Some results on partial ordering and reverse order law of elements of $C^*$-algebras

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Abstract

In this paper we establish some results relating star, left-star, right-star, minus ordering and the reverse order law under certain conditions on Moore-Penrose invertible elements of $C^*$-algebras.

AMS classification: 46L05, 15A09,

Key words: partial ordering; Moore-Penrose inverse; reverse order law; $C^*$-algebras.

1 Introduction

Let $\mathcal{A}$ be a $C^*$-algebra with unit 1. An element $p \in \mathcal{A}$ is said to be a projection if $p = p^2 = p^*$. Let $a \in \mathcal{A}$, consider the equations:

\begin{align*}
(1) \ aba &= a, \\
(2) \ bab &= b, \\
(3) \ (ab)^* &= ab, \\
(4) \ (ba)^* &= ba.
\end{align*}

For any $a \in \mathcal{A}$, let $a\{i, j, \ldots, k\}$ denote the set of elements $b \in \mathcal{A}$ which satisfy equations $(i), (j), \ldots, (k)$ from among equations (1)-(4). In this situation, the element $b$ will be called a $\{i, j, \ldots, k\}$-inverse of $a$. It is well known that $a\{1, 2, 3, 4\}$ is or empty or a singleton and when $a\{1, 2, 3, 4\}$ is a singleton, its unique element is called the Moore-Penrose inverse of $a$, denoted by $a^\dagger$. The subset of $\mathcal{A}$ consisting of elements of $\mathcal{A}$ that have a Moore-Penrose inverse will be denoted by $\mathcal{A}^\dagger$. For an arbitrary $C^*$-algebra $\mathcal{A}$, it may happen that $\mathcal{A} \neq \mathcal{A}^\dagger$.

In [15] it was proved that if $a\{1\} \neq \emptyset$, then $a \in \mathcal{A}^\dagger$ (see also [13]).

The following formulae are well known in the theory of generalized inverses in $C^*$-algebras and they will be useful in the sequel.

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Lemma 1.1. Let $\mathcal{A}$ be a $C^*$-algebra. For any $a \in \mathcal{A}^\dagger$, the following statements are satisfied:

(i) $a^\dagger \in \mathcal{A}^\dagger$ and $(a^\dagger)^\dagger = a$.

(ii) $a^* \in \mathcal{A}^\dagger$ and $(a^*)^\dagger = (a^\dagger)^*$.

(iii) $a^\dagger = a^\dagger (a^\dagger)^* a^* = a^*(a^\dagger)^* a^\dagger$.

(iv) $a^* = a^\dagger a^* = a^* a a^\dagger$.

The set of complex $n \times n$ matrices can be considered a $C^*$-algebra, but let us remark that any complex matrix has a Moore-Penrose inverse. Recall that a matrix $A$ is called EP when $AA^\dagger = A^\dagger A$ and there are many characterizations of EP matrices (see [5, 8]). Recently, many researchers pay their attention to EP elements in $C^*$-algebras and rings and present several equivalent characterizations of elements of a $C^*$-algebra that commute with their Moore-Penrose inverse (see [6, 10, 12]). In this paper, for a $C^*$-algebra $\mathcal{A}$, we will denote $\mathcal{A}^{EP} = \{ a \in \mathcal{A}^\dagger : aa^\dagger = a^\dagger a \}$.

For future use we need the following Theorem 1.1 (see [6, Th 2.1] and [13, Th. 3.1]) and some notation. For any $a \in \mathcal{A}$ we define the nullspace ideals (also called the two annihilators of $a$)

$\circ a = \{ x \in \mathcal{A} : ax = 0 \}$,

$\circ a = \{ x \in \mathcal{A} : xa = 0 \}$.

It is simple to prove from items (iii) and (iv) of Lemma 1.1 that $(a^*)^\circ = (a^\dagger)^\circ$ and $\circ (a^*) = \circ (a^\dagger)$ hold for any $a \in \mathcal{A}^\dagger$.

Theorem 1.1. Let $\mathcal{A}$ be a $C^*$-algebra with unit 1 and $a \in \mathcal{A}$. Then the following conditions are equivalent:

(i) There exists a unique projection $p$ such that $a + p \in \mathcal{A}^{-1}$ and $ap = pa = 0$.

(ii) $a \in \mathcal{A}^{EP}$.

