Some results on partial ordering and reverse order law of elements of C^*-algebras

Xiaoji Liu[a*] Julio Benítez[b†] Jin Zhong[c‡]

[a] College of Mathematics and Computer Science,
Guangxi University for Nationalities, Nanning 530006, China

[b] Departamento de Matemática Aplicada, Instituto de Matemática Multidisciplinar,
Universidad Politécnica de Valencia, Camino de Vera s/n, 46022, Valencia, Spain.

[c] Department of Mathematics,
East China Normal University, Shanghai, 200062, China

Abstract

In this paper we establish some results relating star, left-star, right-star, minus ordering and the reverse order law under certain conditions on Moore-Penrose invertible elements of C^*-algebras.

AMS classification: 46L05, 15A09,

Key words: partial ordering; Moore-Penrose inverse; reverse order law; C^*-algebras.

1 Introduction

Let $\mathscr{A}$ be a C^*-algebra with unit 1. An element $p \in \mathscr{A}$ is said to be a projection if $p = p^2 = p^*$. Let $a \in \mathscr{A}$, consider the equations:

$$(1)\ aba = a, \qquad (2)\ bab = b, \qquad (3)\ (ab)^* = ab, \qquad (4)\ (ba)^* = ba.$$

For any $a \in \mathscr{A}$, let $a\{i, j, \ldots, k\}$ denote the set of elements $b \in \mathscr{A}$ which satisfy equations $(i), (j), \ldots, (k)$ from among equations (1)-(4). In this situation, the element b will be called a $\{i, j, \ldots, k\}$-inverse of a. It is well known that $a\{1, 2, 3, 4\}$ is or empty or a singleton and when $a\{1, 2, 3, 4\}$ is a singleton, its unique element is called the Moore-Penrose inverse of a, denoted by $a^\dagger$. The subset of $\mathscr{A}$ consisting of elements of $\mathscr{A}$ that have a Moore-Penrose inverse will be denoted by $\mathscr{A}^\dagger$. For an arbitrary C^*-algebra $\mathscr{A}$, it may happen that $\mathscr{A} \neq \mathscr{A}^\dagger$. In [15] it was proved that if $a\{1\} \neq \emptyset$, then $a \in \mathscr{A}^\dagger$ (see also [13]).

The following formulae are well known in the theory of generalized inverses in C^*-algebras and they will be useful in the sequel.

*Email: xiaojiliu72@yahoo.com.cn.

†Corresponding author. Email: jbenitez@mat.upv.es

‡Email: zhongjin1984@126.com.

Lemma 1.1. *Let $\mathscr{A}$ be a C^*-algebra. For any $a \in \mathscr{A}^\dagger$, the following statements are satisfied:*

(i) $a^\dagger \in \mathscr{A}^\dagger$ *and* $(a^\dagger)^\dagger = a$.

(ii) $a^* \in \mathscr{A}^\dagger$ *and* $(a^*)^\dagger = (a^\dagger)^*$.

(iii) $a^\dagger = a^\dagger (a^\dagger)^* a^* = a^* (a^\dagger)^* a^\dagger$.

(iv) $a^* = a^\dagger a a^* = a^* a a^\dagger$.

The set of complex $n \times n$ matrices can be considered a C^*-algebra, but let us remark that any complex matrix has a Moore-Penrose inverse. Recall that a matrix A is called EP when $AA^\dagger = A^\dagger A$ and there are many characterizations of EP matrices (see [5, 8]). Recently, many researchers pay their attention to EP elements in C^*-algebras and rings and present several equivalent characterizations of elements of a C^*-algebra that commute with their Moore-Penrose inverse (see [6, 10, 12]). In this paper, for a C^*-algebra $\mathscr{A}$, we will denote $\mathscr{A}^{EP} = \{a \in \mathscr{A}^\dagger : aa^\dagger = a^\dagger a\}$.

For future use we need the following Theorem 1.1 (see [6, Th 2.1] and [13, Th. 3.1]) and some notation. For any $a \in \mathscr{A}$ we define the nullspace ideals (also called the two annihilators of a)

$$a^\circ = \{x \in \mathscr{A} : ax = 0\}, \qquad {}^\circ a = \{x \in \mathscr{A} : xa = 0\}.$$

It is simple to prove from items (iii) and (iv) of Lemma 1.1 that $(a^*)^\circ = (a^\dagger)^\circ$ and ${}^\circ(a^*) = {}^\circ(a^\dagger)$ hold for any $a \in \mathscr{A}^\dagger$.

