linear combination of two orthogonal projectors in a particular case. The main tool used here is the CS decomposition.

Key words. Linear combinations of orthogonal projectors; CS decomposition; principal angles.

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1 Introduction
Let \( \mathbb{C}_{m,n} \) denote the set of \( m \times n \) matrices. A matrix \( A \) is said to be idempotent (also called oblique projector) when \( A^2 = A \). The symbol \( \mathbb{C}^\text{OP}_n \) will stand for the subset of \( \mathbb{C}_{n,n} \) consisting of orthogonal projectors (also called Hermitian idempotent matrices), i.e.,

\[
\mathbb{C}^\text{OP}_n = \{ P \in \mathbb{C}_{n,n} : P^2 = P = P^* \},
\]

where \( P^* \) is the conjugate transpose of \( P \). Moreover, \( I_n \) will be the identity matrix of order \( n \). The symbols \( \text{rk}(K) \) and \( N(K) \) will denote the rank and the null space, respectively, of \( K \in \mathbb{C}_{m,n} \) and \( \sigma(K) \) will denote the set of the eigenvalues of a square matrix \( K \). Further, \( K^\dagger \) will stand for the Moore-Penrose inverse of \( K \in \mathbb{C}_{m,n} \), i.e., the unique matrix satisfying the four equations

\[
KK^\dagger K = K, \quad K^\dagger KK^\dagger = K^\dagger, \quad (KK^\dagger)^* = KK^\dagger, \quad (K^\dagger K)^* = K^\dagger K.
\]

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If $U \in \mathbb{C}^{n,n}$ is unitary (i.e., $U^* = U^{-1}$) and $A \in \mathbb{C}^{n,n}$, then it is obvious that $(UAU^*)^\dagger = UAU^\dagger$. If $B$ and $C$ are square matrices, then it is also evident $(B \oplus C)^\dagger = B^\dagger \oplus C^\dagger$. A square matrix $A$ is said to be $EP$ when $AA^\dagger = A^\dagger A$.

It can be proved that for a matrix $A \in \mathbb{C}^{n,n}$, the set $\{X \in \mathbb{C}^{n,n} : AXA = A, XAX = X, AX =XA\}$ is or empty or a singleton (see [1]). When this set is a singleton, it is usual to denote by $A^#$ its unique element and $A^#$ is called the group inverse of $A$.

For a given subspace $X$ of $\mathbb{C}^{n,1}$, the symbol $X^\perp$ will mean the orthogonal complement of $X$. In addition, for a given nonzero complex number $c$, we shall denote $c^\dagger = c^{-1}$ and $0^\dagger = 0$.

In this paper, we shall investigate several properties of the linear combination

$$X = c_1P_1 + c_2P_2. \quad (1)$$

when $P_1, P_2 \in \mathbb{C}^{OP}_n$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Such linear combinations are important for the combination technique which has repeatedly been shown to be an effective tool for the approximation with sparse grid spaces (see [2]). In [3] it was shown how the spectrum of sums of orthogonal projectors determine the convergence of many parallel iterative algorithms. Moreover, the gap between two equidimensional subspaces $X$ and $Y$ (defined as $\|P_X - P_Y\|$, where $P_X$ and $P_Y$ are the orthogonal projectors onto $X$ and $Y$, respectively) has found many important applications (see, for example, [4, 5, 6]).

A well known fact is that every orthogonal projector is unitarily diagonalizable and its spectrum is contained in $\{0, 1\}$. In other words, if the rank of $P \in \mathbb{C}^{OP}_n$ is $k$, then there exists a unitary matrix $U \in \mathbb{C}^{n,n}$ such that $P = U(I_k \oplus 0)U^*$, wherein some summand may be absent.

The main tool used in this paper is the CS decomposition (see, for example, [7, 5, 2] and for a survey, [8] and references therein). This decomposition is closely related to the principal angles between two subspaces (see [9]). These angles serve as important tool in functional analysis ([4], in perturbation theory ([10, 11]), in statistics ([12, 13]), or in proving identities on the norm of idempotent operators.
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2 Main results

Firstly, we study the easiest situation: when the involved orthogonal projectors occurring in the linear combination (1) commute.

Theorem 2.1. Let $P_1, P_2 \in \mathbb{C}^{n \times n}_{\text{OP}}$ such that $P_1 P_2 = P_2 P_1$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Then

(i) $\sigma(c_1 P_1 + c_2 P_2) \subset \{0, c_1, c_2, c_1 + c_2\}$.

(ii) $\text{rk}(P_1 + P_2) = \text{rk}(P_1) + \text{rk}(P_2) - \text{rk}(P_1 P_2)$.

(iii) $\text{rk}(P_1 - P_2) = \text{rk}(P_1) + \text{rk}(P_2) - 2 \text{rk}(P_1 P_2)$.

(iv) If $c_1 + c_2 \neq 0$, then $\text{rk}(c_1 P_1 + c_2 P_2) = \text{rk}(P_1 + P_2)$.

(v) $(c_1 P_1 + c_2 P_2)^\dagger = [(c_1 + c_2)^\dagger - c_1^{-1} - c_2^{-1}] P_1 P_2 + c_1^{-1} P_1 + c_2^{-1} P_2$.

Proof. Let $x = \text{rk}(P_1 P_2)$, $y = \text{rk}(P_1)$, and $z = \text{rk}(P_2)$. Since $P_1 P_2 = P_2 P_1$, by a simultaneous diagonalization, there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that $P_1 = U (I_x \oplus I_{y-x} \oplus 0 \oplus 0) U^*$ and $P_2 = U (I_x \oplus 0 \oplus I_{z-x} \oplus 0) U^*$. It is clear that

\[
c_1 P_1 + c_2 P_2 = U ((c_1 + c_2) I_x \oplus c_1 I_{y-x} \oplus c_2 I_{z-x} \oplus 0) U^*.
\]

Now, it should be easy to finish the proof. \qed

Let us note that the fourth item of the latter theorem is a particular case of [15, Theorem 2.4]. Moreover, in [16, Corollary 3], it was proved that for two given commuting idempotent matrices $P_1, P_2$, the difference $P_1 - P_2$ is nonsingular if and only if $(P_1 - P_2)^2 = I_n$, which immediately gives the formula for $(P_1 - P_2)^{-1}$ in case $P_1 - P_2$ is nonsingular. Observe that the item (v) of Theorem 2.1 is a generalization of this formula when $P_1$ and $P_2$ are also Hermitian.

