Matrices A such that $AA^\dagger - A^\dagger A$ are nonsingular

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Abstract

In this paper we study the class of square matrices A such that $AA^\dagger - A^\dagger A$ is nonsingular, where $A^\dagger$ stands for the Moore–Penrose inverse of A. Among several characterizations we prove that for a matrix A of order n, the difference $AA^\dagger - A^\dagger A$ is nonsingular if and only if $\mathcal{R}(A) \oplus \mathcal{R}(A^*) = \mathbb{C}_{n,1}$, where $\mathcal{R}(\cdot)$ denotes the range space. Also we study matrices A such that $\mathcal{R}(A)^\perp = \mathcal{R}(A^*)$.

Key words. Co-EP matrices. Moore-Penrose inverse. Idempotent. Orthogonal projectors.

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1 Introduction

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ matrices. A matrix P is said to be a *projector* when $P = P^2$; it is an *orthogonal projector* when $P^2 = P = P^*$, where P^* is the conjugate transpose of P. For a nonsingular matrix M, we shall denote $M^{-*} = (M^*)^{-1} = (M^*)^{-1}$. The symbols $\mathrm{rk}(K)$, $\mathcal{R}(K)$, and $\mathcal{N}(K)$ will denote the rank, the range, and the null space, respectively, of $K \in \mathbb{C}_{m,n}$. For a given subspace $\mathcal{X}$ of $\mathbb{C}_{n,1}$, the symbol $\mathcal{X}^\perp$ will mean the orthogonal complement of $\mathcal{X}$.

Further, $K^\dagger$ will stand for the Moore–Penrose inverse of $K \in \mathbb{C}_{m,n}$, i.e., the unique matrix satisfying the four equations

$$KK^\dagger K = K, \quad K^\dagger K K^\dagger = K^\dagger, \quad (KK^\dagger)^* = KK^\dagger, \quad (K^\dagger K)^* = K^\dagger K. \tag{1}$$

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It is evident that if $K \in \mathbb{C}_{m,n}$, then $KK^\dagger$ and $K^\dagger K$ are orthogonal projectors. Moreover, it can be easily proved that $KK^\dagger$ is the orthogonal projector onto $\mathcal{R}(K)$ and $K^\dagger K$ is the orthogonal projector onto $\mathcal{R}(K^*)$.

We shall use the called CS-decomposition which is now established (see e.g. [1, 2, 3] and for a survey of this decomposition, [4]):

Lemma 1.1 (CS decomposition)**.** *Let $P_1, P_2 \in \mathbb{C}_{n,n}$ be two orthogonal projectors. Then there exists a unitary matrix $U \in \mathbb{C}_{n,n}$ such that*

$$P_1 = U \begin{bmatrix} I & & & & & \\ & 0 & & & & \\ & & I & & & \\ & & & I & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix} U^*, \qquad P_2 = U \begin{bmatrix} \hat{C}^2 & \hat{C}\hat{S} & & & & \\ \hat{C}\hat{S} & \hat{S}^2 & & & & \\ & & I & & & \\ & & & 0 & & \\ & & & & I & \\ & & & & & 0 \end{bmatrix} U^*,$$

where $\hat{C}, \hat{S}$ are positive diagonal real matrices such that $\hat{C}^2 + \hat{S}^2 = I$, the symbol I denotes identity matrices of various sizes, and the corresponding blocks in the two projection matrices are of the same size.

This decomposition is strongly related with the canonical angles (also called principal angles) between two subspaces (see [5]). This angles provide the best available characterization of the relative position of two given subspaces. This concept allows us to characterize or measure, in a natural way, how two subspaces differ, which is the main connection with perturbation theory. In [6, 4, 7] we can find how these angles were discovered and rediscovered again several times. Computation of canonical angles between subspaces is important in many applications including statistics [8, 9, 10], information retrieval [11], perturbation theory [6, 12], and analysis of algorithms [13].

A square matrix A is said to be EP when $AA^\dagger = A^\dagger A$ (for more on EP matrices see for example [14, Chapter 4.4] and [15, Chapter 4]). In some sense, the class of matrices A such that $AA^\dagger - A^\dagger A$ is nonsingular is complementary of the subset of EP matrices. Precisely, the purpose of this paper is to study this class of matrices.

Definition 1.2. *We shall say that a square matrix A is* co-EP *when $AA^\dagger - A^\dagger A$ is nonsingular.*

Recall that $AA^\dagger$ and $A^\dagger A$ are orthogonal projectors, thus the characterization of the nonsingularity of $AA^\dagger - A^\dagger A$ is strongly related with the following problem: find equivalent conditions that ensure the nonsingularity of $P - Q$, where P and Q are orthogonal projectors. In 1964 Vidav [16] found necessary and sufficient conditions for the invertibility of $P - Q$ in the case of Hilbert space operators. Pták [17], apparently unaware of the work of Vidav, gave in 1985 a solution of this problem and applied it to extremal operators. Recently this topic was revisited by Buckholtz [18, 19], Galántai, [20], Wimmer [21, 22], by Koliha and Rakočević [23, 24, 26] and by Koliha, Rakočević, and Straškaba in [25]. In [27] it was used the CS decomposition with the purpose of studying some properties of the linear combination $aP + bQ$, where a, b are nonzero complex numbers and $P, Q \in \mathbb{C}_{n,n}$ are orthogonal projectors.

In [28] it was shown how the spectrum of linear combinations of orthogonal projectors determine the convergence of many parallel iterative algorithms (recall that $\lambda = 0$ belongs to the spectrum of a matrix if and only if this matrix is singular). Moreover, the *gap* between two equidimensional subspaces $\mathcal{X}$ and $\mathcal{Y}$ (defined as $\|P_\mathcal{X} - P_\mathcal{Y}\|$, where $P_\mathcal{X}$ and $P_\mathcal{Y}$ are the orthogonal projectors onto $\mathcal{X}$ and $\mathcal{Y}$, respectively) has found many important applications (see, for example, [2, 29]).

As a further application of our results (see Corollary 2.4) we find the SVD decomposition of $AA^\dagger - A^\dagger A$, where A is co-EP (in particular we obtain $\|AA^\dagger - A^\dagger A\|$ since the Euclidean norm of a matrix is the largest singular value of this matrix). Observe that if $\mathcal{X}$ and $\mathcal{Y}$ are two equidimensional subspaces of $\mathbb{C}^n$, then we can always construct a matrix A such that $\mathcal{R}(A) = \mathcal{X}$ and $\mathcal{R}(A^*) = \mathcal{Y}$ (it is enough to take A as the oblique projector onto $\mathcal{X}$ along $\mathcal{Y}^\perp$, since obviously $\mathcal{R}(A) = \mathcal{X}$ and $\mathcal{R}(A^*) = [\mathcal{N}(A)]^\perp = [\mathcal{Y}^\perp]^\perp = \mathcal{Y}$).

We also provide several expressions for $A^\dagger$ and $AA^\dagger - AA^\dagger$ if A is a co-EP matrix (see e.g., (19), (20), and Corollary 2.10).