Theorem 1.1. *Let $\mathscr{A}$ be a C^*-algebra with unit 1 and $a \in \mathscr{A}$. Then the following conditions are equivalent:*

(i) *There exists a unique projection p such that $a + p \in \mathscr{A}^{-1}$ and $ap = pa = 0$.*

(ii) $a \in \mathscr{A}^{EP}$.

(iii) $a^\circ = (a^*)^\circ$.

(iv) ${}^\circ a = {}^\circ(a^*)$.

Following [12], we denote by a^π the unique projection satisfying condition (i) of Theorem 1.1 for a given $a \in \mathscr{A}^{EP}$. It is proved that

$$a^\pi = 1 - aa^\dagger \quad \text{and} \qquad a^\dagger = (a + a^\pi)^{-1} - a^\pi.$$

The projector a^π will be named the spectral idempotent of a corresponding to 0.

Inspired by matrix theory, for $a \in \mathscr{A}^\dagger$, we will define two projectors a_l^π and a_r^π by

$$a_l^\pi = 1 - a^\dagger a, \qquad a_r^\pi = 1 - aa^\dagger,$$

respectively. Obviously, when $a \in \mathscr{A}^{EP}$, then $a_l^\pi = a_r^\pi$.

Matrix partial orderings have been an area of intense research in the past few years (see [1, 2, 3, 4]). Analogously to the definition introduced by Drazin [11], we define the star ordering in an arbitrary C^*-algebra by

$$a \overset{*}{\leq} b \quad \Longleftrightarrow \quad a^*a = a^*b \ \text{ and } \ aa^* = ba^*.$$

Let us remark that if $a \in \mathscr{A}^\dagger$, then the conditions $a^*a = a^*b$ and $aa^* = ba^*$ are equivalent to $a^\dagger a = a^\dagger b$ and $aa^\dagger = ba^\dagger$, respectively since $(a^*)^\circ = (a^\dagger)^\circ$ and ${}^\circ(a^*) ={}^\circ (a^\dagger)$.

Inspired in a paper of Baksalary and Mitra [1], we define left-star and right-star partial ordering of Moore-Penrose invertible elements a, b of a C^*-algebra by

$$a *\leq b \quad \Longleftrightarrow \quad a^*a = a^*b \;\text{ and }\; b_r^\pi a = 0,$$

and

$$a \leq * \, b \quad \Longleftrightarrow \quad aa^* = ba^* \;\text{ and }\; ab_l^\pi = 0,$$

respectively. It can easily be proved that when A and B are $n \times n$ complex matrices, then $B_r^\pi A = 0$ if and only if $\mathscr{R}(A) \subset \mathscr{R}(B)$; and $AB_l^\pi = 0$ if and only if $\mathscr{R}(A^*) \subset \mathscr{R}(B^*)$, where $\mathscr{R}(\cdot)$ denotes the range space. These inclusions are part of the original definition of the left-star and right-star partial ordering in the set composed of $n \times n$ complex matrices.

Furthermore, we will consider the minus ordering defined in [16]. An extension to $\mathscr{A}^\dagger$ of an equivalent form of this ordering (see [18] or [9]) is the following:

$$a \overset{-}{\leq} b \qquad \Longleftrightarrow \qquad ab^\dagger b = a, \quad bb^\dagger a = a, \quad ab^\dagger a = a.$$

The purpose of this paper is to establish some results on the star, left-star, right-star, and minus orderings of two Moore-Penrose invertible elements of C^*-algebras, when one of them commutes with its Moore-Penrose inverse.

The reverse order law is one of the most important properties of the Moore-Penrose inverse that have been studied, that is under what condition the equation $(ab)^\dagger = b^\dagger a^\dagger$ holds for $a, b \in \mathscr{A}^\dagger$. In [14], T.N.E. Greville gave equivalent conditions on a pair of square complex matrices A and B for $(AB)^\dagger = B^\dagger A^\dagger$ holds. However, it is worth noticing that the proofs work in the more general context of C^*-algebras. An algebraic proof of the reverse order law for the Moore-Penrose inverse (in a ring with involution) is given in [17]. The interested reader can also consult [7, 19].

2 Star ordering and the reverse order law

Next, for two Moore-Penrose invertible elements of a C^*-algebra, say a and b, we study the relation $a \overset{*}{\leq} b$ and the reverse order law for the products ab and ba when a or b commute with its Moore-Penrose inverse.

