Characterizations and linear combinations of k-generalized projectors ⋆

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Abstract

We extend generalized projectors (introduced by Groß and Trenkler in [Linear Algebra Appl. (1997) 264]) to k-generalized projectors and we characterize them obtaining results in the aforesaid paper as a consequence. Moreover, we list all situations when a linear combination of commuting k-generalized projectors is a k-generalized projector. The method for obtaining this result permits to give a revisited version of the main result by Baksalary and Baksalary in [Linear Algebra and its Appl. (Article in press)]. In addition, the case of orthogonal projectors is also analyzed.

Key words: Generalized Projectors; Normal Matrices; Commuting Matrices; Orthogonal Projectors.

1 Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of $m \times n$ complex matrices. For a given matrix $\mathbf{M} \in \mathbb{C}^{m\times n}$, the symbols $\mathbf{M}^*$ and $\overline{\mathbf{M}}$ will stand for the conjugate transpose and the conjugate of $\mathbf{M}$, respectively. A square matrix $\mathbf{A}$ is called a *projector* (also called an idempotent matrix) if $\mathbf{A}^2 = \mathbf{A}$, is called an *orthogonal projector* if $\mathbf{A}^2 = \mathbf{A}$ and $\mathbf{A} = \mathbf{A}^*$ and finally $\mathbf{A}$ is said to be *normal* if $\mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A}$. These matrices have been extensively studied and there are many characterizations of normal matrices (for example see [9]). Results related to normal and EP matrices have recently given by Cheng and Tian in [4]. Using rank equalities,

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Tian and Styan presented in [8] different new characterizations for idempotent matrices and orthogonal projectors.

In [5] the authors introduced the following concept: A square matrix $\mathbf{A}$ is said to be a *generalized projector* if $\mathbf{A}^* = \mathbf{A}^2$. A characterization of generalized projectors (Corollary 2.2 below) was also established in [5]. A characterization of square matrices $\mathbf{A}$ such that $\mathbf{A} = \mathbf{A}^3$ and $\mathbf{A} = \mathbf{A}^*$ is derived by Kathri in [7].

We will use the following notation: for $k \in \mathbb{N}$ and $k > 1$, the set of complex roots of 1 shall be denoted by Ω_k and if we set $\omega_k = \exp(2\pi i/k)$ then $\Omega_k = \{\omega_k^0, \omega_k^1, \ldots, \omega_k^{k-1}\}$. The symbol $\sigma(\mathbf{A})$ will stand for the spectrum of the matrix $\mathbf{A}$.

In this work, we deal with square matrices $\mathbf{A}$ with the property $\mathbf{A}^k = \mathbf{A}^*$ for $k \in \mathbb{N}$ and $k > 1$ which will be called *k-generalized projectors.* Observe that this class of matrices obviously generalizes to the class of generalized projectors. In this paper we characterize this class of matrices and, as simple corollary, we deduce the characterization of generalized projectors presented in [5]. Later, we study the problem of when a linear combination of two nonzero distinct commuting matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ is a k-generalized projector for $k \in \mathbb{N}$ and $k > 1$. The results established herein generalize those results in [2].

In [1] the authors gave a complete solution to the problem of when a linear combination of two different projectors is also a projector. In [2] a complete solution to the same problem for generalized projectors instead of projectors was established. The proof given in [2] is very computational. Here we give a less computational proof of the same result and moreover we simplify the list of all situations in which nonzero complex numbers c_1, c_2 and nonzero generalized projectors $\mathbf{G}_1$, $\mathbf{G}_2$ with $\mathbf{G}_1 \neq \mathbf{G}_2$ satisfy that $c_1\mathbf{G}_1 + c_2\mathbf{G}_2$ is a generalized projector.

Moreover, the technique used throughout this paper seems to be valid for very distinct situations as we show solving the problem of finding all situations when $\mathbf{G} = c_1\mathbf{G}_1 + c_2\mathbf{G}_2$ is a k-generalized projector provided that c_1, c_2 are nonzero complex numbers, $k \in \mathbb{N}$ with $k > 1$ and $\mathbf{G}_1$ and $\mathbf{G}_2$ are nonzero orthogonal projectors such that $\mathbf{G}_1 \neq \mathbf{G}_2$ and $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_2\mathbf{G}_1$.

2 Characterizations of k-generalized projectors

We start this section with a characterization of k-generalized projectors, that is the class of square matrices $\mathbf{A}$ such that $\mathbf{A}^k = \mathbf{A}^*$ for a given integer k greater than 1.