2 Main results

A useful characterization of EP matrices is the following (see [14, Chapter 4.4] or [15, Theorem 4.3.1]):

Theorem 2.1. *Let $A \in \mathbb{C}_{n,n}$ have rank r. The following are equivalent:*

(i) $AA^\dagger = A^\dagger A$.

(ii) *There exist a unitary matrix $U \in \mathbb{C}_{n,n}$ and a nonsingular matrix $K \in \mathbb{C}_{r,r}$, such that $A = U(K \oplus 0)U^*$.*

(iii) $\mathcal{R}(A) = \mathcal{R}(A^*)$.

(iv) $\mathcal{N}(A) = \mathcal{N}(A^*)$.

(v) $\mathbb{C}^n = \mathcal{N}(A) \oplus^\perp \mathcal{R}(A)$, *with the symbol "$\oplus^\perp$" being used to indicate that the two subspaces involved in the direct sum are orthogonal.*

In Theorem 2.3 below, we shall use the CS decomposition in order to find a similar characterization for co-EP matrices.

On the other hand, according to Corollary 6 in [30], every square matrix $A \in \mathbb{C}_{n,n}$ of rank r can be represented in the form

$$A = V \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} V^*, \tag{2}$$

where $V \in \mathbb{C}_{n,n}$ is unitary, $\Sigma = \sigma_1 I_{r_1} \oplus \cdots \oplus \sigma_t I_{r_t}$ is the diagonal matrix of the nonzero singular values of A, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_t > 0$, $r_1 + \cdots + r_t = r$, and $K \in \mathbb{C}_{r,r}, L \in \mathbb{C}_{r,n-r}$ satisfy

$$KK^* + LL^* = I_r. \tag{3}$$

It can be proved, by checking the four conditions in (1) that if A is represented as in (2), then

$$A^\dagger = V \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} V^*. \tag{4}$$

It is straightforward to prove that if a square matrix A is represented as in (2), then $AA^\dagger - A^\dagger A = 0$ (i.e., A is EP) if and only if $L = 0$. In Theorem 2.3 below, we will prove that $AA^\dagger - A^\dagger A$ is nonsingular if and only A can be represented as in (2) with L nonsingular.

The following expression for the Moore–Penrose inverse will be useful in the sequel (see for example [14, Chapter 3.3]).

Theorem 2.2. *For any $A \in \mathbb{C}_{m,n}$ we have*

$$\lim_{t \to 0+} (A^*A + tI_n)^{-1}A^* = A^\dagger.$$

Next result characterizes the class of co-EP matrices.

Theorem 2.3. *Let $A \in \mathbb{C}_{n,n}$ and r be the rank of A. The following are equivalent:*

(i) *$AA^\dagger - A^\dagger A$ is nonsingular.*

(ii) $\mathcal{R}(A) \oplus \mathcal{R}(A^*) = \mathbb{C}_{n,1}$.

(iii) *There exist a unitary matrix $U \in \mathbb{C}_{n,n}$, a nonsingular matrix $M \in \mathbb{C}_{r,r}$, and $\theta_1, \ldots, \theta_r \in]0, \pi/2]$ such that*

$$A = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*, \tag{5}$$

where $C = \mathrm{diag}(\cos\theta_1, \ldots, \cos\theta_r)$, and $S = \mathrm{diag}(\sin\theta_1, \ldots, \sin\theta_r)$.

(iv) *A can be represented as in (2) being L nonsingular.*

(v) *$AA^\dagger + A^\dagger A$ is nonsingular and $\|A(A^\dagger)^2A\| < 1$.*

(vi) *$AA^* + A^*A$ is nonsingular and $\mathcal{R}(A) \cap \mathcal{R}(A^*) = \{0\}$.*

(vii) *$A+A^*$ is nonsingular and there exists a unique projector $P \in \mathbb{C}_{n,n}$ such that $AP = A$, $P^*A = 0$,*

(viii) *$A-A^*$ is nonsingular and there exists a unique projector $P \in \mathbb{C}_{n,n}$ such that $AP = A$, $P^*A = 0$,*

(ix) *$A + A^*$ is nonsingular, $A(A + A^*)^{-1}A = A$, and $A^*(A + A^*)^{-1}A = 0$.*

(x) *$A - A^*$ is nonsingular, $A(A - A^*)^{-1}A = A$, and $A^*(A - A^*)^{-1}A = 0$.*

Proof. (i) $\Leftrightarrow$ (ii): Since $AA^\dagger$ and $A^\dagger A$ are orthogonal projectors, Theorem 6.2 in [24] (see also [19, Th. 1] and [18]) permits affirm that $AA^\dagger - A^\dagger A$ is nonsingular if and only if $\mathcal{R}(AA^\dagger) \oplus \mathcal{R}(A^\dagger A) = \mathbb{C}_{n,1}$. Moreover, having in mind that $AA^\dagger$ is the orthogonal projector onto $\mathcal{R}(A)$ and $A^\dagger A$ is the orthogonal projector onto $\mathcal{R}(A^*)$, then we get that $AA^\dagger - A^\dagger A$ is nonsingular if and only if $\mathcal{R}(A) \oplus \mathcal{R}(A^*) = \mathbb{C}_{n,1}$.

(i) $\Rightarrow$ (iii): Since $AA^\dagger$ and $A^\dagger A$ are orthogonal projectors, by Lemma 1.1, there exist a unitary matrix $U \in \mathbb{C}_{n,n}$ and $p, x, y, z \in \{0\} \cup \mathbb{N}$ such that

$$AA^\dagger = U(T_1 \oplus R_1)U^*, \qquad A^\dagger A = U(T_2 \oplus R_2)U^*, \tag{6}$$

where

$$T_1 = \begin{bmatrix} I_p & \\ & 0 \end{bmatrix} \in \mathbb{C}_{2p,2p}, \quad T_2 = \begin{bmatrix} \hat{C}^2 & \hat{C}\hat{S} \\ \hat{C}\hat{S} & \hat{S}^2 \end{bmatrix} \in \mathbb{C}_{2p,2p}, \tag{7}$$

$$R_1 = I_x \oplus I_y \oplus 0 \oplus 0 \in \mathbb{C}_{n-2p,n-2p}, \quad R_2 = I_x \oplus 0 \oplus I_z \oplus 0 \in \mathbb{C}_{n-2p,n-2p}, \tag{8}$$

and in addition $\hat{C}, \hat{S} \in \mathbb{C}_{p,p}$ have the same meaning as in Lemma 1.1. Let us denote $t = (n-2p) - (x+y+z)$ in order to the last summands in (8) have order t. If $p = 0$, then blocks T_1 and T_2 do not appear in (6). Moreover, some blocks in the representation of R_1 and R_1 in (8) can also be absent. A simple computation shows

$$AA^\dagger - A^\dagger A = U\left((T_1 - T_2) \oplus (0 \oplus I_y \oplus -I_z \oplus 0)\right)U^*.$$

By hypothesis, $AA^\dagger - A^\dagger A$ is nonsingular, thus $x = t = 0$. From representations (6), (7), and (8) we get $\mathrm{rk}(AA^\dagger) = p + y$ and $\mathrm{rk}(A^\dagger A) = p + z$, because $\mathrm{rk}(T_1) = \mathrm{rk}(T_2) = p$. Since $AA^\dagger$ and $A^\dagger A$ are the orthogonal projectors onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively, we have $\mathrm{rk}(AA^\dagger) = \mathrm{rk}(A)$ and $\mathrm{rk}(A^\dagger A) = \mathrm{rk}(A^*)$. Since $\mathrm{rk}(A) = \mathrm{rk}(A^*)$, we deduce $y = z$. Let us define $r = p + z = p + y$, and therefore, $n = 2r$.