Theorem 2.1. *Let $\mathscr{A}$ be a unital C^*-algebra and a, b elements of $\mathscr{A}$ that have a Moore-Penrose inverse. Assume that $a \in \mathscr{A}^{EP}$. The following affirmations are equivalent:*

(i) $a \overset{*}{\leq} b$.

(ii) $ab = ba = a^2$.

(iii) *ab has a Moore-Penrose inverse, $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$ and $aa^\dagger b = a$.*

(iv) *ab has a Moore-Penrose inverse, $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$ and $baa^\dagger = a$.*

Proof. (i) $\Rightarrow$ (ii): From $a^*a = a^*b$ and $aa^* = ba^*$, we have

$$a^*(a-b) = (a-b)a^* = 0.$$

Since $a \in \mathscr{A}^{EP} \iff a^* \in \mathscr{A}^{EP}$ and $a^\pi = (a^*)^\pi$, then by item (i) of [6, Theorem 3.6], we have

$$a^\pi(a-b) = a-b = (a-b)a^\pi.$$

Hence, $a(a-b) = aa^\pi(a-b) = 0$, i.e., $a^2 = ab$ and $(b-a)a = (b-a)a^\pi a = 0$, i.e., $ba = a^2$.

(ii) $\Rightarrow$ (iii): It is easy to see that $a \in \mathscr{A}^{EP}$ implies $a^2 \in \mathscr{A}^\dagger$ and $(a^2)^\dagger = (a^\dagger)^2$. Since $ab = a^2$, then ab has a Moore-Penrose inverse. It is easy to check that $aa^\dagger b = a^\dagger ab = a^\dagger a^2 = aa^\dagger a = a$. Next we will prove that $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$. By using $ab = ba = a^2$ we have

$$a(b-a) = (b-a)a = 0.$$

By item (i) of [6, Theorem 3.6], we have

$$a^\pi b = a^\pi(b-a) = b-a = (b-a)a^\pi = ba^\pi.$$

Thus, we obtain

$$b = a + a^\pi b = a + ba^\pi. \tag{1}$$

From (1) and [6, Lemma 3.5] we get

$$a^\pi b^\dagger = b^\dagger a^\pi. \tag{2}$$

Now, by doing a little algebra we obtain

$$a^\pi ba^\pi b^\dagger a^\pi b = a^\pi b \qquad \text{and} \qquad a^\pi b^\dagger a^\pi ba^\pi b^\dagger = a^\pi b^\dagger.$$

Moreover, recall that a^π is a projection and commutes with b and $b^\dagger$, hence $(a^\pi ba^\pi b^\dagger)^* = (a^\pi bb^\dagger)^* = (bb^\dagger)^* a^\pi = bb^\dagger a^\pi = a^\pi ba^\pi b^\dagger$, and thus, $a^\pi ba^\pi b^\dagger$ is self-adjoint. In the same way, we prove that $a^\pi b^\dagger a^\pi b$ is self-adjoint. We have proved

$$(a^\pi b)^\dagger = a^\pi b^\dagger, \tag{3}$$

in particular $a^\pi b \in \mathscr{A}^\dagger$. Since $aa^\pi b = a^\pi ba = 0$ by item (iv) of [6, Theorem 3.6], we get that $a + a^\pi b$ is Moore-Penrose invertible and $(a + a^\pi b)^\dagger = a^\dagger + (a^\pi b)^\dagger$. Using this last identity, (1), (3), and (2) we obtain

$$b^\dagger = (a + a^\pi b)^\dagger = a^\dagger + (a^\pi b)^\dagger = a^\dagger + a^\pi b^\dagger = a^\dagger + b^\dagger a^\pi$$

Therefore, $b^\dagger - a^\dagger = a^\pi b^\dagger = b^\dagger a^\pi$ and thus

$$a^\dagger(b^\dagger - a^\dagger) = a^\dagger a^\pi b^\dagger = 0, \qquad (b^\dagger - a^\dagger)a^\dagger = b^\dagger a^\pi a^\dagger = 0.$$

Hence, $a^\dagger b^\dagger = b^\dagger a^\dagger = (a^\dagger)^2 = (a^2)^\dagger = (ab)^\dagger$.