THEOREM 2.1 *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *and* $k \in \mathbb{N}$, $k > 1$. *Then the following statements are equivalent:*

(1) $\mathbf{A}$ *is a k-generalized projector (i.e.,* $\mathbf{A}^k = \mathbf{A}^*$*).*
(2) $\mathbf{A}$ *is a normal matrix and* $\sigma(\mathbf{A}) \subseteq \{0\} \cup \Omega_{k+1}$.
(3) $\mathbf{A}$ *is a normal matrix and* $\mathbf{A}^{k+2} = \mathbf{A}$.

PROOF: 1 ⇒ 2. Matrix $\mathbf{A}$ is normal because $\mathbf{A}\mathbf{A}^* = \mathbf{A}\mathbf{A}^k = \mathbf{A}^k\mathbf{A} = \mathbf{A}^*\mathbf{A}$. Since $\mathbf{A}$ is normal then there exist a unitary matrix $\mathbf{U}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$ and so $\mathbf{D}^k = \mathbf{D}^* = \overline{\mathbf{D}}$. Then $\lambda \in \sigma(\mathbf{A})$ if and only if $\lambda^k = \overline{\lambda}$ and taking modulus we get $\lambda = 0$ or $|\lambda| = 1$. If $\lambda = \exp(i\theta)$ for some $\theta \in [0, 2\pi[$, it follows that $\exp(ik\theta) = \exp(-i\theta)$ and thus $\lambda^{k+1} = 1$ holds.

2 ⇒ 3. It follows from Theorem 2.1 in [3].

3 ⇒ 1. As before, there exist a unitary matrix $\mathbf{U}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$. From $\mathbf{A}^{k+2} = \mathbf{A}$ we have $\mathbf{D}^{k+2} = \mathbf{D}$ and so we get that $\lambda^{k+2} = \lambda$ for all $\lambda \in \sigma(\mathbf{A})$. Then $\lambda^k = \overline{\lambda}$ implies that $\mathbf{D}^k = \mathbf{D}^*$. The conclusion follows. □

Note that if $k = 1$ this theorem does not work, however the following related results are well-known:

a) $\mathbf{A}$ is an orthogonal projector if and only if $\mathbf{A}$ is normal and $\sigma(\mathbf{A}) \subseteq \{0, 1\}$.
b) $\mathbf{A}$ is Hermitian if and only if $\mathbf{A}$ is normal and $\sigma(\mathbf{A}) \subset \mathbb{R}$.

By definition of Moore-Penrose inverse ($\mathbf{A}^\dagger$), group inverse ($\mathbf{A}^\#$) and Drazin inverse ($\mathbf{A}^{\mathrm{d}}$) it is easy to see that if $\mathbf{A}$ is a k-generalized projector then

$$\mathbf{A}^\dagger = \mathbf{A}^\# = \mathbf{A}^{\mathrm{d}} = \mathbf{A}^* = \mathbf{A}^k = \mathbf{A}^{m(k+1)+k}, \qquad m \in \mathbb{N}.$$

From Theorem 2.1 we deduce the following results:

COROLLARY 2.1 *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *Hermitian and k-generalized projector for* $k \in \mathbb{N}$ *and* $k > 1$. *If k is even then* $\mathbf{A}$ *is an orthogonal projector. If k is odd then* $\mathbf{A}^3 = \mathbf{A}$ *and* $\mathbf{A}$ *is a normal matrix.*

PROOF: Since $\mathbf{A}$ is Hermitian, $\mathbf{A}$ is normal and $\sigma(\mathbf{A}) \subset \mathbb{R}$. By Theorem 2.1, $\sigma(\mathbf{A}) \subseteq \{0\} \cup \Omega_{k+1}$, so $\sigma(\mathbf{A}) \subseteq \mathbb{R} \cap (\{0\} \cup \Omega_{k+1})$. If k is even then $\mathbb{R} \cap (\{0\} \cup \Omega_{k+1}) = \{0, 1\}$ hence $\mathbf{A}^2 = \mathbf{A} = \mathbf{A}^*$. If k is odd then $\mathbb{R} \cap (\{0\} \cup \Omega_{k+1}) = \{0, 1, -1\}$ hence $\mathbf{A}^3 = \mathbf{A}$. □

COROLLARY 2.2 (TH. 1 [5]) *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $\mathrm{rank}(\mathbf{A}) = r$. *Then the following statements are equivalent:*

(1) $\mathbf{A}^4 = \mathbf{A}$ *and* $\mathbf{A}$ *is normal matrix.*

(2) One has

$$\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}\mathbf{U}^*,$$

where $\mathbf{U}$ is unitary and $\mathbf{D} \in \mathbb{C}^{r\times r}$ is a diagonal matrix with diagonal elements in Ω_3.

(3) $\mathbf{A}^2 = \mathbf{A}^*$.

3 Linear combinations of k-generalized projectors

From now on, we are interested in the problem of when a linear combination of two k-generalized projectors is a k-generalized projector. We study this problem for commuting k-generalized projectors.

