Characterization of consistent completion of reciprocal comparison matrices

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What is the Analytic Hierarchy Process?

Decision making is becoming increasingly complex due to the large number of alternatives and multiple conflicting goals.

The Analytic Hierarchy Process (AHP) has been accepted as a leading multiattribute decision-aiding model.

Decisions arise in every type of application including

- Engineering
- Economy
- Medical management
- Emergency situations
- ...
What is the Analytic Hierarchy Process?

Example: (Pairwise comparisons)
I want to submit a manuscript to one of the following two journals:

- Journal of Superb Mathematics
- International Bulletin of Excellent Theorems

What can I do?

1. Equal importance.
2. Moderate importance of one over another.
3. Strong importance.
4. Very strong importance.
5. Extreme importance.

9 7 5 3 1 3 5 7 9
What is the **Analyty Hierarchy Process**?

**Example: (Reciprocal matrices)**

- Journal of Superb Mathematics
- International Bulletin of Excellent Theorems
- Annals of the Best Mathematical Results

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<table>
<thead>
<tr>
<th>JSB</th>
<th>JBE</th>
<th>ABMR</th>
</tr>
</thead>
<tbody>
<tr>
<td>JSB</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>IBET</td>
<td>1/6</td>
<td>1</td>
</tr>
<tr>
<td>ABMR</td>
<td>1/5</td>
<td>1/3</td>
</tr>
</tbody>
</table>
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What is the Analyty Hierarchy Process?

Definition

A matrix $\mathbf{A} = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}_{n,n}$ is reciprocal when $a_{ij} > 0$ and $a_{ij} = 1/a_{ji}$ for all $i, j \in \{1, \ldots, n\}$. 
What is the **Analyzy Hierarchy Process?**

**Example: (Consistent matrices)**

<table>
<thead>
<tr>
<th></th>
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<tr>
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<td>1/5</td>
<td>1/3</td>
<td>1</td>
</tr>
</tbody>
</table>

I prefer JSB 6 times over IBET.

I prefer IBET 3 times over ABMR.

I prefer JSB 5 times over ABMR.

This is not “rational”. I had to prefer JSB $6 \times 3$ times over ABMR!

$$6 \times 3 \neq 5$$
What is the Analytical Hierarchy Process?

**Definition**

A matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_{n,n}$ is consistent when $a_{ij} > 0$ and $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k \in \{1, \ldots, n\}$.

**Theorem**

Let $A \in \mathbb{R}_{n,n}$ be positive.

- If $A$ is consistent, then $A$ is reciprocal.
- If $A$ is reciprocal and $n = 2$, then $A$ is consistent.
What is the Analyty Hierarchy Process?

T. Saaty (1970) proposed a measure of the inconstency of a reciprocal matrix $A$ based on Perron theory.

**Theorem (O. Perron 1907)**

Let $A \in \mathbb{R}_{n,n}$ be positive. Let $\rho(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$. The following statements hold:

- $\rho(A)$ is an eigenvalue of $A$, and there is a positive vector $x$ such that $Ax = \rho(A)A$.
- If $\lambda$ is an eigenvalue of $A$, then $|\lambda| < \rho(A)$.
- The eigenspace corresponding to $\rho(A)$ has dimension 1.

If $A$ is a matrix “near” to be consistent, the eigenvector corresponding to $\rho(A)$ is called the priority vector.
What is the Analytic Hierarchy Process?

- Is $A$ near to be consistent?
  - Yes: Obtain the priority vector of $A$
  - No: Improve the consistency of $A$
    - Yes: Is $A$ OK for the experts?
    - No: The experts modify $A$
What is the **Analyty Hierarchy Process?**

**Example: (two levels)**

I want to submit a paper to JSB, IBET, or ABMR, but considering several criteria: Quickness of publication, Quality of typesetting, Quality of refereeing, Prestige of the journal.
What is the Analytic Hierarchy Process?

Example: (two levels)
I want to submit a paper to JSB, IBET, or ABMR, but considering several criteria: Quickness of publication, Quality of typesetting, Quality of refereeing, Prestige of the journal.