Now, we consider situations in which orthogonal projectors $P_1$ and $P_2$ occurring in (1) do not commute. In order to study such situations, we shall use the called CS decomposition which is now established (see e.g. [7, 5, 2]):

([14]).
Lemma 2.2 (CS decomposition). Let $P_1,P_2 \in \mathbb{C}^{op}_n$. Then there exists a unitary matrix $U \in \mathbb{C}_{n,n}$ such that

$$P_1 = U \begin{bmatrix} I & 0 \\ 0 & I \\ I & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad P_2 = U \begin{bmatrix} C^2 & CS \\ CS & S^2 \\ I & 0 \\ 0 & I \end{bmatrix} U^*,$$

where $C,S$ are positive diagonal real matrices such that $C^2 + S^2 = I$, the symbol $I$ denotes identity matrices of various sizes, and the corresponding blocks in the two projection matrices are of the same size.

The following theorem gives information on the commutator of two orthogonal projectors (see [17, Theorem 2.7] for some formulae of the rank of the commutator of any two idempotent matrices).

Theorem 2.3. Let $P_1,P_2 \in \mathbb{C}^{op}_n$. Then $P_1P_2 - P_2P_1$ is skew-Hermitian and its rank is even.

Proof. Let $U \in \mathbb{C}_{n,n}$ and $C,S \in \mathbb{C}_{p,p}$ have the same meaning as in Lemma 1. Let us denote

$$T_1 = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}_{2p,2p}, \quad T_2 = \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix} \in \mathbb{C}_{2p,2p},$$

and $R_1,R_2 \in \mathbb{C}_{n-2p,n-2p}$ in such a way that

$$P_i = U(T_i \oplus R_i)U^* \quad (2)$$

for $i = 1,2$. It is trivial

$$R_1R_2 = R_2R_1. \quad (3)$$

A simple computation shows

$$P_1P_2 - P_2P_1 = U(T_1T_2 - T_2T_1 \oplus 0)U^* = U \begin{bmatrix} 0 & CS \\ -CS & 0 \end{bmatrix} \oplus 0 \end{bmatrix} U^*.$$

This immediately proves that $P_1P_2 - P_2P_1$ is skew-Hermitian. Moreover, since

$$\det \begin{bmatrix} 0 & CS \\ -CS & 0 \end{bmatrix} \neq 0,$$

we obviously get $\text{rk}(P_1P_2 - P_2P_1) = 2p$. The proof is finished. $\Box$
Following the above notation, it can be demonstrated that the matrix

\[ \Pi = U(0 \oplus I_{n-2p})U^* \]  

is the orthogonal projector onto \( N(P_1P_2 - P_2P_1) \). In order to prove this, it is sufficient to show \( N(\Pi) = N(P_1P_2 - P_2P_1)^\perp \). In fact, let \( x \in N(\Pi) \) have the representation \( U^*x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \), where \( x_1 \in \mathbb{C}_{2p,1} \) and \( x_2 \in \mathbb{C}_{n-2p,1} \). An obvious computation shows

\[ 0 = \Pi x = U \begin{bmatrix} 0 & 0 \\ 0 & I_{n-2p} \end{bmatrix} U^* x = U \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \]

which implies \( x_2 = 0 \). On the other hand, let \( y \in N(P_1P_2 - P_2P_1) \) have the representation \( U^*y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \), where \( y_1 \in \mathbb{C}_{2p,1} \) and \( y_2 \in \mathbb{C}_{n-2p,1} \). Again, we easily get

\[ 0 = (P_1P_2 - P_2P_1)y = U \begin{bmatrix} T_1T_2 - T_2T_1 & 0 \\ 0 & 0 \end{bmatrix} U^* y = U \begin{bmatrix} (T_1T_2 - T_2T_1)y_1 \\ 0 \end{bmatrix}, \]

which yields to \( (T_1T_2 - T_2T_1)y_1 = 0 \). The nonsingularity of \( T_1T_2 - T_2T_1 \) (proved in Theorem 2.3) implies \( y_1 = 0 \). But now,

\[ x^*y = x^*U^*y = (U^*x)^*(U^*y) = \begin{bmatrix} x_1^* & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = 0. \]

We have proved

\[ N(\Pi) \subset N(P_1P_2 - P_2P_1)^\perp. \]  

But these two latter subspaces have the same dimension:

\[ \dim(N(\Pi)) = n - \text{rk}(\Pi) = n - (n - 2p) = 2p, \]  

and, again by Theorem 2.3,

\[ \dim(N(P_1P_2 - P_2P_1)^\perp) = n - \dim(N(P_1P_2 - P_2P_1)) = \text{rk}(P_1P_2 - P_2P_1) = 2p. \]  

The inclusion (5) together (6) and (7) lead to \( N(\Pi) = N(P_1P_2 - P_2P_1)^\perp \).

Also, the matrix \( I_n - \Pi = U(I_{2p} \oplus 0)U^* \) is the orthogonal projector onto \( N(P_1P_2 - P_2P_2)^\perp \). Furthermore, from (2) and (4) we have \( \Pi P_1 = P_1\Pi \) and \( \Pi P_2 = P_2\Pi \).
Rearranging the entries of $T_1$ and $T_2$, we can suppose by a suitable permutation that
\[
T_1 = \bigoplus_{i=1}^p \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \bigoplus_{i=1}^p \begin{bmatrix} \cos^2 \theta_i & \cos \theta_i \sin \theta_i \\ \cos \theta_i \sin \theta_i & \sin^2 \theta_i \end{bmatrix},
\]
where $\theta_i \in ]0, \pi/2[$. From now on, we shall write $\Theta(P_1, P_2) = \{\theta_1, \ldots, \theta_p\}$ (some $\theta$'s may be equal). These angles are some of the principal angles between $\mathcal{R}(P_1)$ and $\mathcal{R}(P_2)$, where $\mathcal{R}(\cdot)$ denotes the range space. Note that $\Theta(P_1, P_2) = \emptyset$ if and only if $P_1 P_2 = P_2 P_1$. Also, we shall denote
\[
X_i = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} \cos^2 \theta_i & \cos \theta_i \sin \theta_i \\ \cos \theta_i \sin \theta_i & \sin^2 \theta_i \end{bmatrix}
\]
for $i = 1, \ldots, p$. From (8) and the above definition, one evidently has
\[
c_1 T_1 + c_2 T_2 = \bigoplus_{i=1}^p X_i.
\]
Since
\[
c_1 P_1 + c_2 P_2 = U(c_1 T_1 + c_2 T_2 \oplus c_1 R_1 + c_2 R_2) U^*,
\]
the study of the linear combination $c_1 P_1 + c_2 P_2$ reduces to the simultaneous study of $c_1 T_1 + c_2 T_2$ and $c_1 R_1 + c_2 R_2$. Having in mind that $R_1$ and $R_2$ commute, we can apply Theorem 2.1 in order to deal with the linear combination $c_1 R_1 + c_2 R_2$.