THEOREM 3.1 *For nonzero $c_1, c_2 \in \mathbb{C}$ and nonzero k-generalized projectors $\mathbf{G}_1, \mathbf{G}_2 \in \mathbb{C}^{n\times n}$ with $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_2\mathbf{G}_1$, let $\mathbf{G} = c_1\mathbf{G}_1 + c_2\mathbf{G}_2$. If $\mathbf{G}$ is a k-generalized projector then any of the following conditions holds.*

a) $c_1, c_2 \in \Omega_{k+1}$.

b) $c_1 \in \Omega_{k+1}$ *and there exists* $r \in \{0, 1, \dots, k\}$ *such that* $\omega_{k+1}^r c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$.

c) $c_2 \in \Omega_{k+1}$ *and there exists* $s \in \{0, 1, \dots, k\}$ *such that* $c_1 + \omega_{k+1}^s c_2 \in \{0\} \cup \Omega_{k+1}$.

d) *There exist* $r, s \in \{0, 1, \dots, k\}$ *such that any of the following conditions occur.*

   i) $r + s \in \{0, k+1\}$ *and* $\omega_{k+1}^r c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$.

   ii) $r + s$ *is not a multiple of* $k + 1$ *and there exist* $\varphi_1, \varphi_2 \in \{0\} \cup \Omega_{k+1}$ *such that* $\varphi_1\varphi_2 \neq 0$ *and*

$$c_1 = \frac{\varphi_1\omega_{k+1}^s - \varphi_2}{\omega_{k+1}^{r+s} - 1}, \qquad c_2 = \frac{\varphi_2\omega_{k+1}^r - \varphi_1}{\omega_{k+1}^{r+s} - 1}.$$

PROOF: Since $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_2\mathbf{G}_1$ and $\mathbf{G}_1$, $\mathbf{G}_2$ are normal matrices then there exist (Th. 2.5.5 [6]) a unitary matrix $\mathbf{U}$ and diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ such that $\mathbf{G}_i = \mathbf{U}\mathbf{D}_i\mathbf{U}^*$ for $i = 1, 2$. Then $\mathbf{G} = c_1\mathbf{G}_1 + c_2\mathbf{G}_2 = \mathbf{U}(c_1\mathbf{D}_1 + c_2\mathbf{D}_2)\mathbf{U}^*$. Let $\mathbf{D}_1 = \mathrm{diag}(\lambda_{11}, \dots, \lambda_{1n})$ and $\mathbf{D}_2 = \mathrm{diag}(\lambda_{21}, \dots, \lambda_{2n})$.

Since $\mathbf{G}$ is a k-generalized projector, Theorem 2.1 implies that the eigenvalues of $\mathbf{D}_1$, $\mathbf{D}_2$ and $c_1\mathbf{D}_1 + c_2\mathbf{D}_2$ are elements in $\{0\} \cup \Omega_{k+1}$ and hence

$$c_1\lambda_{1i} + c_2\lambda_{2i} \in \{0\} \cup \Omega_{k+1}, \qquad \forall\, i = 1, \dots, n. \tag{1}$$

Since $\mathbf{D}_1 \neq \mathbf{O}$, there exists $j \in \{1, \ldots, n\}$ such that $\lambda_{1j} \neq 0$ and so $\lambda_{1j} \in \Omega_{k+1}$. From (1),

$$c_1 + c_2 \frac{\lambda_{2j}}{\lambda_{1j}} \in \frac{1}{\lambda_{1j}}(\{0\} \cup \Omega_{k+1}) = \{0\} \cup \Omega_{k+1},$$

and moreover $\lambda_{2j}/\lambda_{1j} \in \{0\} \cup \Omega_{k+1}$ because Ω_{k+1} is a multiplicative group. So, there exists $\alpha \in \{0\} \cup \Omega_{k+1}$ such that $c_1 + \alpha c_2 \in \{0\} \cup \Omega_{k+1}$. By applying a similar argument for $\mathbf{D}_2 \neq \mathbf{O}$, there exists $\beta \in \{0\} \cup \Omega_{k+1}$ such that $\beta c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$. We obtain the following table to be studied:

	$c_2 \in \Omega_{k+1}$	$\omega_{k+1}^r c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$
$c_1 \in \Omega_{k+1}$	a)	b)
$c_1 + \omega_{k+1}^s c_2 \in \{0\} \cup \Omega_{k+1}$	c)	d)

Cases a), b) and c) are directly obtained. For case d), let $\varphi_1 = \omega_{k+1}^r c_1 + c_2$ and $\varphi_2 = c_1 + \omega_{k+1}^s c_2$. This case can be obtained by solving this 2×2 linear system where the determinant of the coefficient matrix is $\omega_{k+1}^{r+s} - 1$. If $r + s$ is a multiple of $k+1$ (i.e., $r + s \in \{0, k+1\}$) then the consistent system has not unique solution. If $r + s$ is not a multiple of $k + 1$ then it is easy to compute its solution, which is

$$c_1 = \frac{\varphi_1 \omega_{k+1}^s - \varphi_2}{\omega_{k+1}^{r+s} - 1}, \qquad c_2 = \frac{\varphi_2 \omega_{k+1}^r - \varphi_1}{\omega_{k+1}^{r+s} - 1}.$$