Thus, rearranging the entries of $AA^\dagger$ and $A^\dagger A$ in representations (6), (7), and (8), we can suppose

$$AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U, \qquad A^\dagger A = U \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix} U, \tag{9}$$

where $C = \mathrm{diag}(\cos\theta_1, \ldots, \cos\theta_r)$, $S = \mathrm{diag}(\sin\theta_1, \ldots, \sin\theta_r)$, and $\theta_1, \ldots, \theta_r \in]0, \pi/2]$ (observe that the elements of $\hat{C}$ in (7) are not zero, whereas if $y > 0$, then some entries of C in (9) are zero). Notice that S is nonsingular and $C^2 + S^2 = I_r$. Now, let us represent

$$A = U \begin{bmatrix} X & Y \\ Z & T \end{bmatrix} U^*, \qquad X, Y, Z, T \in \mathbb{C}_{r,r}.$$

From $A = AA^\dagger A$ and the first identity of (9) we have

$$\begin{bmatrix} X & Y \\ Z & T \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}.$$

Hence $Z = 0$ and $T = 0$. From $A = AA^\dagger A$ and the second identity of (9) we obtain

$$\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix}.$$

Therefore, $X = XC^2 + YCS$. Having in mind that $C^2 + S^2 = I_r$ and the invertibility of S we deduce $XS = YC$ (observe that from $Y = XCS + YS^2$ we can not deduce $XS = YC$ because it may happen that C is singular). If we define $M = YS^{-1}$, then we get $Y = MS$ and in view of $SC = CS$, then we easily get $X = YCS^{-1} = MC$.

In order to finish the proof, it remains to prove that M is nonsingular. Let us write

$$A^\dagger = U \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} U^*, \qquad A_1, A_2, A_3, A_4 \in \mathbb{C}_{r,r}. \tag{10}$$

From $A^\dagger = A^\dagger AA^\dagger$ and the first representation of (9) we obtain

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

from which we obtain

$$A_2 = 0, \qquad A_4 = 0. \tag{11}$$

Now, we use the second representation in (9):

$$\begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix} \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix},$$

hence

$$A_1 MS = CS, \qquad A_3 MS = S^2. \tag{12}$$

Since S is nonsingular, from the second relation of (12), we deduce the invertibility of M.

(iii) $\Rightarrow$ (iv): Let $M = X\Sigma Y^*$ be the singular value decomposition of M being X and Y unitary, and Σ diagonal. Let us remark that Σ is nonsingular because M is nonsingular. From (5) we get

$$\begin{aligned} A &= U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} X\Sigma Y^* C & X\Sigma Y^* S \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} X & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} \Sigma Y^* CX & \Sigma Y^* S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X^* & 0 \\ 0 & I_{n-r} \end{bmatrix} U^*. \end{aligned}$$

Let us denote $V = U(X \oplus I_{n-r})$, $K = Y^*CX$, and $L = Y^*S$. Obviously we have

$$A = V \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} V^*,$$

V is unitary (because U and X are unitary) and L is nonsingular (because Y and S are nonsingular). Recalling that C and S are diagonal and real matrices, we get

$$KK^* + LL^* = Y^*CXX^*CY + Y^*SSY = Y^*(C^2 + S^2)Y = I_r.$$

It remains to prove that the nonzero singular values of A are the elements of the diagonal of Σ. The singular values of A are the square roots of the eigenvalues of AA^*. Since

$$AA^* = V \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} V^*V \begin{bmatrix} K^*\Sigma & 0 \\ L^*\Sigma & 0 \end{bmatrix} V^* = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^*,$$

it is clear that the nonzero singular values of A are the square roots of the diagonal of Σ.

(iv) $\Rightarrow$ (i): Assume that A is represented as in (2), with L nonsingular. Since $L \in \mathbb{C}_{r,n-r}$ is nonsingular, we get $r = n - r$. By (2), (3), and (4) we get

$$AA^\dagger - A^\dagger A = V \begin{bmatrix} I_r - K^*K & -K^*L \\ -L^*K & -L^*L \end{bmatrix} V^*. \tag{13}$$

Now, we shall prove

$$\begin{bmatrix} I_r - K^*K & -K^*L \\ -L^*K & -L^*L \end{bmatrix} \begin{bmatrix} I_r & -K^*L^{-*} \\ -L^{-1}K & -I_r \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_r \end{bmatrix}, \tag{14}$$

which in view of (13) permits to prove the nonsingularity of $AA^\dagger - A^\dagger A$. The upper-left and the lower-left entries of (14) are trivial. The other two follow from (3). For the upper-right:

$$\begin{aligned} (I_r - K^*K)(-K^*L^{-*}) + K^*L &= -K^*L^{-*} + K^*KK^*L^{-*} + K^*L \\ &= K^*(-I_r + KK^*)L^{-*} + K^*L \\ &= K^*(-LL^*)L^{-*} + K^*L \\ &= 0. \end{aligned}$$

For the lower-right:

$$(-L^*K)(-K^*L^{-*}) + L^*L = L^*(KK^*)L^{-*} + L^*L = L^*(I_r - LL^*)L^{-*} + L^*L = I_r.$$

(i) $\Rightarrow$ (v): Assume that $AA^\dagger - A^\dagger A$ is nonsingular. By [23, Th. 3.5] we get that $AA^\dagger + A^\dagger A$ is nonsingular. Let us remark $AA^\dagger \neq 0$ (if $AA^\dagger = 0$, postmultiplying by A, then we get $A = AA^\dagger A = 0$, which is unfeasible with the nonsingularity of $AA^\dagger - A^\dagger A$) and analogously, $A^\dagger A \neq 0$. By [24, Th. 6.2] we obtain $\|A(A^\dagger)^2 A\| < 1$.

(v) $\Rightarrow$ (i): Since $AA^\dagger + A^\dagger A$ is nonsingular, in the same way as in the proof of (i) $\Rightarrow$ (v), we get $AA^\dagger \neq 0$ and $A^\dagger A \neq 0$. By [24, Th. 6.2], it remains to prove $\mathbb{C}_{n,1} = \mathcal{R}(AA^\dagger) + \mathcal{R}(A^\dagger A)$. Let $x \in \mathbb{C}_{n,1}$ and let us define $y = (AA^\dagger + A^\dagger A)^{-1} x$. Now

$$x = (AA^\dagger + A^\dagger A)y = AA^\dagger y + A^\dagger A y \in \mathcal{R}(AA^\dagger) + \mathcal{R}(A^\dagger A).$$

(ii) $\Leftrightarrow$ (vi): Let us observe that $\mathcal{R}(AA^* + A^*A) = \mathcal{R}(A) + \mathcal{R}(A^*)$, which follows from [31, Th. 1]. Hence $\mathcal{R}(A) + \mathcal{R}(A^*) = \mathbb{C}_{n,1}$ if and only if $AA^* + A^*A$ is nonsingular.

