

# Achieving Matrix Consistency in AHP through Linearization

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## Abstract

Matrices used in the analytic hierarchy process (AHP) compile expert knowledge as pairwise comparisons among various criteria and alternatives in decision-making problems. Many items are usually considered in the same comparison process and so judgment is not completely consistent - and sometimes the level of consistency may be unacceptable. Different methods have been used in the literature to achieve consistency for an inconsistent matrix. In this paper we use a linearization technique that provides the closest consistent matrix to a given inconsistent matrix using orthogonal projection in a linear space. As a result, consistency can be achieved in a closed form. This is simpler and cheaper compared to methods that relying on optimization, which are iterative by nature. We apply the process to a real-world decision-making problem in water supply management.

**Key words:** analytic hierarchy process, decision-making, linearization, leakage management

**AMS Subject Classification:** 15B48, 90B50, 41A99

## 1 Introduction

The analytic hierarchy process (AHP) [1] provides a useful method to establish relative scales that can be derived by making pairwise comparisons using numerical judgments from an absolute scale of numbers. This approach is essential, for example, when tangible and intangible factors need to be considered within the same pool. The various factors are arranged in a hierarchical or a network structure with the objective(s) at the top, followed by one or more layers of criteria, and finally, the alternatives at the bottom. The ability of alternatives to achieve the objective(s) is measured according to the criteria represented within the structure. To this end, the people involved in the process compare the criteria and the alternatives in pairs, make judgments, and compile the results into matrices (matrices of criteria or matrices of alternatives).

Any two elements, for example criteria  $C_i$  and  $C_j$  are semantically compared. A value  $a_{ij}$  is proposed directly (numerically) or indirectly (verbally) that represents the judgment of the relative importance of the decision element  $C_i$  over  $C_j$ . Among the different approaches for developing such scales [2] the nine-point scale developed by Saaty [3] is one of the most popular. By using the Saaty scale, if the elements  $C_i$  and  $C_j$  are considered to be equally important, then  $a_{ij} = 1$  (homogeneity). If  $C_i$  is preferred to  $C_j$ , then  $a_{ij} > 1$ , with an integer grade ranging from 2 to 9 that respectively corresponds to weak, moderate, ..., until very

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strong, and extreme importance of  $C_i$  over  $C_j$ . Intermediate numerical (decimal) values in the scale may be used to model hesitation between two adjacent judgments [1, 4, 5]. It is assumed that the reciprocal property  $a_{ji} = 1/a_{ij}$  always holds. Homogeneity also implies that  $a_{ii} = 1$  for all  $i = 1, 2, \dots, n$ . In this way, a homogeneous and reciprocal  $n \times n$  matrix of pairwise comparisons  $A$  is compiled. This approach is intended to embody expert know-how regarding a specific problem. Matrices such as  $A$  are positive matrices (matrices with only positive entries) that also exhibit homogeneity and reciprocity.

There are different techniques to extract priority vectors from these comparison matrices [6, 7, 8]. The eigenvector method, proposed by Saaty in his seminal paper [3] in 1977, stands out from the rest. Saaty proved that the Perron eigenvector of the comparison matrix provides the necessary information to deal with complex decisions that involve dependence and feedback - as analyzed in the context of, for example, benefits, opportunities, costs, and risks [9]. The required condition is that the matrix exhibits a minimum level of consistency. Consistency expresses the coherence that should (perhaps) exist between judgments about the elements of a set. Matrix consistency is defined as follows: a positive  $n \times n$  matrix  $A$  is consistent if  $a_{ij}a_{jk} = a_{ik}$ , for  $i, j, k = 1, \dots, n$ . Although different measurements of inconsistency can be developed, in this paper we use the measurement proposed by Saaty [1, 3]. We also use the intrinsic consistency threshold developed by Monsuur [10].

If consistency is unacceptable, it should be improved. Several alternatives, mostly based on various optimization techniques, have been proposed in the literature to help improve consistency, including [9, 12, 13, 24, 14, 15, 16, 17, 18, 19, 20, 21]. For example, Saaty [12] proposes a method based on perturbation theory to find the most inconsistent judgment in the matrix. This action could be followed by the determination of the range of values to which that judgment can be changed and whereby the inconsistency could be improved - and then asking the judge to consider changing the judgment to a plausible value in that range.

In the next section we develop a linearization technique that provides the closest consistent matrix to a given non-consistent matrix by using an orthogonal projection in a given linear space. This way, our method provides a closed form to achieve consistency, while methods relying on optimization, which for this problem is non-linear, are iterative by nature. Section 3 presents an application to a real-world decision-making problem regarding leakage policies in water supply. Finally, the paper closes with conclusions.

## 2 Achieving consistency through linearization

From now on,  $M_{n,m}$  and  $M_{n,m}^+$  will denote the set of  $n \times m$  matrices and the set of  $n \times m$  positive matrices, respectively. It will be assumed that the elements of  $\mathbb{R}^n$  are column vectors. For a given  $A$ , the entry  $(i, j)$  of  $A$  will be denoted by  $[A]_{i,j}$ . Furthermore,  $A^T$  denotes the transposition of the matrix  $A$ . The matrix product component-wise (also called the Hadamard product) of  $A, B$  is the matrix  $A \odot B$  defined by  $[A \odot B]_{i,j} = [A]_{i,j}[B]_{i,j}$ . The (nonlinear) map  $J : M_{n,m} \rightarrow M_{n,m}$  given by  $[J(A)]_{i,j} = 1/[A]_{i,j}$  will be useful. In particular, notice that  $A \in M_{n,n}^+$  is reciprocal if and only if  $J(A) = A^T$ .