The proof is then completed. $\square$

In order to assure that statements a), b), c) and d) of Theorem 3.1 are sufficient conditions for $\mathbf{G}$ being a k-generalized projector, we must check that those statements satisfy the matrix equality $(c_1\mathbf{G}_1 + c_2\mathbf{G}_2)^k = \overline{c_1}\mathbf{G}_1^* + \overline{c_2}\mathbf{G}_2^*$.

Baksalary and Baksalary in [2] presented a complete characterization of when a linear combination of two generalized projectors is again a generalized projector. In the following result we present a revisited formulation of main result given in [2] with a simpler proof when the generalized projectors commute.

Theorem 3.2 *For nonzero $c_1, c_2 \in \mathbb{C}$ and nonzero generalized projectors $\mathbf{G}_1, \mathbf{G}_2 \in \mathbb{C}^{n \times n}$, let $\mathbf{G} = c_1\mathbf{G}_1 + c_2\mathbf{G}_2$ and $\gamma_i = (\overline{c_i} - c_i^2)/(c_1c_2)$ for $i = 1, 2$. When $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_2\mathbf{G}_1$, then $\mathbf{G}$ is a generalized projector if and only if any of the following set of conditions holds.*

a) $c_1, c_2 \in \Omega_3$ *and* $\mathbf{G}_1\mathbf{G}_2 = \mathbf{O}$.
b) $c_1 \in \Omega_3$, *there exists* $r \in \{0, 1, 2\}$ *such that* $\omega_3^r c_1 + c_2 \in \{0\} \cup \Omega_3$ *and* $\mathbf{G}_1\mathbf{G}_2 = \omega_3^r \mathbf{G}_2^*$.
c) $c_2 \in \Omega_3$, *there exists* $s \in \{0, 1, 2\}$ *such that* $c_1 + \omega_3^s c_2 \in \{0\} \cup \Omega_3$ *and* $\mathbf{G}_1\mathbf{G}_2 = \omega_3^s \mathbf{G}_1^*$.

d) *There exist* $r, s \in \{0,1,2\}$ *such that*

i) $r+s \in \{0,3\}$, $\omega_3^r c_1 + c_2 \in \{0\} \cup \Omega_3$ *and* $\mathbf{G}_1\mathbf{G}_2 = \frac{1}{2}(\gamma_1 \mathbf{G}_1^* + \gamma_2 \mathbf{G}_2^*)$.

ii) $r+s$ *is not a multiple of* 3, *there exist* $\varphi_1, \varphi_2 \in \{0\} \cup \Omega_3$ *such that*

$$c_1 = \frac{\varphi_1 \omega_3^s - \varphi_2}{\omega_3^{r+s} - 1}, \qquad c_2 = \frac{\varphi_2 \omega_3^r - \varphi_1}{\omega_3^{r+s} - 1}$$

and

$$\mathbf{G}_1\mathbf{G}_2 = \frac{1}{\omega_3^{r+s} + 1}(\omega_3^s \mathbf{G}_1^* + \omega_3^r \mathbf{G}_2^*).$$

PROOF: It is easy to see that $\mathbf{G}$ is a generalized projector if and only if

$$\mathbf{G}_1\mathbf{G}_2 = \frac{\gamma_1}{2}\mathbf{G}_1^* + \frac{\gamma_2}{2}\mathbf{G}_2^*. \tag{2}$$

In the following we shall use that $\lambda \in \{0\} \cup \Omega_3$ if and only if $\overline{\lambda} - \lambda^2 = 0$, which is easy to check. Suppose that $\mathbf{G}$ is a generalized projector. By using Theorem 3.1 and its notation, we split the proof in the following cases:

a) $c_1, c_2 \in \Omega_3$. Then $\gamma_1 = \gamma_2 = 0$ and by (2) we get $\mathbf{G}_1\mathbf{G}_2 = \mathbf{O}$.

b) $c_1 \in \Omega_3$ and there exists $r \in \{0,1,2\}$ such that $\omega_3^r c_1 + c_2 \in \{0\} \cup \Omega_3$. Since $c_1 \in \Omega_3$ then $\gamma_1 = 0$. Since $(\omega_3^r c_1 + c_2)^2 = \overline{\omega_3^r c_1 + c_2}$, a simple computation shows that $2\omega_3^r = \gamma_2$ for every $r = 0,1,2$. From (2) we get $\mathbf{G}_1\mathbf{G}_2 = \omega_3^r \mathbf{G}_2^*$.