First step: Determine the relative importance of the criteria:

\[
\begin{bmatrix}
1 & 7 & 2 & 1/3 \\
1/7 & 1 & 1/5 & 1/9 \\
1/2 & 5 & 1 & 1/2 \\
3 & 9 & 2 & 1
\end{bmatrix} = A
\]

If \(A\) is near to be consistent, obtain the priority vector

\[
w = (w_1, w_2, w_3, w_4), \quad w_1 + w_2 + w_3 + w_4 = 1.
\]
What is the Analytic Hierarchy Process?

Example: (two levels)
I want to submit a paper to JSB, IBET, or ABMR, but considering several criteria: Quickness of publication, Quality of typesetting, Quality of refereeing, Prestige of the journal

Second step: For each criteria, order the alternatives:

<table>
<thead>
<tr>
<th>Criteria (weight)</th>
<th>JSB</th>
<th>IBET</th>
<th>ABMR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quickness of publication ($w_1$)</td>
<td>$v_{11}$</td>
<td>$v_{12}$</td>
<td>$v_{13}$</td>
</tr>
<tr>
<td>Quality of Typesetting ($w_2$)</td>
<td>$v_{21}$</td>
<td>$v_{22}$</td>
<td>$v_{23}$</td>
</tr>
<tr>
<td>Quality of refereeing ($w_3$)</td>
<td>$v_{31}$</td>
<td>$v_{32}$</td>
<td>$v_{33}$</td>
</tr>
<tr>
<td>Prestige of the journal ($w_4$)</td>
<td>$v_{41}$</td>
<td>$v_{42}$</td>
<td>$v_{43}$</td>
</tr>
</tbody>
</table>

Priority vector: $w_1v_1 + w_2v_2 + w_3v_3 + w_4v_4$
Statement of the problem

Sometimes, the appearing comparison matrices can be incomplete.

- Some experts may not be completely familiar with one or more of the elements.
- It is possible that there exist conflicts of interests.
- Some experts may not want to express their opinion.
- Some data may be lost.
- ...
Sometimes, the appearing reciprocal matrices can be incomplete.

\[
A = \begin{bmatrix}
1 & 2 & 3 & \star \\
1/2 & 1 & 3 & 4 \\
1/3 & 1/3 & 1 & \star \\
\star & 1/4 & \star & 1
\end{bmatrix}
\]

**Problem**

When \( A \) can be completed to be consistent? 

*In case that \( A \) can be completed, find all consistent completions.*
Preliminaries – Linearization

Reciprocal

Consistent

\( (a_{ij}) \)

\( \mathbb{R}^+_{n,n} \)

\( (\exp(b_{ij})) \)

Skew-Hermitian

\( \mathcal{L}_n \)

\( \mathbb{R}_{n,n} \)

\( (b_{ij}) \)

\( \log(a_{ij}) \)

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Notations – Statement of the theorem

Example:

\[ A = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \star \\ \star & 1/4 & \star & 1 \end{bmatrix} \quad \iff \quad L(A) = \begin{bmatrix} 0 & \log 2 & \log 3 & \star \\ -\log 2 & 0 & \log 3 & \log 4 \\ -\log 3 & -\log 3 & 0 & \star \\ \star & -\log 4 & \star & 0 \end{bmatrix} \]

Any skew-Hermitiam completion of \( L(A) \) is of the form

\[ C(\lambda, \mu) = \begin{bmatrix} 0 & \log 2 & \log 3 & \lambda \\ -\log 2 & 0 & \log 3 & \log 4 \\ -\log 3 & -\log 3 & 0 & \mu \\ -\lambda & -\log 4 & -\mu & 0 \end{bmatrix} \]

\[ \text{imm}^2; \quad \text{Fluling} \]
Notations – Statement of the theorem

Example: \( A = \begin{bmatrix} 1 & 2 & 3 & \ast \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \ast \\ \ast & 1/4 & \ast & 1 \end{bmatrix} \)

\[
C(\lambda, \mu) = \begin{bmatrix}
0 & \log 2 & \log 3 & 0 \\
-\log 2 & 0 & \log 3 & \log 4 \\
-\log 3 & -\log 3 & 0 & 0 \\
0 & -\log 4 & 0 & 0 
\end{bmatrix}
\]

known entries

\[
\begin{align*}
+ \lambda \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 
\end{bmatrix} \\
+ \mu \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 
\end{bmatrix}
\end{align*}
\]

entries to fill
Notations – Statement of the theorem

Notations

\[ N_n = \{(i, j) : 1 \leq i < j \leq n\}, \quad B_{ij} = e_i e_j^T - e_j e_i^T, \quad (i, j) \in N_n \]