(iii) $a^\circ = (a^*)^\circ$.

(iv) $\circ a = \circ (a^*)$.

Following [12], we denote by $a^\pi$ the unique projection satisfying condition (i) of Theorem 1.1 for a given $a \in \mathcal{A}^{EP}$. It is proved that

$a^\pi = 1 - aa^\dagger$ and $a^\dagger = (a + a^\pi)^{-1} - a^\pi$.

The projector $a^\pi$ will be named the spectral idempotent of $a$ corresponding to 0.

Inspired by matrix theory, for $a \in \mathcal{A}^\dagger$, we will define two projectors $a^\pi_l$ and $a^\pi_r$ by

$a^\pi_l = 1 - a^\dagger a$, $a^\pi_r = 1 - aa^\dagger$,

respectively. Obviously, when $a \in \mathcal{A}^{EP}$, then $a^\pi_l = a^\pi_r$.

Matrix partial orderings have been an area of intense research in the past few years (see [1, 2, 3, 4]). Analogously to the definition introduced by Drazin [11], we define the star ordering in an arbitrary $C^*$-algebra by

$a \leq b \iff a^* a = a^* b$ and $aa^* = ba^*$.
Let us remark that if \( a \in \mathcal{A}^\dagger \), then the conditions \( a^*a = a^*b \) and \( aa^* = ba^* \) are equivalent to \( a^*a = a^*b \) and \( aa^* = ba^* \), respectively since \((a^*)^\circ = (a^\dagger)^\circ \) and \((a^*)^\circ = (a^\dagger)^\circ \).

Inspired in a paper of Baksalary and Mitra [1], we define left-star and right-star partial ordering of Moore-Penrose invertible elements \( a, b \) of a \( C^* \)-algebra by

\[
a \ast \leq b \iff a^*a = a^*b \text{ and } b^\pi_\gamma a = 0,
\]

and

\[
a \leq b \iff a^*a = ba^* \text{ and } ab^\pi_\gamma = 0,
\]

respectively. It can easily be proved that when \( A \) and \( B \) are \( n \times n \) complex matrices, then \( B^\gamma_\pi A = 0 \) if and only if \( \mathcal{R}(A) \subset \mathcal{R}(B) \); and \( AB^\gamma_\pi = 0 \) if and only if \( \mathcal{R}(A^*) \subset \mathcal{R}(B^*) \), where \( \mathcal{R}(\cdot) \) denotes the range space. These inclusions are part of the original definition of the left-star and right-star partial ordering in the set composed of \( n \times n \) complex matrices.

Furthermore, we will consider the minus ordering defined in [16]. An extension to \( \mathcal{A}^\dagger \) of an equivalent form of this ordering (see [18] or [9]) is the following:

\[
a \preceq b \iff ab^\dagger b = a, \quad bb^\dagger a = a, \quad ab^\dagger a = a.
\]

The purpose of this paper is to establish some results on the star, left-star, right-star, and minus orderings of two Moore-Penrose invertible elements of \( C^* \)-algebras, when one of them commutes with its Moore-Penrose inverse.

The reverse order law is one of the most important properties of the Moore-Penrose inverse that have been studied, that is under what condition the equation \((ab)^\dagger = b^\dagger a^\dagger \) holds for \( a, b \in \mathcal{A}^\dagger \). In [14], T.N.E. Greville gave equivalent conditions on a pair of square complex matrices \( A \) and \( B \) for \((AB)^\dagger = B^\dagger A^\dagger \) holds. However, it is worth noticing that the proofs work in the more general context of \( C^* \)-algebras. An algebraic proof of the reverse order law for the Moore-Penrose inverse (in a ring with involution) is given in [17]. The interested reader can also consult [7, 19].

### 2 Star ordering and the reverse order law

Next, for two Moore-Penrose invertible elements of a \( C^* \)-algebra, say \( a \) and \( b \), we study the relation \( a \preceq b \) and the reverse order law for the products \( ab \) and \( ba \) when \( a \) or \( b \) commute with its Moore-Penrose inverse.

**Theorem 2.1.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra and \( a, b \) elements of \( \mathcal{A} \) that have a Moore-Penrose inverse. Assume that \( a \in \mathcal{A}^{EP} \). The following affirmations are equivalent:

(i) \( a \preceq b \).

(ii) \( ab = ba = a^2 \).