c) $c_2 \in \Omega_3$ and there exists $s \in \{0,1,2\}$ such that $c_1 + c_2\omega_3^s \in \{0\} \cup \Omega_3$. This case is completely similar to the previous one.

d) There exist $r, s \in \{0,1,2\}$ such that any of the following conditions occurs:

i) $r+s$ is a multiple of 3 and $\omega_3^r c_1 + c_2 \in \{0\} \cup \Omega_3$.

ii) $r+s$ is not a multiple of 3 and there exist $\varphi_1, \varphi_2 \in \{0\} \cup \Omega_3$ (i.e., $\varphi_1^2 = \overline{\varphi_1}$ and $\varphi_2^2 = \overline{\varphi_2}$) such that

$$c_1 = \frac{\varphi_1 \omega_3^s - \varphi_2}{\omega_3^{r+s} - 1}, \qquad c_2 = \frac{\varphi_2 \omega_3^r - \varphi_1}{\omega_3^{r+s} - 1}.$$

Since $\varphi_1 = \omega_3^r c_1 + c_2$ and $\varphi_2 = c_1 + \omega_3^s c_2$ then

$$\omega_3^{2r}\gamma_1 + \gamma_2 = 2\omega_3^r, \qquad \gamma_1 + \omega_3^{2s}\gamma_2 = 2\omega_3^s.$$

This linear system, whose unknowns are γ_1 and γ_2, has a unique solution because the determinant of the coefficient matrix is $\omega_3^{2(r+s)} - 1 \neq 0$ (if $\omega_3^{2(r+s)} = 1$ then $2(r+s)$ would be a multiple of 3). The solution is then

$$\gamma_1 = \frac{2\omega_3^s}{\omega_3^{r+s} + 1}, \qquad \gamma_2 = \frac{2\omega_3^r}{\omega_3^{r+s} + 1}.$$

So, equation (2) simplifies to

$$\mathbf{G}_1\mathbf{G}_2 = \frac{1}{\omega_3^{r+s} + 1}(\omega_3^s \mathbf{G}_1^* + \omega_3^r \mathbf{G}_2^*).$$

The sufficiency can be easily obtained by using equation (2). □

For noncommuting generalized projectors $\mathbf{G}_1$ and $\mathbf{G}_2$, Baksalary and Baksalary in [2] proved that $c_1\mathbf{G}_1 + c_2\mathbf{G}_2$ is a generalized projector if and only if $\mathbf{G}_1\mathbf{G}_2 + \mathbf{G}_2\mathbf{G}_1 = \gamma_1\mathbf{G}_1^* + \gamma_2\mathbf{G}_2^*$ and $\gamma_1\gamma_2 = 1$, where $\gamma_i = (\overline{c_i} - c_1^2)/(c_1c_2)$ for $i = 1, 2$. Theorem 3.3 below gives a neccesary condition where the computation of γ_1 and γ_2 is easier than the aforesaid in [2]. This condition will be not sufficient as we show with an example. In order to prove Theorem 3.3 we shall need the following simple lemma.

Lemma 3.1 *Let $\mathbf{G}_1, \mathbf{G}_2 \in \mathbb{C}^{n\times n}$ normal matrices. Then the following statements are equivalent:*

(1) $\mathbf{G}_1$ and $\mathbf{G}_2$ commute.
(2) $\mathbf{G}_1$ and $\mathbf{G}_2^*$ commute.
(3) $\mathbf{G}_1^*$ and $\mathbf{G}_2$ commute.

Proof: We will prove only $1 \Rightarrow 2$ because the other implications are similar. If $\mathbf{G}_1$ and $\mathbf{G}_2$ commute then there exist a unitary matrix $\mathbf{U}$ and two diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ such that $\mathbf{G}_i = \mathbf{U}\mathbf{D}_i\mathbf{U}^*$ for $i = 1, 2$. Now it is easy to prove that $\mathbf{G}_1$ and $\mathbf{G}_2^*$ commute. □

Theorem 3.3 *Let $\mathbf{G}_1, \mathbf{G}_2 \in \mathbb{C}^{n\times n}$ generalized projectors such that $\mathbf{G}_1\mathbf{G}_2 \neq \mathbf{G}_2\mathbf{G}_1$. If $\mathbf{G} = c_1\mathbf{G}_1 + c_2\mathbf{G}_2$ is a generalized projector then $\gamma_1\mathbf{M} = -\mathbf{M}^*$, $\gamma_2\mathbf{M}^* = -\mathbf{M}$, $\gamma_1\gamma_2 = 1$ and $|\gamma_1| = |\gamma_2| = 1$, where $\mathbf{M} = \mathbf{G}_2\mathbf{G}_1^* - \mathbf{G}_1^*\mathbf{G}_2$ and $\gamma_i = (\overline{c_i} - c_i^2)/(c_1c_2)$ for $i = 1, 2$.*