Example: The skew-Hermitian completion considered can be written

\[ C(\lambda, \mu) = C_0 + \lambda B_{14} + \mu B_{34}, \]

where

\[ C_0 = \sum_{(i,j) \in N_4 \setminus \{(1,4),(3,4)\}} \rho_{ij} B_{ij}, \]

and \( \rho_{ij} \in \mathbb{R} \) easily determined from the incomplete matrix \( A \).

- \( C_0 \) is the incomplete skew-Hermitian matrix to be completed.
- \((1,4), (3,4)\) are the void positions that must be filled.
Consistent completion of a reciprocal matrix

The unspecified entries of $A$ will be located at the indices belonging to $I$.

$$A = \begin{bmatrix}
1 & 2 & 3 & \star \\
1/2 & 1 & 3 & 4 \\
1/3 & 1/3 & 1 & \star \\
\star & 1/4 & \star & 1
\end{bmatrix}$$

Theorem

Let $1 \leq i_1, j_1, \ldots, i_k, j_k \leq n$ be indices such that $i_r < j_r$ for $r = 1, \ldots, k$. Denote $I = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ and $J = \mathbb{N}_n \setminus I$. Let $C_0 = \sum_{(i,j) \in J} \rho_{ij} B_{ij}$.

The following statements are equivalent

(i) There exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $C_0 + \sum_{r=1}^{k} \lambda_r B_{i_r j_r} \in \mathcal{L}_n$.

(ii) There exists $w = (w_1, \ldots, w_n)^T \in \mathbb{R}^n$ such that $\rho_{pq} = w_p - w_q$ for any $(p, q) \in J$.

Furthermore, in the case that the statements hold, then

$$\lambda_r = w_{i_r} - w_{j_r}, \quad \forall \ r \in \{1, \ldots, k\}.$$
Consistent completion of a reciprocal matrix

**Theorem**

Let $1 \leq i_1, j_1, \ldots, i_k, j_k \leq n$ be indices such that $i_r < j_r$ for $r = 1, \ldots, k$. Denote $I = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ and $J = \mathbb{N}_n \setminus I$. Let $C_0 = \sum_{(i,j) \in J} \rho_{ij} B_{ij}$. The following statements are equivalent

(i) There exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $C_0 + \sum_{r=1}^{k} \lambda_r B_{irjr} \in \mathcal{L}_n$.

(ii) There exists $w = (w_1, \ldots, w_n)^T \in \mathbb{R}^n$ such that $\rho_{pq} = w_p - w_q$ for any $(p, q) \in J$.

Furthermore, in the case that the statements hold, then

$$\lambda_r = w_{ir} - w_{jr}, \quad \forall \ r \in \{1, \ldots, k\}.$$ 

We have reduced the completion problem to study the linear system occurring in item (ii).

- The coefficients of this system are 0, -1, 1.
- A row has only two non-zero entries (sparsity).
Example

If \( A \) had a consistent completion, then there would exist \( w = (w_1, w_2, w_3, w_4)^T \in \mathbb{R}^4 \) such that

\[
\begin{align*}
\log 2 &= w_1 - w_2 \\
\log 3 &= w_1 - w_3 \\
\log 3 &= w_2 - w_3 \\
\log 4 &= w_2 - w_4
\end{align*}
\]
Example

\[ A = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \star \\ \star & 1/4 & \star & 1 \end{bmatrix} \]