(iii) \( ab \) has a Moore-Penrose inverse, \( (ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger = a \).

(iv) \( ab \) has a Moore-Penrose inverse, \( (ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger = ba^\dagger a = a \).
Proof. (i) $\Rightarrow$ (ii): From $a^*a = a^*b$ and $aa^* = ba^*$, we have
\[ a^*(a - b) = (a - b)a^* = 0. \]
Since $a \in \mathcal{A}EP \iff a^* \in \mathcal{A}EP$ and $a^\pi = (a^*)^\pi$, then by item (i) of [6, Theorem 3.6], we have
\[ a^\pi(a - b) = a - b = (a - b)a^\pi. \]
Hence, $a(a - b) = aa^\pi(a - b) = 0$, i.e., $a^2 = ab$ and $(b - a)a = (b - a)a^\pi a = 0$, i.e., $ba = a^2$.

(ii) $\Rightarrow$ (iii): It is easy to see that $a \in \mathcal{A}EP$ implies $a^2 \in \mathcal{A}$ and $(a^2)^\dagger = (a^\dagger)^2$. Since $ab = a^2$, then $ab$ has a Moore-Penrose inverse. It is easy to check that $aa^\dagger b = a^\dagger ab = a^\dagger a^2 = aa^\dagger a = a$. Next we will prove that $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$. By using $ab = ba = a^2$ we have
\[ a(b - a) = (b - a)a = 0. \]
By item (i) of [6, Theorem 3.6], we have
\[ a^\pi b = a^\pi(b - a) = b - a = (b - a)a^\pi = ba^\pi. \]
Thus, we obtain
\[ b = a + a^\pi b = a + ba^\pi. \tag{1} \]
From (1) and [6, Lemma 3.5] we get
\[ a^\pi b^\dagger = b^\dagger a^\pi. \tag{2} \]
Now, by doing a little algebra we obtain
\[ a^\pi ba^\pi b^\dagger a^\pi b = a^\pi b \quad \text{and} \quad a^\pi b^\dagger a^\pi ba^\pi b^\dagger = a^\pi b^\dagger. \]
Moreover, recall that $a^\pi$ is a projection and commutes with $b$ and $b^\dagger$, hence $(a^\pi ba^\pi b^\dagger)^* = (a^\pi b^\dagger)^* = (bb^\dagger)^*a^\pi = bb^\dagger a^\pi = a^\pi ba^\pi b^\dagger$, and thus, $a^\pi ba^\pi b^\dagger$ is self-adjoint. In the same way, we prove that $a^\pi b^\dagger a^\pi b$ is self-adjoint. We have proved
\[ (a^\pi b)^\dagger = a^\pi b^\dagger, \tag{3} \]
in particular $a^\pi b \in \mathcal{A}$. Since $aa^\pi b = a^\pi ba = 0$ by item (iv) of [6, Theorem 3.6], we get that $a + a^\pi b$ is Moore-Penrose invertible and $(a + a^\pi b)^\dagger = a^\dagger + (a^\pi b)^\dagger$. Using this last identity, (1), (3), and (2) we obtain
\[ b^\dagger = (a + a^\pi b)^\dagger = a^\dagger + (a^\pi b)^\dagger = a^\dagger + a^\pi b^\dagger = a^\dagger + b^\dagger a^\pi \]
Therefore, $b^\dagger - a^\dagger = a^\pi b^\dagger = b^\dagger a^\pi$ and thus
\[ a^\dagger(b^\dagger - a^\dagger) = a^\dagger a^\pi b^\dagger = 0, \quad (b^\dagger - a^\dagger)a^\dagger = b^\dagger a^\pi a^\dagger = 0. \]
Hence, $a^\dagger b^\dagger = b^\dagger a^\dagger = (a^\dagger)^2 = (a^2)^\dagger = (ab)^\dagger$.

(iii) $\Rightarrow$ (i): Noting that $a \in \mathcal{A}EP \iff a^\dagger \in \mathcal{A}EP$ and $(a^\dagger)^\pi = a^\pi$, since $a^\dagger b^\dagger = b^\dagger a^\dagger$, then by [6, Corollary 3.3] we get $b^\dagger a^\pi = a^\pi b^\dagger$. Further, by [6, Lemma 3.5], we also have $ba^\pi = a^\pi b$. By $aa^\dagger b = a$, we have $ba^\pi = a^\pi b = (1 - aa^\dagger)b = b - aa^\dagger b = b - a$. Now, $a^\pi(b - a) = a^\pi a^\pi b = (a^\pi a)^\pi b = 0$, i.e., $a^\pi b = a^\pi a$. In a similar way, from the equality $ba^\pi = b - a$, we get $ba^\pi = aa^\pi$.

(ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i): This has the same proof as (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i), and thus, the theorem is demonstrated. □
Recall that, in addition to the standard properties of involution \( x \mapsto x^* \) we have \( ||x^*|| = ||x|| \) for all \( x \in \mathcal{A} \), and the B*-condition

\[
||x^*x|| = ||x||^2.
\]

**Theorem 2.2.** Let \( \mathcal{A} \) be a unital C*-algebra, \( a \in \mathcal{A} \), and \( b \in \mathcal{A}^{EP} \), then

(i) If \( a^* \leq b \), then \( ab^* = b^*a = 0 \).

(ii) Let \( a \in \mathcal{A}^\dagger \), if \( ab^* = b^*a \), then \( a^*b = b^*a^*, \ a^*b^* = b^*a^*, \ a^\dagger a \in (ab)^\{1,2,3\}, \ a^\dagger b \in (ba)^\{1,2,4\} \).

**Proof.** (i) Since \( a^*b = a^*bb^* = 0 \) we get \( ||ab^*||^2 = ||(ab^*)^*(ab^*)|| = ||b^*a^*ab^*|| = 0 \), and therefore, \( ab^* = 0 \). On the other hand, we have

\[
||b^*a||^2 = ||(b^*a)^*||^2 = ||a^*b^*||^2 = ||(a^*b^*)^*(a^*b^*)|| = ||b^*a^*b^*|| = ||b^*ba^*b^*|| = 0,
\]

which proves \( b^*a = 0 \).

(ii) If \( ab^* = b^*a \), from item (i) of \([6, \text{Lemma 3.5}] \), we have \( a^\dagger b^* = b^*a^\dagger \), and thus, \( aa^\dagger b^* = ab^*a^\dagger = a^\dagger ba^* \), i.e., \( a^\dagger b^* = b^*a^* \). Similarly, from \( a^\dagger ab^* = a^\dagger b^*a = b^*a^\dagger a \), we have \( a^\dagger b^* = b^*a^* \).

Next, we shall prove from \( ab^* = b^*a \) and \( a^\dagger b^* = b^*a^\dagger \) that \( b^\dagger a^\dagger \in (ab)^\{1,2,3\} \).

\[
abb^\dagger a^\dagger ab = bb^\dagger aa^\dagger ab = bb^\dagger b = ab, \ 
\ 
bb^\dagger a^\dagger abb^\dagger a^\dagger = bb^\dagger bba^\dagger = bb^\dagger a^\dagger a = b^\dagger a^\dagger,
\]

and

\[
(aabb^\dagger a^\dagger)^* = (bb^\dagger a^\dagger)^* = (aa^\dagger)^* = bb^\dagger = bb^\dagger a^\dagger = abb^\dagger a^\dagger.
\]

The proof of \( a^\dagger b^\dagger \in (ba)^\{1,2,4\} \) is similar and we will not give it. \( \square \)

**Theorem 2.3.** Let \( \mathcal{A} \) be a unital C*-algebra and \( a, b \in \mathcal{A}^\dagger \). If \( b \in \mathcal{A}^{EP} \) and \( ab^* = b^*a \), then

(i) \( (ab)^\dagger = b^\dagger a^\dagger \) if and only if \( bb^*a^\dagger = a^\dagger bb^* \).

(ii) \( (ba)^\dagger = a^\dagger b^\dagger \) if and only if \( b^*ba^* = a^\dagger b^*b \).