Proof: Since $\mathbf{G}_1$, $\mathbf{G}_2$ and $c_1\mathbf{G}_1 + c_2\mathbf{G}_2$ are generalized projectors then

$$\gamma_1\mathbf{G}_1^* + \gamma_2\mathbf{G}_2^* = \mathbf{G}_1\mathbf{G}_2 + \mathbf{G}_2\mathbf{G}_1. \tag{3}$$

Premultiplying and postmultiplying (3) by $\mathbf{G}_2$ we get $\gamma_1\mathbf{G}_2\mathbf{G}_1^* + \gamma_2\mathbf{G}_2\mathbf{G}_2^* = \mathbf{G}_2\mathbf{G}_1\mathbf{G}_2 + \mathbf{G}_2^*\mathbf{G}_1$ and $\gamma_1\mathbf{G}_1^*\mathbf{G}_2 + \gamma_2\mathbf{G}_2^*\mathbf{G}_2 = \mathbf{G}_1\mathbf{G}_2^* + \mathbf{G}_2\mathbf{G}_1\mathbf{G}_2$. By Theorem 2.1, $\mathbf{G}_2$ is a normal matrix. Substracting the above equations we obtain $\gamma_1(\mathbf{G}_2\mathbf{G}_1^* - \mathbf{G}_1^*\mathbf{G}_2) = \mathbf{G}_2^*\mathbf{G}_1 - \mathbf{G}_1\mathbf{G}_2^*$, which leads to

$$\gamma_1\mathbf{M} = -\mathbf{M}^*. \tag{4}$$

By Theorem 2.1, matrix $\mathbf{G}_1$ is also normal. Analogously, premultiplying and postmultiplying (3) by $\mathbf{G}_1$ we get

$$\gamma_2\mathbf{M}^* = -\mathbf{M}. \tag{5}$$

From equations (4) and (5) we get

$$\mathbf{M}^* = -\gamma_1\mathbf{M} = \gamma_1\gamma_2\mathbf{M}^*. \tag{6}$$

By Lemma 3.1 we get $\mathbf{M} \neq \mathbf{O}$, i.e. $\mathbf{M}^* \neq \mathbf{O}$ and by (6) we obtain $\gamma_1\gamma_2 = 1$. Now, from (4) and (5), it follows that $\overline{\gamma_1}\mathbf{M}^* = -\mathbf{M} = \gamma_2\mathbf{M}^* = \gamma_1^{-1}\mathbf{M}^*$. Since $\mathbf{M}^* \neq \mathbf{O}$ then $|\gamma_1| = 1$ and hence $|\gamma_2| = 1$. The proof is now completed. □

Note that any set of the following conditions:

(c.1) $\gamma_1\mathbf{M} = -\mathbf{M}^*$, $\gamma_1\gamma_2 = 1$.
(c.2) $\gamma_2\mathbf{M}^* = -\mathbf{M}$, $\gamma_1\gamma_2 = 1$.

are equivalent. Then one of these conditions (c.1) or (c.2) may be deleted in Theorem 3.3.

Observe that Theorem 3.3 gives us a procedure in order to find the nonzero numbers $c_1, c_2 \in \mathbb{C}$ such that $c_1\mathbf{G}_1 + c_2\mathbf{G}_2$ is a generalized projector provided that $\mathbf{G}_1$ and $\mathbf{G}_2$ are noncommuting generalized projectors with the same size: i) Compute $\mathbf{M} = \mathbf{G}_2\mathbf{G}_1^* - \mathbf{G}_1^*\mathbf{G}_2$. ii) If there not exists $\gamma_1 \in \mathbb{C}$ such that $\gamma_1\mathbf{M} = -\mathbf{M}^*$ and $|\gamma_1| = 1$ then this problem has not solution. iii) In the other case, let $\gamma_2 = 1/\gamma_1$ and there will be a solution if and only if $\gamma_1\mathbf{G}_1^* + \gamma_2\mathbf{G}_2^* = \mathbf{G}_1\mathbf{G}_2 + \mathbf{G}_2\mathbf{G}_1$. iv) The solutions (if they exist) will satisfy $(\overline{c_i} - c_i^2)/(c_1c_2) = \gamma_i$ for $i = 1, 2$.