(iii) $\Rightarrow$ (vii): Since A is represented as in (5), we get

$$A + A^* = U\left(\begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} CM^* & 0 \\ SM^* & 0 \end{bmatrix}\right)U^* = U\begin{bmatrix} MC + CM^* & MS \\ SM^* & 0 \end{bmatrix}U^*.$$

Since

$$\begin{bmatrix} MC + CM^* & MS \\ SM^* & 0 \end{bmatrix}\begin{bmatrix} 0 & M^{-*}S^{-1} \\ S^{-1}M^{-1} & -S^{-1}CM^{-*}S^{-1} - S^{-1}M^{-1}CS^{-1} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_r \end{bmatrix},$$

we have just proved the nonsigularity of $A + A^*$. Now, let us define $T = CS^{-1}$ and

$$P = U\begin{bmatrix} 0 & 0 \\ T & I_r \end{bmatrix}U^*. \tag{15}$$

It is easy to prove that $P^2 = P$, $AP = A$, and $P^*A = 0$. Let us prove the uniqueness: Since

$$(A + A^*)P = AP + A^*P = AP + (P^*A)^* = A,$$

then $P = (A + A^*)^{-1}A$.

(vii) $\Rightarrow$ (i): As in (iii) $\Rightarrow$ (vii), from $AP = A$, $P^*A = 0$, and the invertibility of $A + A^*$, we deduce $P = (A + A^*)^{-1}A$. Since $AA^\dagger A = A$, premultiplying by $(A + A^*)^{-1}$ we get $PA^\dagger A = P$, or equivalently, since $A^\dagger A$ is Hermitian,

$$A^\dagger A P^* = P^*. \tag{16}$$

Let us remark

$$\begin{aligned}(A+A^*)(I_n-AA^\dagger) &= A(I_n-AA^\dagger)+A^*(I_n-AA^\dagger)\\ &= A(I_n-AA^\dagger)+[(I_n-AA^\dagger)A]^*\\ &= A(I_n-AA^\dagger).\end{aligned}$$

From this, we get $I_n - AA^\dagger = (A+A^*)^{-1}A(I_n-AA^\dagger) = P(I_n-AA^\dagger) = P - PAA^\dagger$, or equivalently, $I_n - P = (I_n-P)AA^\dagger$. The Hermitancy of $AA^\dagger$ implies

$$I_n - P^* = AA^\dagger(I_n-P^*). \tag{17}$$

Now, we will prove

$$A^\dagger P = 0 \tag{18}$$

by using $A^*P = (P^*A)^* = 0$ and Theorem 2.2:

$$A^\dagger P = \left(\lim_{t\to 0+}(A^*A+tI_n)^{-1}A^*\right)P = \lim_{t\to 0}\big((A^*+tI_n)^{-1}A^*P\big) = 0.$$

We use now $AP = A$, (16), (17), and (18) in order to prove $(AA^\dagger - A^\dagger A)(I_n-P-P^*) = I_n$, which leads to the invertibility of $AA^\dagger - A^\dagger A$

$$\begin{aligned}(AA^\dagger - A^\dagger A)(I_n-P-P^*) &= AA^\dagger - AA^\dagger P - AA^\dagger P^* - A^\dagger A + A^\dagger AP + A^\dagger AP^*\\ &= AA^\dagger(I_n-P^*)+P^*\\ &= (I_n-P^*)+P^*\\ &= I_n.\end{aligned}$$

(vii) $\Leftrightarrow$ (viii): Suppose that P is a projector such that $AP = A$ and $P^*A = 0$. Since

$$(A+A^*)(2P-I_n) = 2AP - A + 2A^*P - A^* = A - A^*,$$

and $2P - I_n$ is nonsingular (because $(2P-I_n)^2 = I_n$) we have proved (vii) $\Leftrightarrow$ (viii).

(vii) $\Rightarrow$ (ix): As we have proved in (iii) $\Rightarrow$ (vii), we have $P = (A+A^*)^{-1}A$. From $AP = A$ and $A^*P = 0$ we get $A(A+A^*)^{-1}A = A$ and $A^*(A+A^*)^{-1}A = 0$, respectively.

(ix) $\Rightarrow$ (vii): Let us define $P = (A+A^*)^{-1}A$. From $A(A+A^*)^{-1}A = A$ we easily get the idempotency of P. From $A(A+A^*)^{-1}A = A$ and $A^*(A+A^*)^{-1}A = 0$ we get $AP = A$

and $A^*P = 0$, respectively. The uniqueness of the projector P is guaranteed by the proof of (iii) $\Rightarrow$ (vii).

(viii) $\Leftrightarrow$ (x): This is similar as (vii) $\Rightarrow$ (ix) and (ix) $\Rightarrow$ (vii). This finishes the proof. $\square$

Remarks: Let $A \in \mathbb{C}_{n,n}$ be a co-EP matrix.

(i) It is evident that $n = 2\,\mathrm{rk}(A)$.

(ii) If A is represented as in (5), then the proof of (i) $\Rightarrow$ (iii) shows, from (10), (11), and (12) that

$$A^\dagger = U \begin{bmatrix} CM^{-1} & 0 \\ SM^{-1} & 0 \end{bmatrix} U^*. \tag{19}$$

(iii) If P is the projector given by Theorem 2.3, the proof of (vii) $\Rightarrow$ (i) shows

$$(AA^\dagger - A^\dagger A)^{-1} = I_n - P - P^*. \tag{20}$$

(iv) Let $B \in \mathbb{C}_{n,n}$. We have B is co-EP if and only if B^* is co-EP. Thus, we have double results for Theorem 2.3.

Corollary 2.4. *Let $A \in \mathbb{C}_{n,n}$ be a co-EP matrix. If $\theta_1, \dots, \theta_r$ are the angles given in Theorem 2.3 (iii), then the singular values of $AA^\dagger - A^\dagger A$ are $\sin\theta_1, \dots, \sin\theta_r$, being their multiplicity double as singular values. In particular, we have $\|AA^\dagger - A^\dagger A\| = \max\{\sin\theta_i, i = 1, \dots, r\}$.*

Proof. By using representations (5) and (19) we easily have

$$AA^\dagger - A^\dagger A = U \begin{bmatrix} S^2 & -CS \\ -CS & -S^2 \end{bmatrix} U^*.$$

the proof finishes by checking

$$\begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} 0 & -I_r \\ -I_r & 0 \end{bmatrix} = \begin{bmatrix} S^2 & -CS \\ -CS & -S^2 \end{bmatrix}$$

and by observing that

$$\begin{bmatrix} C & -S \\ S & C \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -I_r \\ -I_r & 0 \end{bmatrix}$$

are unitary. $\square$

If $A, B \in \mathbb{C}_{n,n}$ are EP matrices, then it is known necessary and sufficient conditions that ensure that AB is EP (see [32, 33]). In corollary below, a related result is established for co-EP matrices.