**Proof.** We shall prove the first equivalence, and we will not give the proof of the other because its proof is similar. By Theorem 2.2, we have that \( (ab)^\dagger = b^\dagger a^\dagger \) if and only if \( b^\dagger a^\dagger ab \) is self-adjoint. In order to prove \( (b^\dagger a^\dagger ab)^* = b^\dagger a^\dagger ab \), we will use a consequence of item (ii) of Theorem 2.2, specifically, \( a^\dagger b^* = b^*a^\dagger \). Since \( b^\dagger = (b + b^*)^{-1} - b^* \), we have

\[
b^\dagger a^\dagger ab \text{ is self-adjoint } \iff b^\dagger a^\dagger ab = (b^\dagger a^\dagger ab)^* \iff b^\dagger a^\dagger b = (b^\dagger a^\dagger b)^* \iff b^\dagger a^\dagger b = b^*a^\dagger (b^\dagger)^* \iff [(b + b^*)^{-1} - b^*]a^\dagger b = b^*a^\dagger [(b + b^*)^{-1} - b^*] \iff (b + b^*)^{-1}a^\dagger b = b^*a^\dagger (b + b^*)^{-1} \iff a^\dagger b(b + b^*) = (b + b^*)b^*a^\dagger \iff a^\dagger b(b + b^*) = (b + b^*)b^*a^\dagger \iff a^\dagger bb^* = bb^*a^\dagger.
\]

\( \square \)
3 The left and right star orderings and the reverse order law

In this section we study the relation between \( a * \leq b \) and \( a \leq * b \) and reverse law of \( ab \) and \( ba \) when \( a \) and \( b \) are elements in a \( C^* \)-algebra that have a Moore-Penrose inverse.

**Lemma 3.1.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra. Let \( a \in \mathcal{A}^1 \) and assume that there exists a projection \( p \) such that \( a = pa \), then \( a^\dagger = a^\dagger p \).

**Proof.** It is evident \( a a^\dagger p a = a, a^\dagger p a a^\dagger p = a^\dagger p \), and \( a^\dagger p a = a^\dagger a \) is self-adjoint. Since \( (a a^\dagger)^* = p^* (a a^\dagger)^* = p a a^\dagger = a a^\dagger \) is also self-adjoint, we obtain \( a^\dagger = a^\dagger p \).

The following observation will be useful in the sequel: Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( a \in \mathcal{A}^1 \), \( b \in \mathcal{A} \).

\[
\begin{align*}
  a^* b &= a^* a \iff a^\dagger b = a^\dagger a. 
\end{align*}
\]

This equivalence follows from \( (a^*)^\circ = (a^\dagger)^\circ \).

**Theorem 3.1.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra. Assume that \( a, b \in \mathcal{A}^\dagger \) with \( a \in \mathcal{A}^\text{EP} \). If \( a * \leq b \), then

1. \( ab = a^2 \).
2. \( a^\dagger b^\dagger = (ab)^\dagger \).
3. \( b^\dagger a^\dagger \in (ab)^\{1, 2, 3\} \).
4. \( a^\dagger b^\dagger \in (ba)^\{1, 2, 4\} \).

**Proof.** (i): By using Theorem 1.1 we have \( a^\circ = (a^*)^\circ \). Since \( a^* a = a^* b \), then \( a = (a^*)^\circ = a^\circ \), so \( a(a - b) = 0 \), i.e., \( ab = a^2 \).

(ii): Observe that \( (a^2)^\dagger = (a^\dagger)^2 \) because \( a \in \mathcal{A}^\text{EP} \), and from item (i) of this theorem, it only remains to prove that \( a^\dagger b^\dagger = (a^\dagger)^2 \). Since \( b^\dagger a = 0 \), or equivalently,

\[
bb^\dagger a = a, 
\]

then by Lemma 3.1, we get

\[
a^\dagger = a^\dagger bb^\dagger
\]

and from (4) we have

\[
a^\dagger = a^\dagger ab^\dagger.
\]

Then \( a a^\dagger (b^\dagger - a^\dagger) = a a^\dagger b^\dagger = b^\dagger - a a^\dagger b^\dagger = b^\dagger - a^\dagger \), which implies that \( a a^\dagger (b^\dagger - a^\dagger) = 0 \). Now, premultiplying \( a a^\dagger (b^\dagger - a^\dagger) = 0 \) by \( a^\dagger \), we get \( a^\dagger (b^\dagger - a^\dagger) = 0 \), i.e., \( a^\dagger b^\dagger = (a^\dagger)^2 \).