As we shall see, the following result is the basis for further derivations.

**Theorem 2.4.** Let $P_1, P_2 \in \mathbb{C}_n^{OP}$ and let us denote by $\Pi$ the orthogonal projector onto $\mathcal{N}(P_1 P_2 - P_2 P_2)$. If $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, then the following statements hold:

(i) Let $\theta \in \Theta(P_1, P_2)$ and $z$ be a complex number such that $z^2 = (c_1 + c_2)^2 - 4c_1 c_2 \sin^2 \theta$, then
\[
\frac{c_1 + c_2 + z}{2} \in \sigma(c_1 P_1 + c_2 P_2).
\]
If $\lambda \in \sigma(c_1 P_1 + c_2 P_2) \setminus \{0, c_1, c_2, c_1 + c_2\}$, then there exists $\theta \in \Theta(P_1, P_2)$ and a complex square root of $(c_1 + c_2)^2 - 4c_1 c_2 \sin^2 \theta$, say $z$, such that $\lambda = (c_1 + c_2 + z)/2$. 
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(ii) $\text{rk}(P_1P_2 - P_2P_1) \leq \text{rk}(c_1P_1 + c_2P_2)$. In particular, if $P_1P_2 - P_2P_1$ is non-singular, then $c_1P_1 + c_2P_2$ is nonsingular.

(iii) $\text{rk}((c_1P_1 + c_2P_2)(I - \Pi)) = \text{rk}(P_1P_2 - P_2P_1)$.

(iv) $\text{rk}((c_1P_1 + c_2P_2)\Pi) = n - \text{rk}(c_1P_1 + c_2P_2)$ if and only if $c_1P_1 + c_2P_2$ is nonsingular.

Proof. From (10) and (11), we have

$$\sigma(c_1P_1 + c_2P_2) = \bigcup_{i=1}^{p} \sigma(X_i) \cup \sigma(c_1R_1 + c_2R_2),$$

(12)

where matrices $X_i$ are defined in (9). We easily get

$$\det(X_i - \lambda I_2) = \lambda^2 - (c_1 + c_2)\lambda + c_1c_2 \sin^2 \theta_i.$$  

Thus, if $z$ is any square root of $(c_1 + c_2)^2 - 4c_1c_2 \sin^2 \theta_i$, then

$$\sigma(X_i) = \left\{ \frac{c_1 + c_2 + z}{2}, \frac{c_1 + c_2 - z}{2} \right\}.$$  

Since $\sigma(X_i) \subset \sigma(c_1P_1 + c_2P_2)$ for $i = 1, \ldots, p$, we have proved the first part of item (i). The other part can be proved by using (12) and Theorem 2.1.

We shall prove (ii). It is trivial that $\det(X_i) = c_1c_2 \sin^2 \theta_i \neq 0$. Since

$$c_1P_1 + c_2P_2 = U(X_1 \oplus \cdots \oplus X_p \oplus c_1R_1 + c_2R_2)U^*,$$

we get $\text{rk}(c_1P_1 + c_2P_2) \geq \text{rk}(X_1) + \cdots + \text{rk}(X_p) = 2p$.

Let us prove (iii). Having in mind (4) and (11) we get

$$(c_1P_1 + c_2P_2)(I - \Pi) = U(c_1T_1 + c_2T_2 \oplus 0)U^*.$$  

But the rank of $c_1T_1 + c_2T_2 = \oplus_{i=1}^{p} X_i$ is $2p$.

Finally, let us prove (iv): By (11) and the previous item, we have

$$\text{rk}(c_1P_1 + c_2P_2) = \text{rk}(c_1T_1 + c_2T_2) + \text{rk}(c_1R_1 + c_2R_2) = 2p + \text{rk}(c_1R_1 + c_2R_2).$$  

Since $(c_1P_1 + c_2P_2)\Pi = U(0 \oplus c_1R_1 + c_2R_2)U^*$, we obtain $\text{rk}((c_1P_1 + c_2P_2)\Pi) = n - 2p$ if and only if $\text{rk}(c_1P_1 + c_2P_2) = n$. Thus (iv) has been proved.  

$\square$
In terms of blocks $R_1, R_2, T_1,$ and $T_2$, item (iii) is equivalent to say that $c_1T_1 + c_2T_2$ is always nonsingular and item (iv) expresses that $c_1P_1 + c_2P_2$ is nonsingular if and only if $c_1R_1 + c_2R_2$ is nonsingular.

**Corollary 2.5.** Let $P_1, P_2 \in \mathbb{C}_n^{OP}$ and let $\Pi$ be the orthogonal projector onto $\mathcal{N}(P_1P_2 - P_2P_1)$. Matrix $P_1 - P_2$ is nonsingular if and only if $(P_1 + P_2)\Pi = \Pi$

**Proof.** Following the notation of Theorem 2.4, observe that the nonsingularity of $P_1 - P_2$ is equivalent to the nonsingularity of $R_1 - R_2$. Since $R_1$ and $R_2$ are two commuting idempotent matrices, we can apply [16, Corollary 3] to assure that $R_1 - R_2$ is nonsingular if and only if $R_1 + R_2 = I_{n-2p}$. Since $(P_1 + P_2)\Pi = U(0 \oplus R_1 + R_2)U^*$ and $\Pi = U(0 \oplus I_{n-2p})U^*$, the conclusion follows. \qed

**Note 1.** Let $A \in \mathbb{C}_{n,n}$, $\mathcal{X}$ a vector subspace of $\mathbb{C}_{n,1}$, and $P_{\mathcal{X}}$ the orthogonal projector onto $\mathcal{X}$. It can be easily verified that $AP_{\mathcal{X}} = 0$ if and only if $\mathcal{X} \subset \mathcal{N}(A)$. Moreover, recall that $\Pi P_1 = P_1 \Pi$ and $\Pi P_2 = P_2 \Pi$ hold. Hence Corollary 2.5 can be reformulated without mentioning the orthogonal projector onto $\mathcal{N}(P_1P_2 - P_2P_1)$ as follows: Let $P_1, P_2 \in \mathbb{C}_n^{OP}$, then $P_1 - P_2$ is nonsingular if and only if $\mathcal{N}(P_1P_2 - P_2P_1) \subset \mathcal{N}(P_1 + P_2 - I)$.

The following result permits to compute the Moore-Penrose inverse of a linear combination of two non commuting orthogonal projectors in a simple case.