Corollary 2.5. *Let $A, B \in \mathbb{C}_{n,n}$ be co-EP matrices. Then AB is co-EP if and only if*

(1) $\mathcal{R}(AB) = \mathcal{R}(A)$.

(2) $\mathcal{N}(AB) = \mathcal{N}(B)$.

(3) $\mathcal{R}(A) \cap \mathcal{R}(B^*) = \{0\}$.

Proof. If AB is co-EP, then $\mathcal{R}(AB) \oplus \mathcal{R}(B^*A^*) = \mathbb{C}_{n,1}$ and $\mathrm{rk}(AB) = \mathrm{rk}(B^*A^*) = n/2$. Now, $\mathcal{R}(AB) \subset \mathcal{R}(A)$, $\mathcal{R}(B^*A^*) \subset \mathcal{R}(B^*)$, and $\mathrm{rk}(A) = \mathrm{rk}(B^*) = n/2$ imply (1), $\mathcal{R}(B^*A^*) = \mathcal{R}(B^*)$ (thus we have (2)) and (3).

Suppose that (1), (2), and (3) hold. By (3) we have

$$\{0\} \subset \mathcal{R}(AB) \cap \mathcal{R}(B^*A^*) \subset \mathcal{R}(A) \cap \mathcal{R}(B^*) = \{0\}.$$

From (2) we have $\mathcal{R}(B^*A^*) = \mathcal{R}(B^*)$. Thus, $\mathrm{rk}(AB) = \mathrm{rk}(A) = n/2$ and $\mathrm{rk}(B^*A^*) = \mathrm{rk}(B^*) = n/2$, and therefore,

$$\dim\left(\mathcal{R}(AB) + \mathcal{R}(B^*A^*)\right) = \dim \mathcal{R}(AB) + \dim \mathcal{R}(B^*A^*) - \dim\left(\mathcal{R}(AB) \cap \mathcal{R}(B^*A^*)\right) = n,$$

what ensures $\mathcal{R}(AB) + \mathcal{R}(B^*A^*) = \mathbb{C}_{n,1}$. Thus we have $\mathcal{R}(AB) \oplus \mathcal{R}((AB)^*) = \mathbb{C}_{n,1}$ and the proof is complete. □

Remark: Observe that for $A, B \in \mathbb{C}_{n,n}$ we have

$$\mathcal{N}(A^*) + \mathcal{N}(B) = \mathbb{C}_{n,1} \quad \Leftrightarrow \quad \mathcal{R}(A) \cap \mathcal{R}(B)^{\perp} = \{0\} \quad \Leftrightarrow \quad \mathcal{R}(A) \cap \mathcal{R}(B^*) = \{0\}.$$

In view of the spectral theorem for normal matrices and characterization (i) $\Leftrightarrow$ (ii) of Theorem 2.1, it is easily seen that if A is a normal matrix, then A is an EP matrix. In the following corollary we state a similar result concerned with co-EP matrices.

Corollary 2.6. *If $A \in \mathbb{C}_{n,n}$ is a co-EP matrix, then*

$$(AA^\dagger - A^\dagger A)^{-1} = (A + A^*)^{-1}(A^*A - AA^*)(A + A^*)^{-1}$$

and

$$(AA^\dagger - A^\dagger A)^{-1} = (A - A^*)^{-1}(AA^* - A^*A)(A - A^*)^{-1}.$$

In particular, $AA^* - A^*A$ *is nonsingular.*

Proof. Let P be the projector given by Theorem 2.3. By the proof of Theorem 2.3, (iii) $\Rightarrow$ (vii) we get $P = (A + A^*)^{-1}A$. Matrix $(A + A^*)^{-1}$ is Hermitian since $A + A^*$ is Hermitian. Therefore

$$\begin{aligned} I_n - P - P^* &= I_n - (A + A^*)^{-1}A - A^*(A + A^*)^{-1} \\ &= (A + A^*)^{-1}\left[(A + A^*) - A - (A + A^*)A^*(A + A^*)^{-1}\right] \\ &= (A + A^*)^{-1}\left[A^* - (A + A^*)A^*(A + A^*)^{-1}\right] \\ &= (A + A^*)^{-1}\left[A^*(A + A^*) - (A + A^*)A^*\right](A + A^*)^{-1} \\ &= (A + A^*)^{-1}(A^*A - AA^*)(A + A^*)^{-1}. \end{aligned}$$

Combining this equality with (20) proves the first formula of this corollary. Next, we prove the second one: By subtracting $AP = A$ and $A^*P = 0$ we get $(A - A^*)P = A$; hence $P = (A - A^*)^{-1}A$ (the nonsingularity of $A - A^*$ is guaranteed by Theorem 2.3, (viii)). Let us remark that $A - A^*$ is skew Hermitian, and therefore, $(A - A^*)^{-1}$ is also skew Hermitian. Thus

$$\begin{aligned} I_n - P - P^* &= I_n - (A - A^*)^{-1}A - ((A - A^*)^{-1}A)^* \\ &= I_n - (A - A^*)^{-1}A + A^*(A - A^*)^{-1} \\ &= (A - A^*)^{-1}\left[(A - A^*) - A + (A - A^*)A^*(A - A^*)^{-1}\right] \\ &= (A - A^*)^{-1}\left[-A^* + (A - A^*)A^*(A - A^*)^{-1}\right] \\ &= (A - A^*)^{-1}\left[-A^*(A - A^*) + (A - A^*)A^*\right](A - A^*)^{-1} \\ &= (A - A^*)^{-1}(AA^* - A^*A)(A - A^*)^{-1}. \end{aligned}$$

This finishes the proof. □

Examples: In Theorem 2.3 (vi), (vii), (viii), and Corollary 2.6 it is seen that if A is a co-EP matrix, then $AA^* + A^*A$, $A + A^*$, $A - A^*$, and $AA^* - A^*A$ are nonsingular matrices.

In the following example we will see these implications are not equivalences. Let

$$A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix},$$

being a and b two nonzero real numbers. Trivially, A is nonsingular, hence A is not a co-EP matrix. Evidently, $AA^* + A^*A$ is nonsingular because $a^2 + b^2 \neq 0$. If $a + b \neq 0$, then $A + A^*$ is nonsingular. If $a \neq b$, then $A - A^*$ is nonsingular. And if $a^2 \neq b^2$, then $AA^* - A^*A$ is nonsingular.

As we have seen in the proof of the last corollary, the projector P given in Theorem 2.3 plays an important role. In next corollary we describe exactly this projector.

Corollary 2.7. *Let A be a co-EP matrix and P the projector given by Theorem 2.3. Then*

(i) *P^* is the projector onto $\mathcal{R}(A^*)$ along $\mathcal{R}(A)$.*

(ii) *P is the projector onto $\mathcal{N}(A^*)$ along $\mathcal{N}(A)$.*

Proof. (i) We will show $\mathcal{R}(P^*) = \mathcal{R}(A^*)$ and $\mathcal{N}(P^*) = \mathcal{R}(A)$. Since $A^* = P^*A^*$ we get $\mathcal{R}(A^*) = \mathcal{R}(P^*A^*) \subset \mathcal{R}(P^*)$; moreover, $\dim \mathcal{R}(A^*) = \mathrm{rk}(A^*) = n/2$ and from representation (15) we have $\dim \mathcal{R}(P^*) = \mathrm{rk}(P^*) = n/2$; hence $\mathcal{R}(A^*) = \mathcal{R}(P^*)$. On the other hand, from $P^*A = 0$ we easily obtain $\mathcal{R}(A) \subset \mathcal{N}(P^*)$; moreover, $\dim \mathcal{R}(A) = \mathrm{rk}(A) = n/2$ and $\dim \mathcal{N}(P^*) = n - \mathrm{rk}(P^*) = n/2$, and therefore, $\mathcal{R}(A) = \mathcal{N}(P^*)$.