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\end{align*}
\]
Example

$$A = \begin{bmatrix}
1 & 2 & 3 & \star \\
1/2 & 1 & 3 & 4 \\
1/3 & 1/3 & 1 & \star \\
\star & 1/4 & \star & 1 \\
\end{bmatrix}$$

If $A$ had a consistent completion, then there would exist $w = (w_1, w_2, w_3, w_4)^T \in \mathbb{R}^4$ such that

$$\log 2 = w_1 - w_2$$
$$\log 3 = w_1 - w_3$$
$$\log 3 = w_2 - w_3$$
$$\log 4 = w_2 - w_4$$
Example

\[
A = \begin{bmatrix}
1 & 2 & 3 & * \\
1/2 & 1 & 3 & 4 \\
1/3 & 1/3 & 1 & * \\
* & 1/4 & * & 1
\end{bmatrix}
\]

If \( A \) had a consistent completion, then there would exist \( w = (w_1, w_2, w_3, w_4)^T \in \mathbb{R}^4 \) such that

\[
\begin{align*}
\log 2 &= w_1 - w_2 \\
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\log 3 &= w_2 - w_3 \\
\log 4 &= w_2 - w_4
\end{align*}
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Example

\[ A = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 3 & 4 \\ 1/3 & 1/3 & 1 & \star \\ \star & 1/4 & \star & 1 \end{bmatrix} \]

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\begin{align*}
\log 2 &= w_1 - w_2 \\
\log 3 &= w_1 - w_3 \\
\log 3 &= w_2 - w_3 \\
\log 4 &= w_2 - w_4
\end{align*}
\]

This linear system has no solution.
Another example

\[ A = \begin{bmatrix}
1 & \ast & 1/3 \\
\ast & 1 & 2/3 \\
3 & 3/2 & 1 \\
\end{bmatrix} \]
Another example

\[ A = \begin{bmatrix}
1 & \star & 1/3 \\
\star & 1 & 2/3 \\
3 & 3/2 & 1
\end{bmatrix} \]

By taking logarithms, we construct

\[ C_0 = \begin{bmatrix}
0 & 0 & - \log 3 \\
0 & 0 & \log 2 - \log 3 \\
\log 3 & \log 3 - \log 2 & 0
\end{bmatrix}. \]
Another example

\[
\mathbf{A} = \begin{bmatrix}
1 & 1/3 \\
\star & 1 & 2/3 \\
3 & 3/2 & 1
\end{bmatrix}
\]

By taking logarithms, we construct

\[
\mathbf{C}_0 = \begin{bmatrix}
0 & 0 & -\log 3 \\
0 & 0 & \log 2 - \log 3 \\
\log 3 & \log 3 - \log 2 & 0
\end{bmatrix}
\]

If there were a consistent completion, then there would exist \( \mathbf{w} = (w_1, w_2, w_3)^T \in \mathbb{R}^3 \) such that

\[
-\log 3 = w_1 - w_3 \\
\log 2 - \log 3 = w_2 - w_3
\]
Another example

$$A = \begin{bmatrix} 1 & * & 1/3 \\ * & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$$

By taking logarithms, we construct

$$C_0 = \begin{bmatrix} 0 & 0 & - \log 3 \\ 0 & 0 & \log 2 - \log 3 \\ \log 3 & \log 3 - \log 2 & 0 \end{bmatrix}.$$ 

If there were a consistent completion, then there would exist

$$w = (w_1, w_2, w_3)^T \in \mathbb{R}^3$$

such that

$$- \log 3 = w_1 - w_3 \quad \log 2 - \log 3 = w_2 - w_3$$
Another example

\[ A = \begin{bmatrix} 1 & \star & 1/3 \\ \star & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix} \]

By taking logarithms, we construct

\[ C_0 = \begin{bmatrix} 0 & 0 & -\log 3 \\ 0 & 0 & \log 2 - \log 3 \\ \log 3 & \log 3 - \log 2 & 0 \end{bmatrix}. \]

If there were a consistent completion, then there would exist \( w = (w_1, w_2, w_3)^T \in \mathbb{R}^3 \) such that

\[-\log 3 = w_1 - w_3 \quad \log 2 - \log 3 = w_2 - w_3\]