**Theorem 2.6.** Let $P_1, P_2 \in \mathbb{C}_n^{OP}$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Assume that the cardinal of $\Theta(P_1, P_2)$ is exactly 1. Then

$$(c_1P_1 + c_2P_2)^\dagger = \left[ \frac{c_1 + c_2}{c_1c_2\sin^2\theta} I_n - \frac{1}{c_1c_2\sin^2\theta} (c_1P_1 + c_2P_2) \right] (I_n - \Pi) + \left[ ((c_1 + c_2)^\dagger - c_1^{-1} - c_2^{-1})P_1P_2 + c_1^{-1}P_1 + c_2^{-1}P_2 \right] \Pi,$$

where $\Pi$ is the orthogonal projector onto $\mathcal{N}(P_1P_2 - P_2P_1)$ and $\{\theta\} = \Theta(P_1, P_2)$.

**Proof.** From (11) and item (iii) of Theorem 2.4, we have

$$(c_1P_1 + c_2P_2)^\dagger = U[(c_1T_1 + c_1T_2)^{-1} \oplus (c_1R_1 + c_1R_2)^\dagger]U^*.$$
Let $\theta$ be the unique angle in $\Theta(P_1, P_2)$. From (8), we have
\[
c_1 T_1 + c_1 T_2 = \bigoplus_{i=1}^p X_i, \quad X_i = \begin{bmatrix} c_1 + c_2 \cos^2 \theta & c_2 \cos \theta \sin \theta \\ c_2 \cos \theta \sin \theta & c_2 \sin^2 \theta \end{bmatrix}.
\] (13)

Since $\det(X_i - \lambda I_2) = \lambda^2 - (c_1 + c_2)\lambda + c_1 c_2 \sin^2 \theta$, by the Cayley-Hamilton Theorem, one has $X_i^2 - (c_1 + c_2)X_i + c_1 c_2 \sin^2 \theta I_2 = 0$, which implies
\[
X_i^{-1} = \frac{1}{c_1 c_2 \sin^2 \theta} ((c_1 + c_2)I_2 - X_i).
\] (14)

Hence (13) and (14) entail
\[
(c_1 T_1 + c_1 T_2)^{-1} = \bigoplus_{i=1}^p X_i^{-1} = \bigoplus_{i=1}^p \frac{1}{c_1 c_2 \sin^2 \theta} ((c_1 + c_2)I_2 - X_i)
\]
\[
= \frac{1}{c_1 c_2 \sin^2 \theta} \left[ (c_1 + c_2)I_{2p} - \bigoplus_{i=1}^p X_i \right]
\]
\[
= \frac{1}{c_1 c_2 \sin^2 \theta} \left[ (c_1 + c_2)I_{2p} - (c_1 T_1 + c_1 T_2) \right].
\]

Since $I_n - \Pi = U(I_{2p} \oplus 0)U^*$, we get
\[
U((c_1 T_1 + c_1 T_2)^{-1} \oplus 0)U^* = \left[ \frac{c_1 + c_2}{c_1 c_2 \sin^2 \theta} I_n - \frac{1}{c_1 c_2 \sin^2 \theta} (c_1 P_1 + c_1 P_2) \right] (I_n - \Pi).
\]

On the other hand, by applying item (v) of Theorem 2.1, we get
\[
U(0 \oplus (c_1 R_1 + c_2 R_2)^\dagger)U^* = \left[ [(c_1 + c_2)^\dagger - c_1^{-1} - c_2^{-1}]P_1 P_2 + c_1^{-1}P_1 + c_2^{-1}P_2 \right] \Pi.
\]

Adding the last two equations completes the proof of the theorem.

The following result is an easy consequence of Theorem 2.4.

**Corollary 2.7.** Let $P_1, P_2 \in \mathbb{C}_{\text{OP}}^n$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Then $c_1 P_1 + c_2 P_2$ is an EP matrix.

**Proof.** We shall use representation (11). Matrix $c_1 R_1 + c_2 R_2$ is an EP matrix because it is diagonal. Matrix $c_1 T_1 + c_2 T_2$ is an EP matrix because it is nonsingular (by item (iii) of Theorem 2.4).
This latter corollary proves that for given $P_1, P_2 \in \mathbb{C}_n^{OP}$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, the group inverse of $c_1 P_1 + c_2 P_2$ exists and it equals to $(c_1 P_1 + c_2 P_2)^{\dagger}$.

Next result permits to relate the eigenvalues of the product $P_1 P_2$ with the eigenvalues of the linear combination $c_1 P_1 + c_2 P_2$ for two noncommuting $P_1, P_2 \in \mathbb{C}_n^{OP}$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Observe that if $P_1 P_2 = P_2 P_1$, then $P_1 P_2$ is again an orthogonal projector, hence $\sigma(P_1 P_2) \subset \{0, 1\}$.

**Theorem 2.8.** Let $P_1, P_2 \in \mathbb{C}_n^{OP}$ with $P_1 P_2 \neq P_2 P_1$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If
\[ \lambda \in \mathbb{C} \setminus \{0, c_1, c_2, c_1 + c_2\}, \]
then
\[ \lambda \in \sigma(c_1 P_1 + c_2 P_2) \iff \frac{\lambda^2 - \lambda(c_1 + c_2) + c_1 c_2}{c_1 c_2} \in \sigma(P_1 P_2). \]

**Proof.** Let $P_1$ and $P_2$ have the representation (2), where $T_1$ and $T_2$ are written in the form (8). Evidently, one has
\[ T_1 T_2 = \bigoplus_{i=1}^{p} \begin{bmatrix} \cos^2 \theta_i & \cos \theta_i \sin \theta_i \\ 0 & 0 \end{bmatrix}, \] (15)

hence $\sigma(T_1 T_2) = \{\cos^2 \theta : \theta \in \Theta(P_1, P_2)\} \cup \{0\}$.

Pick any $\lambda \in \sigma(c_1 P_1 + c_2 P_2) \setminus \{0, c_1, c_2, c_1 + c_2\}$. By the first item of Theorem 2.4, there exists $z \in \mathbb{C}$ and $\theta \in \Theta(P_1, P_2)$ such that
\[ z^2 = (c_1 + c_2)^2 - 4 c_1 c_2 \sin^2 \theta, \quad \lambda = \frac{c_1 + c_2 + z}{2}. \] (16)

It is simple to check $(c_1 + c_2)^2 - 4 c_1 c_2 \sin^2 \theta = (c_1 - c_2)^2 + 4 c_1 c_2 \cos^2 \theta$. Therefore, there exists $\mu \in \sigma(T_1 T_2)$ such that $z^2 = (c_1 - c_2)^2 + 4 c_1 c_2 \mu$. The second relation of (16) yields to
\[ (2 \lambda - (c_1 + c_2))^2 = (c_1 - c_2)^2 + 4 c_1 c_2 \mu, \]
which reduces to
\[ \mu = \frac{\lambda^2 - \lambda(c_1 + c_2) + c_1 c_2}{c_1 c_2}. \] (17)

Because $\sigma(T_1 T_2) \subset \sigma(P_1 P_2)$ and $\mu \in \sigma(T_1 T_2)$, relation (17) proves the “$\Rightarrow$” part.