(ii) It follows from $\mathcal{N}(P) = \mathcal{R}(P^*)^\perp = \mathcal{R}(A^*)^\perp = \mathcal{N}(A)$ and $\mathcal{R}(P) = \mathcal{N}(P^*)^\perp = \mathcal{R}(A)^\perp = \mathcal{N}(A^*)$. This finishes the proof. □

The range projection of an idempotent $Q \in \mathbb{C}_{n,n}$ (in [34, 26] this concept was studied in the more general setting of C^*-algebras) is the orthogonal projector $Q^\perp$ onto the range of Q. It is easily checked that $Q^\perp Q = Q$ and $QQ^\perp = Q^\perp$.

Corollary 2.8. *Let $A \in \mathbb{C}_{n,n}$ be a co-EP matrix and P the projector given in Theorem 2.3. Then*

$$(P^*)^\perp = A^\dagger A, \qquad (I_n - P^*)^\perp = AA^\dagger, \qquad (I_n - P)^\perp = I_n - A^\dagger A, \qquad P^\perp = I_n - AA^\dagger.$$

Proof. Let us recall that $AA^\dagger$ is the orthogonal projector onto $\mathcal{R}(A)$ and $A^\dagger A$ is the orthogonal projector onto $\mathcal{R}(A^*)$.

By definition, $(P^*)^\perp$ is the orthogonal projector onto $\mathcal{R}(P^*)$; but Corollary 2.7 gives $\mathcal{R}(P^*) = \mathcal{R}(A^*)$. This proves $(P^*)^\perp = A^\dagger A$.

It is well known that for every idempotent $Q \in \mathbb{C}_{n,n}$ we have $\mathcal{R}(I_n - Q) = \mathcal{N}(Q)$, thus $(I_n - P^*)^\perp$ is the orthogonal projector onto $\mathcal{R}(I_n - P^*) = \mathcal{N}(P^*)$; but Corollary 2.7 gives $\mathcal{N}(P^*) = \mathcal{R}(A)$. Hence $(I_n - P^*)^\perp = AA^\dagger$.

The other expressions come from Proposition 1.4 of [34]. The proof is finished. □

Observe Figure 1 for a better understanding of the above results for a 2×2 co-EP matrix A. In this figure, it is assumed that $x \in \mathbb{C}_{2,1}$ does not belong to $\mathcal{R}(A) \cup \mathcal{R}(A^*) \cup \mathcal{N}(A) \cup \mathcal{N}(A^*)$.

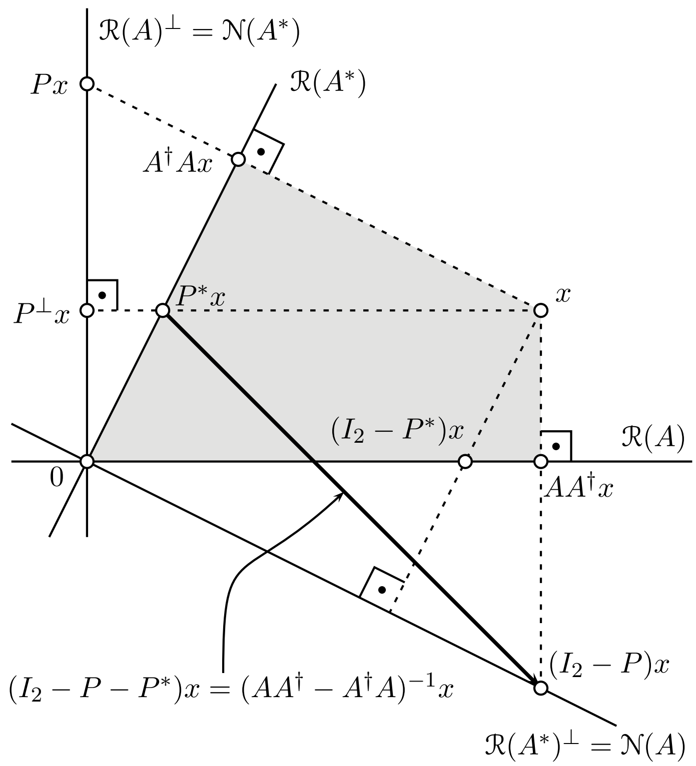

Figure 1: The geometry of a 2×2 co-EP matrix.

The problem of the existence of an idempotent h in a unital C^*-algebra $\mathcal{A}$ satisfying $h^\perp = p$ and $(1-h)^\perp = q$, where p and q are given nonzero self-adjoint idempotents in $\mathcal{A}$ has been studied many times (see [34, 26] and references therein). We denote such an idempotent h by $\pi(p,q)$. In Theorem 4.1 of [26], the authors proved (among other things) that the idempotent $\pi(p,q)$ exists if and only if $p-q$ is invertible.

When the above result is restricted to the C^*-algebra $\mathbb{C}_{n,n}$ of all $n\times n$ complex matrices we get that for a co-EP matrix A, the idempotent $\pi(A^\dagger A, AA^\dagger)$ exists. Precisely, Corollary 2.8 leads to

$$\pi(A^\dagger A, AA^\dagger) = P^* \qquad \text{or} \qquad \pi(AA^\dagger, A^\dagger A) = I_n - P^*.$$

As we pointed, for a given matrix $A \in \mathbb{C}_{n,n}$, one has that A is co-EP if and only if $\mathcal{R}(A) \oplus \mathcal{R}(A^*) = \mathbb{C}_{n,1}$. But, what happens if $\mathcal{R}(A) \oplus^\perp \mathcal{R}(A^*) = \mathbb{C}_{n,1}$?

We can see in figure 1 that if A is a 2×2 co-EP matrix with $\mathcal{R}(A) \oplus^\perp \mathcal{R}(A^*) = \mathbb{C}_{2,1}$, then the projector P given by Theorem 2.3 is orthogonal, hence $P = P^*$. Thus, it is instructive to compare items (ii), (vii), and (viii) of Theorem 2.3 with items (viii), (ii), and (iii) of Theorem 2.9, respectively. Moreover, when the angle between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ is $\pi/2$, for $x \in \mathbb{C}_{2,1}$, the quadrilateral whose vertices are 0, x, $AA^\dagger x$, and $A^\dagger A x$ is parallelogram, therefore, it is reasonable that $AA^\dagger + A^\dagger A = I_n$ when $A \in \mathbb{C}_{n,n}$ satisfies $\mathcal{R}(A) \oplus^\perp \mathcal{R}(A^*) = \mathbb{C}_{n,1}$

Next result characterizes this class of matrices.