Note that the above conditions in Theorem 3.3 are necessary but not sufficient as we can see in the following example. Let $\gamma_1 = \gamma_2 = 1$,

$$\mathbf{G}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} x & z \\ y & t \end{pmatrix}, \quad \mathbf{G}_2 = \mathbf{U}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{U}^*$$

such that matrix $\mathbf{U}$ is unitary and $xy \neq 0$. It is clear that $\mathbf{G}_1$ and $\mathbf{G}_2$ are generalized projectors and a simple computation shows that

$$\mathbf{G}_1\mathbf{G}_2 = \begin{pmatrix} 0 & 0 \\ y\overline{x} & |y|^2 \end{pmatrix}, \qquad \mathbf{G}_2\mathbf{G}_1 = \begin{pmatrix} 0 & x\overline{y} \\ 0 & |y|^2 \end{pmatrix},$$

which gives $\mathbf{G}_1\mathbf{G}_2 \neq \mathbf{G}_2\mathbf{G}_1$. Computing the matrix $\mathbf{M} = \mathbf{G}_2\mathbf{G}_1^* - \mathbf{G}_1^*\mathbf{G}_2$ we have

$$\mathbf{M} = \begin{pmatrix} 0 & x\overline{y} \\ -y\overline{x} & 0 \end{pmatrix}.$$

It is clear that $\mathbf{M}^* = -\mathbf{M}$ but however $c_1\mathbf{G}_1 + c_2\mathbf{G}_2$ is not a generalized projector since $\gamma_1\mathbf{G}_1^* + \gamma_2\mathbf{G}_2^* \neq \mathbf{G}_1\mathbf{G}_2 + \mathbf{G}_2\mathbf{G}_1$.

The same technique used in Theorem 3.1 can be used in order to obtain the following important particular case of orthogonal projectors:

Theorem 3.4 *For nonzero $c_1, c_2 \in \mathbb{C}$, $k \in \mathbb{N}$ with $k > 1$ and nonzero orthog-*

onal projectors $\mathbf{G}_1, \mathbf{G}_2 \in \mathbb{C}^{n\times n}$ such that $\mathbf{G}_1 \neq \mathbf{G}_2$, let $\mathbf{G} = c_1\mathbf{G}_1 + c_2\mathbf{G}_2$. If $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_2\mathbf{G}_1$ then $\mathbf{G}$ is a k-generalized projector if and only if any of the following disjoint sets of conditions holds:

a) $c_1, c_2 \in \Omega_{k+1}$.
  i) $(c_1 + c_2)^k = c_1^k + c_2^k$.
  ii) $(c_1 + c_2)^k \neq c_1^k + c_2^k$ *and* $\mathbf{G}_1\mathbf{G}_2 = \mathbf{O}$.

b) $c_1 \in \Omega_{k+1}$, $c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$, $c_2 \notin \Omega_{k+1}$.
  i) $c_1 + c_2 = 0$, *k is even and* $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_2$.
  ii) $c_1 + c_2 \neq 0$ *and* $((c_1 + c_2)^k - (c_1^k + c_2^k))\mathbf{G}_1\mathbf{G}_2 = (\overline{c_2} - c_2^k)\mathbf{G}_2$.

c) $c_2 \in \Omega_{k+1}$, $c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$, $c_1 \notin \Omega_{k+1}$.
  i) $c_1 + c_2 = 0$, *k is even and* $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_1$.
  ii) $c_1 + c_2 \neq 0$ *and* $((c_1 + c_2)^k - (c_1^k + c_2^k))\mathbf{G}_1\mathbf{G}_2 = (\overline{c_1} - c_1^k)\mathbf{G}_1$.

d) $c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$, $c_1 \notin \Omega_{k+1}$ *and* $c_2 \notin \Omega_{k+1}$.
  i) $c_1 + c_2 = 0$, *k is even and* $2\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_1 + \mathbf{G}_2 - \overline{c_1}c_1^{-k}(\mathbf{G}_1 - \mathbf{G}_2)$.
  ii) $c_1 + c_2 \neq 0$ *and* $((c_1 + c_2)^k - (c_1^k + c_2^k))\mathbf{G}_1\mathbf{G}_2 = (\overline{c_1} - c_1^k)\mathbf{G}_1 + (\overline{c_2} - c_2^k)\mathbf{G}_2$.

Proof: The following observation will be useful: if $k \in \mathbb{N}$ and $k > 1$ then $\lambda \in \{0\} \cup \Omega_{k+1}$ if and only if $\overline{\lambda} = \lambda^k$. Since matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ are projectors and $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_2\mathbf{G}_1$ then by applying the binomial theorem it is easy to check that

$$\mathbf{G}^k = (c_1\mathbf{G}_1 + c_2\mathbf{G}_2)^k = c_1^k\mathbf{G}_1 + ((c_1 + c_2)^k - (c_1^k + c_2^k))\mathbf{G}_1\mathbf{G}_2 + c_2^k\mathbf{G}_2.$$

So, we get

$$\mathbf{G}^k = \mathbf{G}^* \iff ((c_1+c_2)^k - (c_1^k + c_2^k))\mathbf{G}_1\mathbf{G}_2 = (\overline{c_1} - c_1^k)\mathbf{G}_1 + (\overline{c_2} - c_2^k)\mathbf{G}_2, \quad (7)$$

because $\mathbf{G}_i^* = \mathbf{G}_i$ for $i = 1, 2$.