Now assume that $\mu = (\lambda^2 - \lambda(c_1 + c_2) + c_1 c_2)/(c_1 c_2) \in \sigma(P_1 P_2)$, where $\lambda \in \mathbb{C} \setminus \{0, c_1, c_2, c_1 + c_2\}$. It is easily seen that $\mu = 0$ or $\mu = 1$ implies $\lambda \in \{c_1, c_2\}$ or $\lambda \in \{0, c_1 + c_2\}$, respectively, which is unfeasible. Since $\sigma(P_1 P_2) \setminus \{0, 1\} = \sigma(T_1 T_2) \setminus \{0, 1\}$, the theorem is proved.
\[ \sigma(T_1T_2) \setminus \{0\}, \] we have \( \mu \in \sigma(T_1T_2) \setminus \{0\} \). From (15), there exists \( i \in \{1, \ldots, p\} \) such that \( \mu = \cos^2 \theta_i \). If we denote \( z = 2\lambda - (c_1 + c_2) \), then it is verified that \( z^2 = (c_1 + c_2)^2 - 4c_1c_2\sin^2 \theta_i \), and finally, by applying item (i) of Theorem 2.4, we obtain \( \lambda \in \sigma(c_1P_1 + c_2P_2) \).

By giving particular values in Theorem 2.8, we get simple expressions. Let \( P_1, P_2 \in \mathbb{C}^{n}_{OP} \) with \( P_1P_2 \neq P_2P_1 \). Then

1. If \( \lambda \in \mathbb{C} \setminus \{0, 1, 2\} \), then \( \lambda \in \sigma(P_1 + P_2) \iff (\lambda - 1)^2 \in \sigma(P_1P_2) \).

2. If \( \lambda \in \mathbb{C} \setminus \{0, 1, -1\} \), then \( \lambda \in \sigma(P_1 - P_2) \iff 1 - \lambda^2 \in \sigma(P_1P_2) \).

Note that the second equivalence was proved in [18, Lemma 2.4] in the more general setting of \( C^*-\)algebras.

**Note 2.** It is straightforward to check that if \( P_1, P_2 \in \mathbb{C}^{n}_{OP} \) and \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \), then

\[
(c_1P_1 + c_2P_2)(c_1P_1 + c_2P_2)^* - (c_1P_1 + c_2P_2)^*(c_1P_1 + c_2P_2) = (c_1\overline{c}_2 - \overline{c}_1c_2)(P_1P_2 - P_2P_1),
\]

which immediately proves that \( c_1P_1 + c_2P_2 \) is normal if and only if \( c_1/c_2 \in \mathbb{R} \) or \( P_1P_2 = P_2P_1 \).

The following result characterizes the diagonalizability of \( c_1P_1 + c_2P_2 \) when \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \) and \( P_1, P_2 \) are non commuting orthogonal projectors. Observe that if \( P_1, P_2 \in \mathbb{C}^{n}_{OP} \) and \( P_1P_2 = P_2P_1 \), then, by a unitarily simultaneous diagonalization (or by the previous note), the linear combination \( c_1P_1 + c_2P_2 \) is a normal matrix, hence diagonalizable.

**Corollary 2.9.** Let \( P_1, P_2 \in \mathbb{C}^{n}_{OP} \) with \( P_1P_2 \neq P_2P_1 \) and \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \). Matrix \( c_1P_1 + c_2P_2 \) is diagonalizable if and only if \( (c_1 + c_2)^2 \neq 4c_1c_2\sin^2 \theta \) for all \( \theta \in \Theta(P_1, P_2) \).

**Proof.** Matrix \( c_1P_1 + c_2P_2 \) is diagonalizable if and only if matrices \( c_1R_1 + c_2R_2 \) and \( c_1T_1 + c_2T_2 \) involved in (11) are both diagonalizable (see [19, Lemma 1.3.10]). Since
$c_1 \mathbf{R}_1 + c_2 \mathbf{R}_2$ is diagonal, the diagonalizability of $c_1 \mathbf{P}_1 + c_2 \mathbf{P}_2$ is equivalent to the diagonalizability of $c_1 \mathbf{T}_1 + c_2 \mathbf{T}_2$. In view of (10), the diagonalizability of $c_1 \mathbf{T}_1 + c_2 \mathbf{T}_2$ is equivalent to the diagonalizability of $\mathbf{X}_i$ for all $i = 1, \ldots, p$. The characteristic polynomial of $\mathbf{X}_i$ is $\lambda^2 - (c_1 + c_2) \lambda + c_1 c_2 \sin^2 \theta_i$, which has two distinct roots if and only if $(c_1 + c_2)^2 \neq 4 c_1 c_2 \sin^2 \theta_i$. Hence, if $(c_1 + c_2)^2 \neq 4 c_1 c_2 \sin^2 \theta_i$, then $\mathbf{X}_i$ is diagonalizable. If $(c_1 + c_2)^2 = 4 c_1 c_2 \sin^2 \theta_i$ then $\mathbf{X}_i$ has only one eigenvalue, say $\lambda$. If $\mathbf{X}_i$ were diagonalizable, then $\mathbf{X}_i = \lambda \mathbf{I}_2$, which is irreconcilable with, for example, the upper-right entry of $\mathbf{X}_i$ in (13).

It can be easily proved that

$$ (c_1 + c_2)^2 = 4 c_1 c_2 \sin^2 \theta \iff \frac{c_1}{c_2} = -\cos(2\theta) \pm i \sin(2\theta). $$

Hence we can characterize the diagonalizability of $c_1 \mathbf{P}_1 + c_2 \mathbf{P}_2$ in terms of $c_1/c_2$.

A matrix $\mathbf{X} \in \mathbb{C}_{n,n}$ is said tripotent when satisfies $\mathbf{X}^3 = \mathbf{X}$; involutive when $\mathbf{X}^2 = \mathbf{I}_n$; nilpotent when there exists $k \in \mathbb{N}$ such that $\mathbf{X}^k = \mathbf{0}$, generalized projector when $\mathbf{X}^2 = \mathbf{X}^*$, and hypergeneralized projector when $\mathbf{X}^2 = \mathbf{X}^\dagger$.