This linear system is solvable. Thus, exists a consistent completion of \( A \).
Another example (continuation)

\[
\mathbf{A} = \begin{bmatrix}
1 & * & 1/3 \\
* & 1 & 2/3 \\
3 & 3/2 & 1
\end{bmatrix} \quad \mathbf{C}_0 = \begin{bmatrix}
0 & 0 & -\log 3 \\
0 & 0 & \log 2 - \log 3 \\
\log 3 & \log 3 - \log 2 & 0
\end{bmatrix}
\]

The solution of the prior system is

\[
w_1 = -\log 3 + \alpha, \quad w_2 = \log 2 - \log 3 + \alpha, \quad w_3 = \alpha, \quad \alpha \in \mathbb{R}.
\]

If \( \mathbf{X} \) is any consistent completion of \( \mathbf{A} \), then

\[
L(\mathbf{X}) = \mathbf{C}_0 + (w_1 - w_2)\mathbf{B}_{12} = \mathbf{C}_0 - \log 2 \cdot \mathbf{B}_{12}.
\]

There is a unique consistent completion of \( \mathbf{A} \):

\[
\mathbf{X} = \begin{bmatrix}
1 & 1/2 & 1/3 \\
2 & 1 & 2/3 \\
3 & 3/2 & 1
\end{bmatrix}
\]
Connection with the graph theory

\[
\begin{bmatrix}
1 & 2 & 3 & \star \\
1/2 & 1 & 3 & 4 \\
1/3 & 1/3 & 1 & \star \\
\star & 1/4 & \star & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix} \begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{bmatrix} = \begin{bmatrix}
\log 2 \\
\log 3 \\
\log 3 \\
\log 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & \star & 1/3 \\
\star & 1 & 2/3 \\
3 & 3/2 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1
\end{bmatrix} \begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix} = \begin{bmatrix}
-\log 3 \\
\log 2 - \log 3
\end{bmatrix}
\]

The transpose of the matrices are the incidence matrix of the graphs.
Connection with the graph theory

\[
\begin{bmatrix}
1 & 2 & 3 & \star \\
1/2 & 1 & 3 & 4 \\
1/3 & 1/3 & 1 & \star \\
\star & 1/4 & \star & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{bmatrix}
= \begin{bmatrix}
\log 2 \\
\log 3 \\
\log 3 \\
\log 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & \star & 1/3 \\
\star & 1 & 2/3 \\
3 & 3/2 & 1
\end{bmatrix}
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\begin{bmatrix}
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0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3
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Connection with the graph theory

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\begin{bmatrix}
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1/2 & 1 & 3 & 4 \\
1/3 & 1/3 & 1 & \star \\
\star & 1/4 & \star & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{bmatrix}
= 
\begin{bmatrix}
\log 2 \\
\log 3 \\
\log 3 \\
\log 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & \star & 1/3 \\
\star & 1 & 2/3 \\
3 & 3/2 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix}
= 
\begin{bmatrix}
-\log 3 \\
\log 2 - \log 3
\end{bmatrix}
\]

The transpose of the matrices are the incidence matrix of the graphs
Notations

\[ A = \begin{bmatrix} 1 & \ast & 1/3 \\ \ast & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix} \]

\[
\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -\log 3 \\ \log 2 - \log 3 \end{bmatrix}
\]

\[ A \rightarrow \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} \]

\[ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \]

\[ G_A \]
Some consequences

Theorem

If $A$ is an incomplete reciprocal matrix, then $A$ can be completed to be a consistent matrix if and only if the system $M_A^T w = b_A$ is consistent.
Some consequences

**Theorem**

*If $A$ is an incomplete reciprocal matrix, then $A$ can be completed to be a consistent matrix if and only if the system $M_A^T w = b_A$ is consistent.*

**Theorem**

*Let $A \in \mathbb{R}_{n,n}$ be a reciprocal incomplete matrix and $2k$ be the number of void entries (located up and down the main diagonal of $A$). If $G_A$ has $p$ connected components and $2k \geq n^2 - 3n + 2p$, then $A$ can be completed to be consistent.*

\[
A = \begin{bmatrix}
1 & \star & 1/3 \\
\star & 1 & 2/3 \\
3 & 3/2 & 1
\end{bmatrix}
\]