(iii) $\Rightarrow$ (i): Noting that $a \in \mathscr{A}^{EP} \iff a^\dagger \in \mathscr{A}^{EP}$ and $(a^\dagger)^\pi = a^\pi$, since $a^\dagger b^\dagger = b^\dagger a^\dagger$, then by [6, Corollary 3.3] we get $b^\dagger a^\pi = a^\pi b^\dagger$. Further, by [6, Lemma 3.5], we also have $ba^\pi = a^\pi b$. By $aa^\dagger b = a$, we have $ba^\pi = a^\pi b = (1 - aa^\dagger)b = b - aa^\dagger b = b - a$. Now, $a^*(b-a) = a^*a^\pi b = (a^\pi a)^* b = 0$, i.e., $a^*b = a^*a$. In a similar way, from the equality $ba^\pi = b - a$, we get $ba^* = aa^*$.

(ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i): This has the same proof as (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i), and thus, the theorem is demonstrated. $\square$

Recall that, in addition to the standard properties of involution $x \mapsto x^*$ we have $\|x^*\| = \|x\|$ for all $x \in \mathscr{A}$, and the B*-condition

$$\|x^*x\| = \|x\|^2.$$

Theorem 2.2. *Let $\mathscr{A}$ be a unital C^*-algebra, $a \in \mathscr{A}$, and $b \in \mathscr{A}^{EP}$, then*

(i) *If $a \overset{*}{\leq} b$, then $ab^\pi = b^\pi a = 0$.*

(ii) *Let $a \in \mathscr{A}^\dagger$, if $ab^\pi = b^\pi a$, then $a_l^\pi b^\pi = b^\pi a_l^\pi$, $a_r^\pi b^\pi = b^\pi a_r^\pi$, $b^\dagger a^\dagger \in (ab)\{1,2,3\}$, $a^\dagger b^\dagger \in (ba)\{1,2,4\}$.*

Proof. (i) Since $a^*ab^\pi = a^*bb^\pi = 0$ we get $\|ab^\pi\|^2 = \|(ab^\pi)^*(ab^\pi)\| = \|b^\pi a^*ab^\pi\| = 0$, and therefore, $ab^\pi = 0$. On the other hand, we have

$$\|b^\pi a\|^2 = \|(b^\pi a)^*\|^2 = \|a^*b^\pi\|^2 = \|(a^*b^\pi)^*(a^*b^\pi)\| = \|b^\pi aa^*b^\pi\| = \|b^\pi ba^*b^\pi\| = 0,$$

which proves $b^\pi a = 0$.

(ii) If $ab^\pi = b^\pi a$, from item (i) of [6, Lemma 3.5], we have $a^\dagger b^\pi = b^\pi a^\dagger$, and thus, $aa^\dagger b^\pi = ab^\pi a^\dagger = b^\pi aa^\dagger$, i.e, $a_r^\pi b^\pi = b^\pi a_r^\pi$. Similarly, from $a^\dagger ab^\pi = a^\dagger b^\pi a = b^\pi a^\dagger a$, we have $a_l^\pi b^\pi = b^\pi a_l^\pi$.

Next, we shall prove from $ab^\pi = b^\pi a$ and $a^\dagger b^\pi = b^\pi a^\dagger$ that $b^\dagger a^\dagger \in (ab)\{1,2,3\}$.

$$abb^\dagger a^\dagger ab = bb^\dagger aa^\dagger ab = bb^\dagger ab = abb^\dagger b = ab,$$

$$b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger bb^\dagger aa^\dagger = b^\dagger bb^\dagger a^\dagger aa^\dagger = b^\dagger a^\dagger,$$

and

$$(abb^\dagger a^\dagger)^* = (bb^\dagger aa^\dagger)^* = (aa^\dagger)^*(bb^\dagger)^* = aa^\dagger bb^\dagger = bb^\dagger aa^\dagger = abb^\dagger a^\dagger.$$

The proof of $a^\dagger b^\dagger \in (ba)\{1,2,4\}$ is similar and we will not give it. □

Theorem 2.3. *Let $\mathscr{A}$ be a unital C^*-algebra and $a, b \in \mathscr{A}^\dagger$. If $b \in \mathscr{A}^{EP}$ and $ab^\pi = b^\pi a$, then*

(i) *$(ab)^\dagger = b^\dagger a^\dagger$ if and only if $bb^*a_l^\pi = a_l^\pi bb^*$.*

(ii) *$(ba)^\dagger = a^\dagger b^\dagger$ if and only if $b^*ba_r^\pi = a_r^\pi b^*b$.*