For two given nonzero orthogonal projectors $\mathbf{P}_1, \mathbf{P}_2$ that commute and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, the idempotency and the tripotency of $c_1 \mathbf{P}_1 + c_2 \mathbf{P}_2$ was studied in [20] and [21], respectively. Concerning the involutiveness of $c_1 \mathbf{P}_1 + c_2 \mathbf{P}_2$ is sufficient to observe that a matrix is involutive if and only if it is nonsingular and tripotent. Finally, the characterization of $(c_1 \mathbf{P}_1 + c_2 \mathbf{P}_2)^2 = (c_1 \mathbf{P}_1 + c_2 \mathbf{P}_2)^*$ was made in [22, Theorem 3.4].

**Theorem 2.10.** Let $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{C}_{n,n}^{OP}$ such that $\mathbf{P}_1 \mathbf{P}_2 = \mathbf{P}_2 \mathbf{P}_1$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Then

1. $c_1 \mathbf{P}_1 + c_2 \mathbf{P}_2$ is nilpotent if and only if $c_1 + c_2 = 0$ and $\mathbf{P}_1 = \mathbf{P}_2$.

2. $c_1 \mathbf{P}_1 + c_2 \mathbf{P}_2$ is a hypergeneralized projector if and only if any of the following condition holds:
   a) $c_1, c_2 \in \sqrt[3]{1}$ and $\mathbf{P}_1 \mathbf{P}_2 = \mathbf{0}$.
   b) $c_2 \in \sqrt[3]{1}$, $c_1 + c_2 \in \{0\} \cup \sqrt[3]{1}$, and $\mathbf{P}_1 \mathbf{P}_2 = \mathbf{P}_1$. 

c) \( c_1 \in \sqrt{n}, c_1 + c_2 \in \{0\} \cup \sqrt{n}, \text{ and } P_1 P_2 = P_2 \).

\[ c_1 + c_2 \in \{0\} \cup \sqrt{n} \text{ and } P_1 = P_2. \]

Proof. As in the proof of Theorem 2.1, we can write \( P_1 = U(I_x \oplus I_y \oplus 0 \oplus 0)U^* \) and \( P_2 = U(I_x \oplus 0 \oplus I_z \oplus 0)U^* \) for some unitary matrix \( U \in \mathbb{C}_{n,n} \), where \( x, y, z \) is the rank of \( P_1 P_2, P_1, \) and \( P_2 \), respectively. In particular, we deduce that \( c_1 P_1 + c_2 P_2 \) is diagonalizable. Moreover, the following simple observation will be useful: a complex number \( c \) satisfies \( c^* = c \) if and only if \( c \in \{0\} \cup \sqrt{n} \).

Note that the sufficiency of the conditions revealed in the theorem follows by direct verifications and by Theorem 2.1, item (v).

(i) Assume that \( c_1 P_1 + c_2 P_2 \) is nilpotent. Since \( c_1 P_1 + c_2 P_2 \) is also diagonalizable, we get \( c_1 P_1 + c_2 P_2 = 0 \). Multiplying by \( P_1 \) and by \( P_2 \) leads respectively to \( P_1 P_2 = P_2 \) and \( P_1 P_2 = P_1 \), hence \( P_1 = P_2 \). Since \( c_1 P_1 + c_2 P_2 = 0 \) and \( P_1 \neq 0 \), we get \( c_1 + c_2 = 0 \).

(ii) Assume that \( c_1 P_1 + c_2 P_2 \) is a hypergeneralized projector. If \( y - x > 0 \), then \( c_1 \in \sqrt{n} \). Now, alternative \( z - x > 0 \) leads to \( c_2 \in \sqrt{n} \), hence \( c_1 + c_2 \notin \{0\} \cup \sqrt{n} \), which implies \( x = 0 \), hence case a) of the theorem has been obtained. Whereas alternative \( z - x = 0 \) leads, as is easy to see, to case c). Now, suppose \( y - x = 0 \). Since \( P_1 \neq 0 \), we have \( x > 0 \) and depending on \( z - x > 0 \) or \( z - x = 0 \), we get cases b) and c), respectively, of the theorem.

Theorem 2.4 allows us to deduce the following result. The first item was proved in another way in [20] and the second one corresponds to the Hermitian case (not mentioned) of the main result in [21].

**Theorem 2.11.** Let \( P_1, P_2 \in \mathbb{C}^{OP}_{n} \) such that \( P_1 P_2 \neq P_2 P_1 \) and \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \). Let us denote by \( \Pi \) the orthogonal projector onto \( N(P_1 P_2 - P_2 P_1) \). Then the following statements hold:

(i) \( c_1 P_1 + c_2 P_2 \) is not an idempotent matrix.
(ii) $c_1 P_1 + c_2 P_2$ is a tripotent matrix if and only if $\Theta(P_1, P_2) = \{\theta\}$ and
\[
c_1 = \pm \frac{1}{\sin \theta}, \quad c_2 = \mp \frac{1}{\sin \theta}, \quad P_1 \Pi = P_2 \Pi, \quad (P_1 - P_2)^2 (I_n - \Pi) = \sin^2 \theta (I_n - \Pi). \tag{18}
\]

(iii) $c_1 P_1 + c_2 P_2$ is an involutive matrix if and only if $\Theta(P_1, P_2) = \{\theta\}$, $\Pi = 0$, and
\[
c_1 = \pm \frac{1}{\sin \theta}, \quad c_2 = \mp \frac{1}{\sin \theta}, \quad (P_1 - P_2)^2 = \sin^2 \theta I_n. \tag{19}
\]

(iv) $c_1 P_1 + c_2 P_2$ is not a nilpotent matrix.

(v) $c_1 P_1 + c_2 P_2$ is not a generalized projector.

(vi) $c_1 P_1 + c_2 P_2$ is a hypergeneralized projector if and only $\Theta(P_1, P_2) = \{\theta\}$,
\[
P_1 \Pi = P_2 \Pi = 0, \quad (c_1 P_1 + c_2 P_2)^3 (I_n - \Pi) = I_n - \Pi, \quad \text{and}
\]
\[
c_1 + c_2 = 1 + \eta \quad \text{or} \quad \frac{c_1 + c_2}{c_1 c_2 \sin^2 \theta} = \frac{1 + \eta}{\eta} \quad \text{or} \quad \frac{c_1 + c_2}{c_1 c_2 \sin^2 \theta} = 1
\]
\[
\text{where } \eta = \exp(i \pi/3).
\]

Proof. We follow the notation used in Theorem 2.4. First of all, notice that in representation (11) the blocks corresponding to matrices $T_1$ and $T_2$ must appear, since otherwise $P_1 P_2 = P_2 P_1$.