Suppose that $\mathbf{G}$ is a k-generalized projector. Analogously, as in the proof of Theorem 3.1, there exist a unitary matrix $\mathbf{U}$ and diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ such that $\mathbf{G}_i = \mathbf{U}\mathbf{D}_i\mathbf{U}^*$ for $i = 1, 2$ and $\mathbf{G} = \mathbf{U}(c_1\mathbf{D}_1 + c_2\mathbf{D}_2)\mathbf{U}^*$. Let $\mathbf{D}_1 = \mathrm{diag}(\lambda_{11}, \ldots, \lambda_{1n})$ and $\mathbf{D}_2 = \mathrm{diag}(\lambda_{21}, \ldots, \lambda_{2n})$ with $\lambda_{ij} \in \{0, 1\}$ for $i = 1, 2$ and $j = 1, \ldots, n$. By Theorem 2.1, the eigenvalues of $c_1\mathbf{D}_1 + c_2\mathbf{D}_2$ are elements in $\{0\} \cup \Omega_{k+1}$ and hence

$$c_1\lambda_{1j} + c_2\lambda_{2j} \in \{0\} \cup \Omega_{k+1}, \qquad \forall\, j = 1, \ldots, n. \quad (8)$$

Since $\mathbf{D}_1 \neq \mathbf{O}$, there exists $j \in \{1, \ldots, n\}$ such that $\lambda_{1j} \neq 0$ and so $\lambda_{1j} = 1$. From (8), we get $c_1 + c_2\lambda_{2j} \in \{0\} \cup \Omega_{k+1}$. Since $\lambda_{2j} \in \{0, 1\}$ then $c_1 \in \Omega_{k+1}$ or $c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$. By applying a similar argument for $\mathbf{D}_2 \neq \mathbf{O}$ we obtain that $c_2 \in \Omega_{k+1}$ or $c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$. So, we split the study in the following disjoint cases:

a) $c_1, c_2 \in \Omega_{k+1}$. From (7) we get $((c_1 + c_2)^k - (c_1^k + c_2^k))\mathbf{G}_1\mathbf{G}_2 = \mathbf{O}$. So, case a) of the theorem has just been obtained.

b) $c_1 \in \Omega_{k+1}$, $c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$ and $c_2 \notin \Omega_{k+1}$. We split this case depending on the value of $c_1 + c_2$ and the parity of k:

   i) If $c_1 + c_2 = 0$ and k is even then $c_1^k = c_2^k$ and since $c_1^{k+1} = 1$ then $c_2^{k+1} = -1$, that is $c_2^k = -1/c_2$ and $|c_2| = 1$. Replacing in (7) we get $-2c_2^k \mathbf{G}_1\mathbf{G}_2 = (\overline{c_2} - c_2^k)\mathbf{G}_2$. This equation yields to $2\mathbf{G}_1\mathbf{G}_2 = (1 - \overline{c_2}c_2^{-k})\mathbf{G}_2 = (1 + \overline{c_2}c_2)\mathbf{G}_2 = 2\mathbf{G}_2$. The case $c_1 + c_2 = 0$ and k odd yields to a contradiction because $c_1^k = -c_2^k$ and $1 = c_1^{k+1} = c_2^{k+1}$.

   ii) If $c_1 + c_2 \neq 0$, from (7) we get $((c_1+c_2)^k - (c_1^k + c_2^k))\mathbf{G}_1\mathbf{G}_2 = (\overline{c_2} - c_2^k)\mathbf{G}_2$.

c) $c_2 \in \Omega_{k+1}$, $c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$ and $c_1 \notin \Omega_{k+1}$. This case is completely similar to the previous one.

d) $c_1 + c_2 \in \{0\} \cup \Omega_{k+1}$, $c_1 \notin \Omega_{k+1}$ and $c_2 \notin \Omega_{k+1}$. We split this case depending on the value of $c_1 + c_2$ and the parity of k:

   i) If $c_1 + c_2 = 0$ and k is even, similarly to case a), $c_1^k = c_2^k$ and from (7) we get $2\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_1 + \mathbf{G}_2 - \overline{c_1}c_1^{-k}(\mathbf{G}_1 - \mathbf{G}_2)$. If $c_1 + c_2 = 0$ and k odd, we obtain a contradicition because equation (7) gives $(\overline{c_1} - c_1^k)(\mathbf{G}_1 - \mathbf{G}_2) = \mathbf{O}$.

   ii) $c_1 + c_2 \neq 0$. No further simplification can be made in (7).

The sufficiency follows by a direct computation. □

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