Proof. We shall prove the first equivalence, and we will not give the proof of the other because its proof is similar. By Theorem 2.2, we have that $(ab)^\dagger = b^\dagger a^\dagger$ if and only if $b^\dagger a^\dagger ab$ is self-adjoint. In order to prove $(b^\dagger a^\dagger ab)^* = b^\dagger a^\dagger ab$, we will use a consequence of item (ii) of Theorem 2.2, specifically, $a_l^\pi b^\pi = b^\pi a_l^\pi$. Since $b^\dagger = (b + b^\pi)^{-1} - b^\pi$, we have

$$\begin{aligned}
b^\dagger a^\dagger ab \text{ is self-adjoint} \quad &\iff \quad b^\dagger a^\dagger ab = (b^\dagger a^\dagger ab)^* \\
&\iff \quad b^\dagger a_l^\pi b = (b^\dagger a_l^\pi b)^* \\
&\iff \quad b^\dagger a_l^\pi b = b^* a_l^\pi (b^\dagger)^* \\
&\iff \quad [(b + b^\pi)^{-1} - b^\pi] a_l^\pi b = b^* a_l^\pi [(b + b^\pi)^{-*} - b^\pi] \\
&\iff \quad (b + b^\pi)^{-1} a_l^\pi b = b^* a_l^\pi (b + b^\pi)^{-*} \\
&\iff \quad a_l^\pi b (b + b^\pi)^* = (b + b^\pi) b^* a_l^\pi \\
&\iff \quad a_l^\pi b (b^* + b^\pi) = (b + b^\pi) b^* a_l^\pi \\
&\iff \quad a_l^\pi bb^* = bb^* a_l^\pi.
\end{aligned}$$

□

3 The left and right star orderings and the reverse order law

In this section we study the relation between $a \ast\!\leq b$ and $a \leq\!\ast\, b$ and reverse law of ab and ba when a and b are elements in a C^*-algebra that have a Moore-Penrose inverse.

Lemma 3.1. *Let $\mathscr{A}$ be a unital C^*-algebra. Let $a \in \mathscr{A}^\dagger$ and assume that there exists a projection p such that $a = pa$, then $a^\dagger = a^\dagger p$.*

Proof. It is evident $aa^\dagger pa = a$, $a^\dagger paa^\dagger p = a^\dagger p$, and $a^\dagger pa = a^\dagger a$ is self-adjoint. Since $(aa^\dagger p)^* = p^*(aa^\dagger)^* = paa^\dagger = aa^\dagger$ is also self-adjoint, we obtain $a^\dagger = a^\dagger p$. □

The following observation will be useful in the sequel: Let $\mathscr{A}$ be a C^*-algebra and $a \in \mathscr{A}^\dagger$, $b \in \mathscr{A}$.

$$a^* b = a^* a \qquad \Longleftrightarrow \qquad a^\dagger b = a^\dagger a. \tag{4}$$

This equivalence follows from $(a^*)^\circ = (a^\dagger)^\circ$.

Theorem 3.1. *Let $\mathscr{A}$ be a unital C^*-algebra. Assume that $a, b \in \mathscr{A}^\dagger$ with $a \in \mathscr{A}^{EP}$. If $a \ast\!\leq b$, then*

(i) $ab = a^2$.

(ii) $a^\dagger b^\dagger = (ab)^\dagger$.

(iii) $b^\dagger a^\dagger \in (ab)\{1, 2, 3\}$.

(iv) $a^\dagger b^\dagger \in (ba)\{1, 2, 4\}$.

Proof. (i): By using Theorem 1.1 we have $a^\circ = (a^*)^\circ$. Since $a^*a = a^*b$, then $(a-b) \in (a^*)^\circ = a^\circ$, so $a(a-b) = 0$, i.e., $ab = a^2$.

(ii): Observe that $(a^2)^\dagger = (a^\dagger)^2$ because $a \in \mathscr{A}^{EP}$, and from item (i) of this theorem, it only remains to prove that $a^\dagger b^\dagger = (a^\dagger)^2$. Since $b_r^\pi a = 0$, or equivalently,

$$bb^\dagger a = a, \tag{5}$$

then by Lemma 3.1, we get

$$a^\dagger = a^\dagger bb^\dagger \tag{6}$$

and from (4) we have

$$a^\dagger = a^\dagger ab^\dagger. \tag{7}$$

Then $a^\pi(b^\dagger - a^\dagger) = a^\pi b^\dagger = b^\dagger - aa^\dagger b^\dagger = b^\dagger - a^\dagger$, which implies that $aa^\dagger(b^\dagger - a^\dagger) = 0$. Now, premultiplying $aa^\dagger(b^\dagger - a^\dagger) = 0$ by $a^\dagger$, we get $a^\dagger(b^\dagger - a^\dagger) = 0$, i.e., $a^\dagger b^\dagger = (a^\dagger)^2$.