(i) Let us suppose that $c_1 P_1 + c_2 P_2$ is idempotent. By (11), matrix $c_1 T_1 + c_2 T_2$ is idempotent and by item (iii) of Theorem 2.4, matrix $c_1 T_1 + c_2 T_2$ is nonsingular. Thus we get $c_1 T_1 + c_2 T_2 = I_{2p}$, which can not occur because $T_1 T_2 \neq T_2 T_1$.

(ii) Suppose that conditions (18) hold. Since
\[
c_1 P_1 + c_2 P_2 = \pm \frac{1}{\sin \theta} (P_1 - P_2) = \pm \frac{1}{\sin \theta} [(P_1 - P_2) \Pi + (P_1 - P_2)(I_n - \Pi)]
\]
\[
= \pm \frac{1}{\sin \theta} (P_1 - P_2)(I_n - \Pi),
\]
and recalling $P_i \Pi = \Pi P_i$ for $i = 1, 2$, we get
\[
(c_1 P_1 + c_2 P_2)^3 = \pm \frac{1}{\sin^3 \theta} (P_1 - P_2)^3 (I_n - \Pi)^3
\]
\[
= \pm \frac{1}{\sin^3 \theta} (P_1 - P_2)(P_1 - P_2)^2 (I_n - \Pi)
\]
\[
= \pm \frac{1}{\sin \theta} (P_1 - P_2)(I_n - \Pi),
\]
which proves the tripotency of $c_1P_1 + c_2P_2$.

Assume that $c_1P_1 + c_2P_2$ is tripotent. By (11), matrix $c_1T_1 + c_2T_2$ is tripotent and the nonsingularity of $c_1T_1 + c_2T_2$ implies $(c_1T_1 + c_2T_2)^2 = I_{2p}$. Hence, matrices $X_i$ defined in (9) satisfy $X_i^2 = I_2$, or in other words, matrices $X_i$ are diagonalizable (a square matrix is diagonalizable if and only if its minimal polynomial has distinct linear factors, see [19, Corollary 3.3.10]) and $\sigma(X_i) \subset \{1, -1\}$. If the spectrum of $X_i$ were a singleton, then $X_i = \lambda I_2$ for some $\lambda \in \{1, -1\}$, which can not happen in view of the upper-right entry of $X_i$ in (9). Thus, $\sigma(X_i) = \{1, -1\}$. Since $\det(X_i - \lambda I_2) = \lambda^2 - (c_1 + c_2)\lambda + c_1c_2\sin^2 \theta_i$, by Vieta’s formulas, we get $c_1 + c_2 = 0$ and $c_1c_2\sin^2 \theta_i = -1$. Thus

$$c_1 = \pm \frac{1}{\sin \theta_i}, \quad c_2 = \mp \frac{1}{\sin \theta_i}. \quad (20)$$

By observing that $\Theta(P_1, P_2) \subset [0, \pi/2]$, expression (20) permits to assure that the cardinal of $\Theta(P_1, P_2)$ is exactly one. From $(c_1T_1 + c_2T_2)^2 = I_{2p}$ and (20) we get

$$(T_1 - T_2)^2 = \sin^2 \theta I_{2p}. \quad (21)$$

Now, let us study the linear combination $c_1R_1 + c_2R_2$ occurring in (11). By definition, $R_1 = I \oplus I \oplus 0 \oplus 0$ and $R_2 = I \oplus 0 \oplus I \oplus 0$, wherein the symbol $I$ denotes identity matrices of various sizes, the corresponding blocks in $R_1$ and $R_2$ are of the same size, and some summands can be absent. From (20),

$$c_1R_1 + c_2R_2 = (c_1 + c_2)I \oplus c_1I \oplus c_2I \oplus 0 = 0 \oplus c_1I \oplus c_2I \oplus 0.$$

If the second or the third summand in the above representation appears, having in mind that $c_1R_1 + c_2R_2$ is tripotent, then $c_1 \in \{0, -1, 1\}$ or $c_2 \in \{0, -1, 1\}$; which is irreconcilable with (20). Thus $R_1 = R_2$. This, together with (21), finishes the proof.

(iii) It is evident that conditions (19) imply the involutiveness of $c_1P_1 + c_2P_2$. Now, let us suppose that $(c_1P_1 + c_2P_2)^2 = I_n$. It is evident that $c_1P_1 + c_2P_2$ is tripotent, and thus, by applying item (ii) of this theorem, it is enough to prove $\Pi = 0$. If $\Pi \neq 0$, then summand $c_1R_1 + c_2R_2$ appears in (11). Moreover, applying
again item (ii) of this theorem we get $\Pi P_1 = \Pi P_2$, or in other words, $R_1 = R_2$. Hence $c_1 R_1 + c_2 R_2 = (c_1 + c_2)R_1$. Since $R_1$ is idempotent and $c_1 R_1 + c_2 R_2$ is involutive, then $(c_1 + c_2)^2 R_1 = I_{n-2p}$. This latter equality is in contradiction with the values of $c_1$ and $c_2$ in (19).

(iv) If $c_1 P_1 + c_2 P_2$ were a nilpotent matrix, then $c_1 T_1 + c_2 T_2$ would be also nilpotent. In particular $c_1 T_1 + c_2 T_2$ would be a singular matrix. This last affirmation contradicts item (iii) of Theorem 2.4.

(v) If $c_1 P_1 + c_2 P_2$ were a generalized projector, then matrices $X_i$ would be also generalized projectors for $i = 1, \ldots, p$. By [23, Theorem 1] we get that $X_i$ are normal matrices (and thus, $c_1 = \alpha c_2$ for some $\alpha \in \mathbb{R}$ by Note 2) and $X_i^2 = X_i$, hence $X_i^3 = I_2$ because $X_i$ is nonsingular. Moreover, the minimal polynomial of $X_i$ has linear factors (since $X_i^3 = I_2$), hence matrix $X_i$ is diagonalizable and following as in the proof of item (ii) of this theorem, matrix $X_i$ has exactly two distinct eigenvalues, say $\lambda$ and $\mu$. The characteristic polynomial of $X_i$ is $\lambda^2 - (c_1 + c_2) \lambda + c_1 c_2 \sin^2 \theta_i$. By Vieta’s formulas, $\lambda + \mu = c_1 + c_2$ and $\lambda \mu = c_1 c_2 \sin^2 \theta_i$. Using $c_1 = \alpha c_2$ yields $\lambda \mu / (\lambda + \mu)^2 = \alpha \sin^2 \theta_i / (\alpha + 1)^2$. Taking into account that $\lambda \neq \mu$ and $\lambda, \mu \in \mathbb{R}$, we have $\lambda \mu = (\lambda + \mu)^2$, and thus, $1 = \alpha \sin^2 \theta_i / (\alpha + 1)^2$. But the quadratic equation $\alpha^2 + (2 - \sin^2 \theta_i) \alpha + 1 = 0$ has no real solutions since its discriminant is

$$(2 - \sin^2 \theta_i)^2 - 4 = 4 + \sin^4 \theta_i - 4 \sin^2 \theta_i - 4 = \sin^2 \theta_i (\sin^2 \theta_i - 4) < 0.$$  