(iii) We shall prove this item by the definition of \( (ab)^\{1, 2, 3\} \): Recall that one hypothesis is \( a a^\dagger = a^\dagger a \). By using (5)

\[
ab b^\dagger a^\dagger ab = ab b^\dagger a^\dagger ab = a a^\dagger b = ab.
\]

Now we use (6)

\[
b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger aa^\dagger bb^\dagger a^\dagger = b^\dagger aa^\dagger a^\dagger = b^\dagger a^\dagger,
\]

and finally, from (5)

\[
ab b^\dagger a^\dagger = ab b^\dagger a^\dagger = ab b^\dagger a^\dagger = a a^\dagger a^\dagger = a a^\dagger = a a^\dagger \text{ is self-adjoint.}
\]

(iv) The proof is similar as in (iii), and we will not give it. 

\[ 6 \]
Next result characterizes the reverse law for the product $ab$ when $a$ commutes with its Moore-Penrose inverse and $a \ast \leq b$. It is remarkable that one of these equivalent conditions is $a \ast \leq b$.

**Theorem 3.2.** Let $\mathcal{A}$ be a unital $C^*$-algebra. Assume that $a, b \in \mathcal{A}^\dagger$ with $a \in \mathcal{A}^{EP}$. If $a \ast \leq b$, then the following affirmations are equivalent:

(i) $b^\dagger a^\dagger = (ab)^\dagger$.

(ii) $ab = ba$.

(iii) $a \ast \leq b$.

(iv) $a^\dagger b^\dagger = (ba)^\dagger$.

**Proof.** By item (i) of Theorem 3.1, if $a \in \mathcal{A}^{EP}$ and $a \ast \leq b$, then $ab = a^2$. Also recall $(a^2)^\dagger = (a^\dagger)^2$ because $aa^\dagger = a^\dagger a$.

(i) $\Rightarrow$ (ii): The hypothesis $b^\dagger a^\dagger = (ab)^\dagger$ implies $0 = (b^\dagger - a^\dagger)a^\dagger = (b^\dagger - a^\dagger)[(a + a^\ast)^{-1} - a^\ast]$, then $(b^\dagger - a^\dagger)(a + a^\ast)^{-1} = b^\dagger a^\ast$ and thus, $b^\dagger - a^\dagger = b^\dagger a^\ast (a + a^\ast) = b^\dagger a^\ast$. Now, we have

$$a^\dagger = b^\dagger (1 - a^\ast) = b^\dagger aa^\dagger$$

Postmultiplying the equality $(8)$ by $a^2$, we get $a = b^\dagger a^2$, premultiplying by $b$ and using $bb^\dagger a = a$ (obtained in (5)) we get $ba = bb^\dagger a^2 = a^2 = ab$.

(ii) $\Rightarrow$ (iii): By the definitions of the different orderings involved in this implication, it is enough to prove $aa^\ast = ba^\ast$. For the proof of $aa^\ast = ba^\ast$, we will use item (iv) of Theorem 1.1:

$$a^2 = ab = ba \Rightarrow (a - b)a = 0 \Rightarrow a - b \in \circ a = \circ (a^\ast) \Rightarrow (a - b)a^\ast = 0 \Rightarrow aa^\ast = ba^\ast.$$  

(iii) $\Rightarrow$ (iv): By items (iii) and (iv) of Theorem 1.1 we get

$$a \ast \leq b \iff \{ a^\ast a = a^\ast b \} \iff \{ a - b \in \circ a = \circ (a^\ast) \} \iff \{ a - b \in \circ a \iff a^2 = ab = ba. $$

By using item (ii) of Theorem 3.1 we have $(ba)^\dagger = (ab)^\dagger = a^\dagger b^\dagger$.

(iv) $\Rightarrow$ (ii): From item (ii) of Theorem 3.1 and hypothesis we have $(ba)^\dagger = a^\dagger b^\dagger = (ab)^\dagger$. Now, the conclusion follows from item (i) of Lemma 1.1.

(ii) $\Rightarrow$ (i): By item (iii) of Theorem 3.1, it is enough to prove that $b^\dagger a^\dagger ab$ is self-adjoint. By [6, Corollary 3.3] and [6, Lemma 3.5] we get $a^\ast b^\dagger = b^\dagger a^\ast$. Moreover, we will need $a^\dagger b = a^\dagger a$ (obtained in the observation given in (4)), and the relation (7). Thus

$$b^\dagger a^\dagger ab = b^\dagger aa^\dagger b = b^\dagger aa^\dagger a = b^\dagger a = (1 - a^\ast)b^\dagger a = a^\dagger ab^\dagger a = a^\dagger a,$$

which proves that $b^\dagger a^\dagger ab$ is self-adjoint. \hfill \Box

**Theorem 3.3.** Let $\mathcal{A}$ be a unital $C^*$-algebra. Assume that $a, b \in \mathcal{A}^\dagger$ with $b \in \mathcal{A}^{EP}$. If $a \ast \leq b$, then

(i) $a^\dagger b^\dagger \in (ba)\{1, 2, 4\}$.