(iii) We shall prove this item by the definition of $(ab)\{1, 2, 3\}$: Recall that one hypothesis is $aa^\dagger = a^\dagger a$. By using (5)

$$abb^\dagger a^\dagger ab = abb^\dagger aa^\dagger b = aaa^\dagger b = ab.$$

Now we use (6)

$$b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger aa^\dagger bb^\dagger a^\dagger = b^\dagger aa^\dagger a^\dagger = b^\dagger a^\dagger,$$

and finally, from (5)

$$abb^\dagger a^\dagger = abb^\dagger a^\dagger aa^\dagger = abb^\dagger aa^\dagger a^\dagger = aaa^\dagger a^\dagger = aa^\dagger \text{ is self-adjoint.}$$

(iv) The proof is similar as in (iii), and we will not give it. □

Next result characterizes the reverse law for the product ab when a commutes with its Moore-Penrose inverse and $a \ast\!\leq b$. It is remarkable that one of these equivalent conditions is $a \overset{*}{\leq} b$.

Theorem 3.2. *Let $\mathscr{A}$ be a unital C^*-algebra. Assume that $a, b \in \mathscr{A}^\dagger$ with $a \in \mathscr{A}^{EP}$. If $a \ast\!\leq b$, then the following affirmations are equivalent:*

(i) $b^\dagger a^\dagger = (ab)^\dagger$.

(ii) $ab = ba$.

(iii) $a \overset{*}{\leq} b$.

(iv) $a^\dagger b^\dagger = (ba)^\dagger$.

Proof. By item (i) of Theorem 3.1, if $a \in \mathscr{A}^{EP}$ and $a \ast\!\leq b$, then $ab = a^2$. Also recall $(a^2)^\dagger = (a^\dagger)^2$ because $aa^\dagger = a^\dagger a$.

(i) $\Rightarrow$ (ii): The hypothesis $b^\dagger a^\dagger = (ab)^\dagger$ implies $0 = (b^\dagger - a^\dagger)a^\dagger = (b^\dagger - a^\dagger)[(a + a^\pi)^{-1} - a^\pi]$, then $(b^\dagger - a^\dagger)(a + a^\pi)^{-1} = b^\dagger a^\pi$ and thus, $b^\dagger - a^\dagger = b^\dagger a^\pi (a + a^\pi) = b^\dagger a^\pi$. Now, we have

$$a^\dagger = b^\dagger(1 - a^\pi) = b^\dagger a a^\dagger \tag{8}$$

Postmultiplying the equality (8) by a^2, we get $a = b^\dagger a^2$, premultiplying by b and using $bb^\dagger a = a$ (obtained in (5)) we get $ba = bb^\dagger a^2 = a^2 = ab$.

(ii) $\Rightarrow$ (iii) By the definitions of the different orderings involved in this implication, it is enough to prove $aa^* = ba^*$. For the proof of $aa^* = ba^*$, we will use item (iv) of Theorem 1.1:

$$a^2 = ab = ba \;\Rightarrow\; (a-b)a = 0 \;\Rightarrow\; a - b \in {}^\circ a = {}^\circ(a^*) \;\Rightarrow\; (a-b)a^* = 0 \;\Rightarrow\; aa^* = ba^*.$$

(iii) $\Rightarrow$ (iv): By items (iii) and (iv) of Theorem 1.1 we get

$$a \overset{*}{\leq} b \iff \begin{cases} a^*a = a^*b \\ aa^* = ba^* \end{cases} \iff \begin{cases} a - b \in (a^*)^\circ \\ a - b \in {}^\circ(a^*) \end{cases} \iff \begin{cases} a - b \in a^\circ \\ a - b \in {}^\circ a \end{cases} \iff a^2 = ab = ba.$$

By using item (ii) of Theorem 3.1 we have $(ba)^\dagger = (ab)^\dagger = a^\dagger b^\dagger$.

(iv) $\Rightarrow$ (ii): From item (ii) of Theorem 3.1 and hypothesis we have $(ba)^\dagger = a^\dagger b^\dagger = (ab)^\dagger$. Now, the conclusion follows from item (i) of Lemma 1.1.