Theorem 2.9. *Let $A \in \mathbb{C}_{n,n}$ and r be the rank of A. The following are equivalent:*

(i) *There exist a unitary matrix $U \in \mathbb{C}_{n,n}$, a nonsingular matrix $M \in \mathbb{C}_{r,r}$ such that*

$$A = U \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} U^*. \tag{21}$$

(ii) *$A + A^*$ is nonsingular and there exists a unique orthogonal projector P such that $AP = A$ and $PA = 0$.*

(iii) *$A - A^*$ is nonsingular and there exists a unique orthogonal projector P such that $AP = A$ and $PA = 0$.*

(iv) *$AA^\dagger + A^\dagger A = I_n$.*

(v) *$(AA^\dagger - A^\dagger A)^2 = I_n$.*

(vi) *$\mathcal{N}(A) = \mathcal{R}(A)$.*

(vii) $\mathcal{N}(A^*) = \mathcal{R}(A^*)$.

(viii) $\mathcal{R}(A) \oplus^\perp \mathcal{R}(A^*) = \mathbb{C}_{n,1}$.

(ix) *There exists a unitary matrix* $U \in \mathbb{C}_{n,n}$ *such that* $AA^\dagger = U(I_r \oplus 0)U^*$ *and* $A^\dagger A = U(0 \oplus I_r)U^*$.

(x) $A^2 = 0$ *and* $2\,\mathrm{rk}(A) = n$.

(xi) A *can be represented as in (2) being* L *unitary and* $K = 0$.

(xii) $\mathcal{N}(A^\dagger) = \mathcal{R}(A^\dagger)$.

Proof. (i) $\Rightarrow$ (ii): Obviously, $A + A^*$ is nonsingular since M is nonsingular. Also, it is evident that the orthogonal projector defined by $P = U(0 \oplus I_r)U^*$ satisfies $AP = A$ and $PA = 0$. Since $PA = 0$ and $P = P^*$ we get $A^*P = 0$, hence $(A+A^*)P = A$, and therefore, $P = (A + A^*)^{-1}A$. This proves the uniqueness of the orthogonal projector P satisfying $PA = 0$ and $AP = A$.

(ii) $\Leftrightarrow$ (iii): If P is an orthogonal projector satisfying $AP = A$ and $PA = 0$, then it is easily seen that

$$(A + A^*)(2P - I_n) = 2AP + 2A^*P - A - A^* = A - A^*,$$

because $A^*P = 0$ is obtained from $PA = 0$. Moreover, since $(2P - I_n)^2 = I_n$, we get that $A + A^*$ is nonsingular if and only if $A - A^*$ is nonsingular.

(ii) $\Rightarrow$ (i): Let $s = \mathrm{rk}(P)$ and let us write the orthogonal projector as $P = U(0 \oplus I_s)U^*$, where $U \in \mathbb{C}_{n,n}$ is unitary. Let us represent A as

$$A = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}, \qquad X \in \mathbb{C}_{n-s,n-s},\ T \in \mathbb{C}_{s,s}.$$

From $AP = A$ we get $X = 0$ and $Z = 0$. The equality $PA = 0$ leads to $T = 0$. Next, we are going to prove that Y is nonsingular. Since $A + A^*$ is nonsingular, there exists blocks A_1, A_2, A_3, and A_4 of suitable sizes such that

$$\begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} I_{n-s} & 0 \\ 0 & I_s \end{bmatrix}, \quad \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix} = \begin{bmatrix} I_{n-s} & 0 \\ 0 & I_s \end{bmatrix}. \tag{22}$$

From the first equality of (22) we can get $YA_3 = I_{n-s}$ and from the second one of (22) we can get $A_3Y = I_s$. Therefore, Y is nonsingular and $n - s = s$. Now, it is clear that $\mathrm{rk}(A) = \mathrm{rk}(Y) = s$.

(i) ⇒ (iv): By checking the four conditions in (1), representation (21) leads to

$$A^\dagger = U \begin{bmatrix} 0 & 0 \\ M^{-1} & 0 \end{bmatrix} U^*.$$

It is trivial to prove

$$AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^* \qquad \text{and} \qquad A^\dagger A = U \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} U^*.$$

Therefore $AA^\dagger + A^\dagger A = I_n$ holds.

(iv) ⇒ (v): Let us define the orthogonal projectors $P_1 = AA^\dagger$ and $P_2 = A^\dagger A$. From $P_1 + P_2 = I_n$, we get, premultiplying by P_1 and P_2, the identities $P_1P_2 = P_2P_1 = 0$. Hence $(P_1 - P_2)^2 = P_1 + P_2 = I_n$.

(v) ⇒ (i): From $(AA^\dagger - A^\dagger A)^2 = I_n$ we obtain the nonsingularity of $AA^\dagger - A^\dagger A$. By the equivalence (i) ⇔ (iii) of Theorem 2.3, there exists a unitary matrix U, a nonsingular matrix $M \in \mathbb{C}_{r,r}$ and $\theta_1, \ldots, \theta_r \in]0, \pi/2]$ such that

$$A = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*, \tag{23}$$

where $C = \mathrm{diag}(\cos\theta_1, \ldots, \cos\theta_r)$ and $S = \mathrm{diag}(\sin\theta_1, \ldots, \sin\theta_r)$. It was shown in the second remark after Theorem 2.3 that $A^\dagger$ can be expressed as in (19). A simple computation shows

$$(AA^\dagger - A^\dagger A)^2 = U \begin{bmatrix} S^2 & 0 \\ 0 & S^2 \end{bmatrix} U^*.$$

Since $(AA^\dagger - A^\dagger A)^2 = I_n$ we obtain $S^2 = I_r$. Recalling that $\theta_i \in]0, \pi/2]$ for $i = 1, \cdots, r$ we get $\theta_i = \pi/2$ for all $i = 1, \cdots, r$. Hence $S = I_r$ and $C = 0$. Now, representation (23) proves this implication.

(vi) ⇔ (vii): It should be obvious in view of the well known identities $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ and $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp$.

(vii) ⇒ (viii): It follows from

$$\mathbb{C}_{n,1} = \mathcal{R}(A) \oplus^\perp \mathcal{R}(A)^\perp = \mathcal{R}(A) \oplus^\perp \mathcal{N}(A^*).$$

(viii) $\Rightarrow$ (i): Since $\mathcal{R}(A) \oplus \mathcal{R}(A^*) = \mathbb{C}_{n,1}$, by Theorem 2.3, equivalence (ii) $\Leftrightarrow$ (iii), matrix A can be written as in (5). Since $\mathcal{R}(A) \perp \mathcal{R}(A^*)$ we get $A^*x \perp Ax$ for every $x \in \mathbb{C}_{n,1}$, or equivalently, $(A^*x)^*(Ax) = 0$ for every $x \in \mathbb{C}_{n,1}$, which implies $A^2 = 0$. From representation (5) we get $0 = MCMS$ and the nonsingularity of M and S leads to $C = 0$, which implies $S = 0$ and representation (5) permits write A as in (21).