(ii) $b^\dagger a^\dagger \in (ab)\{1, 2, 3\}$.
a b† = (ba)† if and only if b* b commutes with aa†. Moreover, b† a† = (ab)† if and only if bb* commutes with a† a.

Proof. (i): Note that b ∈ 𝒜EP implies b† a = b* a = 0, i.e., bb† a = b† b a = a. Now,

ba a b a = b a b a = b a b a = a a b a = b a b a,

a† b† b a = a a b a = a a b a = b a b a,

and thus

∥ab∥2 = ∥b* a* ab∥ = ∥b* a* b∥ = 0,

which, obviously implies ab = 0, or equivalently, a = ab b†.

(ii): We have

a b a b a = a b a b a = b a a b a = b a a b a = b a a b a = b a a b a = a a b a = b a b a

a† b† a∗ = a a b a = b a a b a = b a a b a = b a a b a = a a b a = b a b a

(iii): It is a trivial consequence of Theorem 2.3 since b* a = ab* = 0. Having in mind that a * ≤ b ⇐⇒ a* ≤ b*, we can obtain similar results for the right-star ordering.

Theorem 3.4. Let 𝒜 be a unital C*-algebra. Assume that a, b ∈ 𝒜† with a ∈ 𝒜EP. If a ≤∗ b, then

(i) ba = a².

(ii) b† a† = (ba)†.

(iii) a† b† ∈ (ba)†{1, 2, 4}.

(iv) b† a† ∈ (ab)†{1, 2, 3}.

Theorem 3.5. Let 𝒜 be a unital C*-algebra. Assume that a, b ∈ 𝒜† with a ∈ 𝒜EP. If a ≤∗ b, then the following affirmations are equivalent:

(i) a† b† = (ba)†.

(ii) ab = ba.

(iii) a ≤ b.

(iv) b† a† = (ab)†.

Theorem 3.6. Let 𝒜 be a unital C*-algebra. Assume that a, b ∈ 𝒜† with b ∈ 𝒜EP. If a ≤∗ b, then

(i) a† b† ∈ (ba)†{1, 2, 4}.

(ii) b† a† ∈ (ab)†{1, 2, 3}.

(iii) a† b† = (ba)† if and only if b* b commutes with aa†. Moreover, b† a† = (ab)† if and only if bb* commutes with a† a.
4 The minus ordering and the reverse order law

As we made in the previous sections, we link the minus ordering with the reverse law.

Firstly, let us remark that if \( \mathscr{A} \) is a unital \( C^*-\)algebra, and \( a \in \mathscr{A} \), \( b \in \mathscr{A}^\dagger \) satisfy \( ab^\dagger a = a \), then [15, Theorem 6] assures that \( a \in \mathscr{A}^\dagger \).

**Theorem 4.1.** Let \( \mathscr{A}^\dagger \) be a unital \( C^*-\)algebra and \( a \in \mathscr{A} \), \( b \in \mathscr{A}^\dagger \) satisfy \( \overline{a} \leq b \). Then

(i) \( a^\dagger b^\dagger \in (ba)\{1, 2, 4\} \).

(ii) If \( b \in \mathscr{A}^{EP} \), then \( b^\dagger a^\dagger \in (ab)\{1, 2, 3\} \).

**Proof.** (i): The equalities \( ba a^\dagger b^\dagger = ba \), \( a^\dagger b^\dagger b a a^\dagger = a^\dagger b^\dagger \), and \( a^\dagger b^\dagger b a = a^\dagger a \) follow directly from \( b^\dagger b a = a \).

(ii): The equalities \( ab b^\dagger a^\dagger ab = ab \), \( b^\dagger a^\dagger ab b^\dagger a^\dagger = b^\dagger a^\dagger \), and \( ab b^\dagger a^\dagger = a a^\dagger \) follow from \( ab^\dagger b = a \) and \( b^\dagger b = b^\dagger b \). \( \square \)

**References**