(vi) Let us see that $P_1 \Pi = P_2 \Pi = 0$ and $(c_1 P_1 + c_2 P_2)^3(I_n - \Pi) = I_n - \Pi$ imply that $c_1 P_1 + c_2 P_2$ is a hypergeneralized projector: We decompose $c_1 P_1 + c_2 P_2$ as in (11). Condition $P_1 \Pi = P_2 \Pi = 0$ is equivalent to $R_1 = R_2 = 0$ and condition $(c_1 P_1 + c_2 P_2)^3(I_n - \Pi) = I_n - \Pi$ is equivalent to $(c_1 T_1 + c_2 T_2)^3 = I_{2p}$. Now it is evident

$$(c_1 P_1 + c_2 P_2)^2$$

$$= U((c_1 T_1 + c_2 T_2)^2 \oplus 0) U^* = U((c_1 T_1 + c_2 T_2)^{-1} \oplus 0) U^* = (c_1 P_1 + c_2 P_2)^\dagger,$$

which proves that $c_1 P_1 + c_2 P_2$ is a hypergeneralized projector.

Assume that $c_1 P_1 + c_2 P_2$ is a hypergeneralized projector. Hence $c_1 R_1 + c_2 R_2$ and $X_i$ for $i = 1, \ldots, p$ are also hypergeneralized projectors. The nonsingularity of
$X_i$ entails $X^3 = I_2$ and as in the proof of latter item (v), $X_i$ has exactly two distinct eigenvalues, say $\lambda_i, \mu_i \in \sqrt[3]{1}$, which satisfy

$$\lambda_i + \mu_i = c_1 + c_2 \quad \text{and} \quad \lambda_i \mu_i = c_1 c_2 \sin^2 \theta_i. \quad (22)$$

Since $\lambda_i \neq \mu_i$ and $\lambda_i^3 = \mu_i^3$, it can be seen that $\lambda_i \mu_i = (\lambda_i + \mu_i)^2$, hence $c_1 c_2 \sin^2 \theta_i = (c_1 + c_2)^2$, which implies that $\Theta(P_1, P_2)$ is a singleton. Let $\theta$ be the unique element of $\Theta(P_1, P_2)$.

Now, we are going to study the block corresponding to $c_1 R_1 + c_2 R_2$. By definition, $R_1 = I \oplus I \oplus 0 \oplus 0$ and $R_2 = I \oplus 0 \oplus I \oplus 0$, wherein the symbol $I$ denotes identity matrices of various sizes, the corresponding blocks in $R_1$ and $R_2$ are of the same size, and some summands can be absent. Therefore

$$c_1 R_1 + c_2 R_2 = (c_1 + c_2) I \oplus c_1 I \oplus c_2 I \oplus 0. \quad (23)$$

Since $P_1 P_2 \neq P_2 P_1$, there exists $i \in \{1, \ldots, p\}$ such that $\theta_i \in \Theta(P_1, P_2)$. Having in mind that for $c \in \mathbb{C}$ one has $c^\dagger = c^2 \Leftrightarrow c \in \{0\} \cup \sqrt[3]{1}$, we can deduce that the first summand in (23) must be absent because from (22), we have $c_1 + c_2 = \lambda_i + \mu_i \notin \{0\} \cup \sqrt[3]{1}$. If the second summand in (23) were present, then $c_1 \in \sqrt[3]{1}$, which can not happen: let $\{\xi_i\} = \sqrt[3]{1} \setminus \{\lambda_i, \mu_i\}$. If $c_1 \in \{\lambda_i, \mu_i\}$, the first equation of (22) would imply $c_2 \in \sqrt[3]{1}$ and now,

$$1 = |\lambda_i \mu_i| = |c_1 c_2 \sin^2 \theta| = |c_1||c_2| \sin^2 \theta = \sin^2 \theta < 1,$$

which is unfeasible. If $c_1 = \xi_i$, from the first relation of (22), we get $\xi_i + c_2 = \lambda_i + \mu_i$, hence $2\xi_i + c_2 = \lambda_i + \mu_i + \xi_i = 0$, and thus $c_2 = -2\xi_i$. Now, let us use the second relation of (22):

$$\sin^2 \theta = \frac{\lambda_i \mu_i}{c_1 c_2} = \frac{\lambda_i \mu_i}{-2\xi_i^2} = \frac{\lambda_i \mu_i}{-2(\lambda_i - \mu_i)^2} = \frac{\lambda_i \mu_i}{-2(\lambda_i + \mu_i)^2} = -\frac{1}{2},$$

which is clearly a contradiction. In a similar way, the third summand in (23) must be absent, and therefore, $R_1 = R_2 = 0$, or in other words, $P_1 \Pi = P_1 \Pi = 0$. Thus the necessity has been proved. \qed
Note 3. In the spirit of Note 1, we can reformulate some conditions of this latter theorem without mentioning the orthogonal projector $\Pi$. For example, condition $\Pi P_1 = \Pi P_2$ is equivalent to say $N(P_1 P_2 - P_2 P_1) \subseteq N(P_1 - P_2)$ or condition $\Pi = 0$ is equivalent to the nonsingularity of $P_1 P_2 - P_2 P_1$.

Note 4. It is instructive to see how the number of situations when $c_1 P_1 + c_2 P_2$ is tripotent provided $P_1$ and $P_2$ are idempotent presented in [21], decreases when $P_1$ and $P_2$ are also Hermitian.

Note 5. It can be verified that if $A, B, C \in \mathbb{C}_{n,n}$ satisfy $AB = BA = 0$ and $(I_n - A)C = 0$, then $BC = 0$. We can apply this observation in order to obtain weaker conditions than the obtained in the previous theorem, but easier to check: Conditions (18) imply $(P_1 - P_2)[(P_1 - P_2)^2 \sin^2 \theta I] = 0$, whereas matrix conditions in item (vi) imply $P_1[(c_1 P_1 + c_2 P_2)^3 - I] = 0$ and $P_2[(c_1 P_1 + c_2 P_2)^3 - I] = 0$.

References