(ii) $\Rightarrow$ (i): By item (iii) of Theorem 3.1, it is enough to prove that $b^\dagger a^\dagger ab$ is self-adjoint. By [6, Corollary 3.3] and [6, Lemma 3.5] we get $a^\pi b^\dagger = b^\dagger a^\pi$. Moreover we will need $a^\dagger b = a^\dagger a$ (obtained in the observation given in (4)), and the relation (7). Thus

$$b^\dagger a^\dagger ab = b^\dagger a a^\dagger b = b^\dagger a a^\dagger a = b^\dagger a = (1 - a^\pi) b^\dagger a = a^\dagger a b^\dagger a = a^\dagger a,$$

which proves that $b^\dagger a^\dagger ab$ is self-adjoint. □

Theorem 3.3. *Let $\mathscr{A}$ be a unital C^*-algebra. Assume that $a, b \in \mathscr{A}^\dagger$ with $b \in \mathscr{A}^{EP}$. If $a \ast\!\leq b$, then*

(i) $a^\dagger b^\dagger \in (ba)\{1, 2, 4\}$.

(ii) $b^\dagger a^\dagger \in (ab)\{1, 2, 3\}$.

(iii) $a^\dagger b^\dagger = (ba)^\dagger$ *if and only if* b^*b *commutes with* $aa^\dagger$. *Moreover,* $b^\dagger a^\dagger = (ab)^\dagger$ *if and only if* bb^* *commutes with* $a^\dagger a$.

Proof. (i): Note that $b \in \mathscr{A}^{EP}$ implies $b_r^\pi a = b^\pi a = 0$, i.e., $bb^\dagger a = b^\dagger ba = a$. Now,

$$baa^\dagger b^\dagger ba = baa^\dagger a = ba, \quad a^\dagger b^\dagger baa^\dagger b^\dagger = a^\dagger aa^\dagger b^\dagger = a^\dagger b^\dagger, \quad a^\dagger b^\dagger ba = a^\dagger a \text{ is self-adjoint.}$$

For the rest of the proof we will need $ab^\pi = 0$. In fact, since $a \ast\!\leq b$ we have $a^*a = a^*b$, and thus

$$\|ab^\pi\|^2 = \|b^\pi a^* ab^\pi\| = \|b^\pi a^* bb^\pi\| = 0,$$

which, obviously implies $ab^\pi = 0$, or equivalently, $a = abb^\dagger$.

(ii): We have

$$abb^\dagger a^\dagger ab = aa^\dagger ab = ab, \quad b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger aa^\dagger = b^\dagger a^\dagger, \quad abb^\dagger a^\dagger = aa^\dagger \text{ is self-adjoint.}$$

(iii): It is a trivial consequence of Theorem 2.3 since $b^\pi a = ab^\pi = 0$. □

Having in mind that $a \ast\!\leq b \Longleftrightarrow a^* \leq\!\ast\, b^*$, we can obtain similar results for the right-star ordering.

Theorem 3.4. *Let* $\mathscr{A}$ *be a unital* C^**-algebra. Assume that* $a, b \in \mathscr{A}^\dagger$ *with* $a \in \mathscr{A}^{EP}$. *If* $a \leq\!\ast\, b$, *then*

(i) $ba = a^2$.

(ii) $b^\dagger a^\dagger = (ba)^\dagger$.

(iii) $a^\dagger b^\dagger \in (ba)\{1, 2, 4\}$.

(iv) $b^\dagger a^\dagger \in (ab)\{1, 2, 3\}$.

Theorem 3.5. *Let* $\mathscr{A}$ *be a unital* C^**-algebra. Assume that* $a, b \in \mathscr{A}^\dagger$ *with* $a \in \mathscr{A}^{EP}$. *If* $a \leq\!\ast\, b$, *then the following affirmations are equivalent:*

(i) $a^\dagger b^\dagger = (ba)^\dagger$.

(ii) $ab = ba$.

(iii) $a \overset{*}{\leq} b$.

(iv) $b^\dagger a^\dagger = (ab)^\dagger$.

Theorem 3.6. *Let* $\mathscr{A}$ *be a unital* C^**-algebra. Assume that* $a, b \in \mathscr{A}^\dagger$ *with* $b \in \mathscr{A}^{EP}$. *If* $a \leq\!\ast\, b$, *then*

(i) $a^\dagger b^\dagger \in (ba)\{1, 2, 4\}$.

(ii) $b^\dagger a^\dagger \in (ab)\{1, 2, 3\}$.

(iii) $a^\dagger b^\dagger = (ba)^\dagger$ *if and only if* b^*b *commutes with* $aa^\dagger$. *Moreover,* $b^\dagger a^\dagger = (ab)^\dagger$ *if and only if* bb^* *commutes with* $a^\dagger a$.