$1 \rightarrow 2$ (with $n = 3$)
$1 \rightarrow 3$ (with $k = 1$)
$3 \rightarrow 2$ (with $p = 1$)
Example (disconnected graph)

\[
A = \begin{bmatrix}
1 & \star & \star \\
\star & 1 & a \\
\star & 1/a & 1
\end{bmatrix}
\]

\[M_A = \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix}\]
Example (disconnected graph)

\[
A = \begin{bmatrix}
1 & \ast & \ast \\
\ast & 1 & a \\
\ast & 1/a & 1
\end{bmatrix}
\]

\[
M_A = \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix}
\]

Any consistent completion of \( A \) is of the following form

\[
X = \begin{bmatrix}
1 & b & ab \\
1/b & 1 & a \\
1/(ab) & 1/a & 1
\end{bmatrix}.
\]

The completion of \( A \) is not unique.
A result on the uniqueness

Theorem

Let $\mathbf{A} \in \mathbb{R}_{n,n}$ be a reciprocal incomplete matrix. If $G_{\mathbf{A}}$ is connected and exists a consistent completion of $\mathbf{A}$, then this completion is unique.
Let us recall

**Theorem**

*If \( A \) is an incomplete reciprocal matrix, then \( A \) can be completed to be a consistent matrix if and only if the system \( M_A^T w = b_A \) is consistent.*
Let us recall

**Theorem**

*If $A$ is an incomplete reciprocal matrix, then $A$ can be completed to be a consistent matrix if and only if the system $M_A^T w = b_A$ is consistent.*

\[ M_A^T w = b_A \text{ is consistent } \iff b_A \in \mathcal{R}(M_A^T) \]
Let us recall

**Theorem**

*If \( \mathbf{A} \) is an incomplete reciprocal matrix, then \( \mathbf{A} \) can be completed to be a consistent matrix if and only if the system \( M_A^T \mathbf{w} = \mathbf{b}_A \) is consistent.*

\[
M_A^T \mathbf{w} = \mathbf{b}_A \text{ is consistent} \iff \mathbf{b}_A \in \mathcal{R}(M_A^T) = \mathcal{N}(M_A)^\perp
\]
Let us recall

**Theorem**

*If $A$ is an incomplete reciprocal matrix, then $A$ can be completed to be a consistent matrix if and only if the system $M_A^T w = b_A$ is consistent.*

$$M_A^T w = b_A \text{ is consistent } \iff b_A^T x = 0 \text{ for any } x \in \mathcal{N}(M_A)$$
More matrix algebra and graph theory

Let us recall

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\]

The null space of $M_A$ corresponds to the cycles of $G_A$.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0

-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0

0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0

0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0

0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1

0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0

0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1

0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
0

-1

1

0

0

1

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-1

1
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1 \\
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1 \\
0 \\
1
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**Theorem**

Let $G$ be a planar graph and $M$ its incidence matrix. If $x_1, \ldots, x_f$ correspond to the faces, then $\{x_1, \ldots, x_f\}$ is a basis of $\mathcal{N}(M)$.
Theorem

If $\mathbf{A}$ is an incomplete reciprocal matrix, then $\mathbf{A}$ can be completed to be a consistent matrix if and only if $\mathbf{b}_A^T \mathbf{x} = 0$ for any $\mathbf{x} \in \mathcal{N}(M_A)$.

Let $G$ be a planar graph and $M$ its incidence matrix. If $x_1, \ldots, x_f$ correspond to the faces, then $\{x_1, \ldots, x_f\}$ is a basis of $\mathcal{N}(M)$.

Corollary

Let $\mathbf{A}$ be an incomplete reciprocal matrix. If $G_A$ is planar and has no faces, then there exists a consistent completion of $\mathbf{A}$.
Many thanks for your kind attention.

These slides are available at

http://personales.upv.es/jbenitez/investigacion.html