(i) $\Rightarrow$ (ix) Since A has the representation (21), we clearly obtain

$$A^\dagger = U \begin{bmatrix} 0 & 0 \\ K^{-1} & 0 \end{bmatrix} U^*, \tag{24}$$

and thus,

$$AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*, \qquad A^\dagger A = U \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} U^*.$$

(ix) $\Rightarrow$ (x): By hypothesis, we trivially get $(A^\dagger A)(AA^\dagger) = 0$. Pre and postmultiplying by A and using $AA^\dagger A = A$, we obtain $A^2 = 0$. Now, since $AA^\dagger$ is the orthogonal projector onto $\mathcal{R}(A)$ we get $\operatorname{rk}(A) = \operatorname{rk}(AA^\dagger) = r = n/2$.

(x) $\Rightarrow$ (vi): Since $A^2 = 0$ we get $\mathcal{R}(A) \subset \mathcal{N}(A)$. Moreover, we have

$$\frac{n}{2} = \operatorname{rk}(A) = \dim \mathcal{R}(A) = n - \dim \mathcal{N}(A),$$

hence $\dim \mathcal{N}(A) = n/2 = \dim \mathcal{R}(A)$. We conclude $\mathcal{R}(A) = \mathcal{N}(A)$.

(xi) $\Rightarrow$ (i): It is trivial.

(i) $\Rightarrow$ (xi): Let $M = X\Sigma Y$ be the singular value decomposition of M, Now, we have

$$A = U \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} 0 & X\Sigma Y \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} X & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} 0 & \Sigma Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X^* & 0 \\ 0 & I_r \end{bmatrix} U^*.$$

In order to write A as in (2), it is enough to take $V = U(X \oplus I_r)$, $M = 0$, and $L = Y$. Moreover, since

$$A^*A = V \begin{bmatrix} 0 & \Sigma Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ Y^*\Sigma & 0 \end{bmatrix} V^* = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^*,$$

and having in mind that AA^* and A^*A have the same nonzero eigenvalues, we can conclude that the nonzero singular values of A are the entries of the diagonal of Σ.

(vii) $\Leftrightarrow$ (xii): Let us recall that for every matrix X, we have $(X^\dagger)^\dagger = X$, $\mathcal{R}(X) = \mathcal{R}(XX^\dagger)$, $\mathcal{N}(X) = \mathcal{R}(X^*)^\perp$, and $\mathcal{R}(X^*) = \mathcal{R}(X^\dagger X)$. Now, (vii) $\Leftrightarrow$ (xii) should be evident in view of

$$\mathcal{R}(A^\dagger) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^*)$$

and

$$\mathcal{N}(A^\dagger) = \mathcal{R}((A^\dagger)^*)^\perp = \mathcal{R}(AA^\dagger)^\perp = \mathcal{R}(A)^\perp = \mathcal{N}(A^*).$$

The proof is finished. □

The well known Akhiezer-Glazman equality (see e.g., [35, Lemma 1 (i)]) says that if p, q are two projections in a C^*-algebra, then $\|p-q\| = \max\{\|p(1-q)\|, \|q(1-p)\|\}$, which implies that $\|p-q\| \leq 1$, and thus the gap between two equidimensional subspaces cannot be greater than 1. We remark that for a co-EP matrix A satisfying Theorem 2.9 we have that the gap between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ is exactly 1.

Corollary 2.10. *Let $A \in \mathbb{C}_{n,n}$ satisfy any of the conditions of Theorem 2.9. Then*

(i) $A^\dagger = (A + A^*)^{-1} A (A + A^*)^{-1}$.

(ii) $A^\dagger = (A - A^*)^{-1} A (A - A^*)^{-1}$.

(iii) $A^\dagger + (A^\dagger)^* = (A + A^*)^{-1}$.

(iv) $A^\dagger - (A^\dagger)^* = (A - A^*)^{-1}$.

(v) $A^\dagger = \frac{1}{2}[(A + A^*)^{-1} + (A - A^*)^{-1}]$.

Proof. If we write matrix A as in (21), then $A^\dagger$ can be represented as in (24). The five formulae of this corollary are trivial to prove. Alternatively, (iii) can be proved by using item (i) of this corollary and the fact that $(A + A^*)^{-1}$ is Hermitian:

$$\begin{aligned} A^\dagger + (A^\dagger)^* &= (A + A^*)^{-1} A (A + A^*)^{-1} + [(A + A^*)^{-1} A (A + A^*)^{-1}]^* \\ &= (A + A^*)^{-1} A (A + A^*)^{-1} + (A + A^*)^{-1} A^* (A + A^*)^{-1} \\ &= (A + A^*)^{-1}. \end{aligned}$$

The proof of (iv) works in the same way as in (iii), and the proof of (v) follows from items (iii) and (iv). □

It is evident that if A is a normal matrix, then A^n is also normal. Next result is concerned with a similar property but for co-EP matrices and matrices satisfying Theorem 2.9.

Theorem 2.11. *Let $A \in \mathbb{C}_{n,n}$ and $m \in \mathbb{N}$.*

(i) *If A^m is co-EP and $\|A(A^\dagger)^2A\| < 1$, then A is a co-EP matrix.*

(ii) *If A^m satisfy Theorem 2.9 and $\mathrm{rk}(A) \le n/2$, then A satisfy Theorem 2.9.*

Proof. (i) Since A^m is co-EP, by Theorem 2.3, equivalence (i) $\Leftrightarrow$ (ii), we have $\mathcal{R}(A^n) \oplus \mathcal{R}((A^n)^*) = \mathbb{C}_{n,1}$. But, since that for every pair of matrices X and Y such that XY exists we have $\mathcal{R}(XY) \subset \mathcal{R}(X)$, we obtain $\mathcal{R}(A) + \mathcal{R}(A^*) = \mathbb{C}_{n,1}$. Since $\mathcal{R}(A) = \mathcal{R}(AA^\dagger)$ and $\mathcal{R}(A^*) = \mathcal{R}(A^\dagger A)$ we obtain $\mathcal{R}(AA^\dagger) + \mathcal{R}(A^\dagger A) = \mathbb{C}_{n,1}$. Since for every pair of orthogonal projectors P and Q we have $\mathcal{R}(P + Q) = \mathcal{R}(P) + \mathcal{R}(Q)$ (see [31, Corollary 2]), we have $\mathcal{R}(AA^\dagger + A^\dagger A) = \mathbb{C}_{n,1}$. Therefore $AA^\dagger + A^\dagger A$ is nonsingular. By Theorem 2.3 (equivalence (i) $\Leftrightarrow$ (v)) we have that A is co-EP.

(ii) The relations $\mathcal{N}(X) \subset \mathcal{N}(X^k)$ and $\mathcal{R}(X^k) \subset \mathcal{R}(X)$ are satisfied for every matrix $X \in \mathbb{C}_{n,n}$. Since A^k satisfy condition (vi) of Theorem 2.9, we get $\mathcal{N}(A) \subset \mathcal{R}(A)$. But by hypothesis we get $\dim \mathcal{N}(A) = n - \mathrm{rk}(A) \ge n - n/2 = n/2 \ge \mathrm{rk}(A)$. Hence $\mathcal{N}(A) = \mathcal{R}(A)$ and therefore, A satisfy condition (vi) of Theorem 2.9. This finishes the proof. $\square$

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