4 The minus ordering and the reverse order law

As we made in the previous sections, we link the minus ordering with the reverse law.

Firstly, let us remark that if $\mathscr{A}$ is a unital C^*-algebra, and $a \in \mathscr{A}$, $b \in \mathscr{A}^\dagger$ satisfy $ab^\dagger a = a$, then [15, Theorem 6] assures that $a \in \mathscr{A}^\dagger$.

Theorem 4.1. *Let $\mathscr{A}^\dagger$ be a unital C^*-algebra and $a \in \mathscr{A}$, $b \in \mathscr{A}^\dagger$ satisfy $a \overset{-}{\leq} b$. Then*

(i) $a^\dagger b^\dagger \in (ba)\{1,2,4\}$.

(ii) *If $b \in \mathscr{A}^{EP}$, then $b^\dagger a^\dagger \in (ab)\{1,2,3\}$.*

Proof. (i): The equalities $baa^\dagger b^\dagger ba = ba$, $a^\dagger b^\dagger baa^\dagger b^\dagger = a^\dagger b^\dagger$, and $a^\dagger b^\dagger ba = a^\dagger a$ follow directly from $b^\dagger ba = a$.

(ii): The equalities $abb^\dagger a^\dagger ab = ab$, $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$, and $abb^\dagger a^\dagger = aa^\dagger$ follow from $ab^\dagger b = a$ and $bb^\dagger = b^\dagger b$. □

References

[1] J.K. Baksalary, S.K. Mitra, Left-star and right-star partial orderings, Linear Algebra Appl. 149 (1991) 73-89 .

[2] J.K. Baksalary, O.M. Baksalary, X. Liu, Further properties of the star, left-star, right-star, and minus partial orderings, Linear Algebra Appl. 375 (2003) 83-94.

[3] J.K. Baksalary, O.M. Baksalary, X. Liu, Further relationships between certain partial orders of matrices and their squares, Linear Algebra Appl., 375, 171-180 (2003).

[4] J.K. Baksalary, J. Hauke, X. Liu, S. Liu, Relationships between partial orders of matrices and their powers, Linear Algebra Appl. 379 (2004) 277-287.

[5] O.M. Baksalary, G. Trenkler, Characterizations of EP, normal, and Hermitian matrices, Linear Multilinear Algebra 56 (2008) 299-304.

[6] J. Benítez, Moore Penrose inverses and commuting elements of C^*-algebras, J. Math. Anal. Appl. 345 (2008) 766-770.

[7] E. Boasso, On the Moore-Penrose Inverse in C^*-algebras, Extracta Math. 21 (2006) 93-106.

[8] S. Cheng, Y. Tian, Two sets of new characterizations for normal and EP matrices, Linear Algebra Appl. 375 (2003) 181-195.

[9] R.E. Cline, R.E Funderlic, The rank of a difference of matrices and associated generalized inverses, Linear Algebra Appl. 24 (1979) 185-215.

[10] D. S. Djordjević, J. J. Koliha, I. Straškraba, Factorization of EP elements in C^*-algebras, Linear Multilinear Algebra 57 (2009) 587-594.

[11] M.P. Drazin, Natural structures on semigroups with involution, Bull. Amer. Math. Soc. 84 (1978) 139-141.

[12] J. J. Koliha, Elements of C^*-algebras commuting with their Moore-Penrose inverse, Studia Math. 139 (2000) 81-90.

[13] J. J. Koliha, The Drazin and Moore-Penrose inverse in C^*-algebras, Math. Proc. R. Ir. Acad. 99A (1999) 17-27.

[14] T.N.E. Greville, Note on the generalized inverse of a matrix product, J. Soc. Indust. Appl. Math. 9 (1966) 109-115.

[15] R. Harte, M. Mbekhta, On generalized inverses in C^*-algebras, Studia Math. 103 (1992) 71-77.

[16] R. E. Hartwig, How to partially order regular elements, Math. Japon 25 (1980) 1-13.

[17] J.J. Koliha, D. Djordjević, D. Cvetković, Moore-Penrose inverse in rings with involution, Linear Algebra Appl. 426, (2007) 371-381.

[18] G. Marsaglia, G.P.H. Styan, Equalities and inequalities for ranks of matrices, Linear Multilinear Algebra 2 (1974) 269-292.

[19] Z. Xiong, B. Zheng, The reverse order laws for $\{1, 2, 3\}$- and $\{1, 2, 4\}$-inverses of a two-matrix product. Appl. Math. Lett. 21 (2008) 649-655.