In the following lines we linearize the problem of finding a consistent matrix close to a given positive matrix. The mathematical tool to measure the closeness of two given matrices is the concept of matrix norm (see e.g. [22, section 5.2]). Here we use the Frobenius norm

because of its simplicity. Such a norm is defined as

$$\|A\|_F = \left( \sum_{i,j} [A]_{i,j}^2 \right)^{1/2} = [\text{trace}(A^T A)]^{1/2}, \quad A \in M_{n,n}.$$

Furthermore, let us define the following map:

$$L : M_{n,n}^+ \rightarrow M_{n,n}, \quad [L(X)]_{i,j} = \log([X]_{i,j}).$$

Obviously, this map is bijective (one to one) and satisfies  $L(X \odot Y) = L(X) + L(Y)$  for all  $X, Y \in M_{n,n}^+$ . The following map is the inverse of  $L$ :

$$E : M_{n,n} \rightarrow M_{n,n}^+, \quad [E(X)]_{i,j} = \exp([X]_{i,j}).$$

This map satisfies  $E(X + Y) = E(X) \odot E(Y)$  for all  $X, Y \in M_{n,n}$ .

We can characterize reciprocal and consistent matrices by using the map  $L$ . Observe that in [13] it is proven that a matrix  $A \in M_{n,n}^+$  is consistent if, and only if, there exists  $\mathbf{w} = (w_1, \dots, w_n)^T \in M_{n,1}^+$  such that  $[A]_{i,j} = w_i/w_j$  for all  $1 \leq i, j \leq n$ .

**Theorem 2.1.** *Let  $A \in M_{n,n}^+$ .*

- (i)  *$A$  is reciprocal if, and only if,  $L(A)$  is skew-Hermitian.*
- (ii)  *$A$  is consistent if, and only if, there exists  $\mathbf{v} = (v_1, \dots, v_n)^T \in \mathbb{R}^n$  such that  $[L(A)]_{i,j} = v_i - v_j$  for all  $i, j \in \{1, 2, \dots, n\}$ .*

Let  $\mathcal{A}_n$  and  $\mathcal{S}_n$  denote the subsets of  $M_{n,n}$  composed of skew-Hermitian matrices and Hermitian matrices, respectively. In the following result we give the key properties of the set

$$\mathcal{L}_n = \{L(A) : A \in M_{n,n}^+, A \text{ is consistent}\}.$$

To this end, it will be useful to define the following map:

$$\phi : \mathbb{R}^n \rightarrow M_{n,n}, \quad [\phi(\mathbf{x})]_{i,j} = x_i - x_j, \quad \mathbf{x} = (x_1, \dots, x_n)^T. \quad (1)$$

**Theorem 2.2.** *The set  $\mathcal{L}_n$  is a linear subspace of  $M_{n,n}$  whose dimension equals  $n - 1$ .*

*Proof.* The map defined in (1) is obviously linear and  $\text{Im} \phi = \mathcal{L}_n$ . Also, it should be evident that  $\ker \phi = \text{span}\{(1, \dots, 1)^T\}$ . Thus,  $\dim \mathcal{L}_n = \dim \text{Im} \phi = \dim \mathbb{R}^n - \dim \ker \phi = n - 1$ .  $\square$

The vector of  $\mathbb{R}^n$  with all its coordinates equal to 1 will play an important role from now on. Thus we introduce a special symbol for this vector:  $\mathbf{1}_n = (1, \dots, 1)^T \in \mathbb{R}^n$ .

The main idea of using the map  $L$  and the subspace  $\mathcal{L}_n$  instead of the subset composed of consistent matrices is that we can use methods of linear algebra to solve approximation problems.

Let us recall that if we define in  $M_{n,n}$  the inner product  $\langle A, B \rangle = \text{trace}(A^T B)$  (see e.g., [22, Pg. 286]), then  $\|A\|_F^2 = \langle A, A \rangle$  holds for any  $A \in M_{n,n}$ .

If we consider this inner product, we have (see e.g., [22, Pg. 436])  $\mathcal{A}_n^\perp = \mathcal{S}_n$ . Since any matrix  $A \in M_{n,n}$  can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix by writing

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T),$$

the orthogonal projection of  $A$  onto  $\mathcal{S}_n$  is  $(A + A^T)/2$  and the orthogonal projection of  $A$  onto  $\mathcal{A}_n$  is  $(A - A^T)/2$ . Hence, for a given matrix  $A \in M_{n,n}$ , we find a reciprocal matrix  $C$  ‘close to’  $A$ . By using standard theory of linear algebra (see e.g., [22, Pg. 436]) the closest skew-Hermitian matrix  $X$  to  $L(A)$  is the orthogonal projection of  $L(A)$  onto  $\mathcal{A}_n$ , i.e.,  $X = (L(A) - L(A)^T)/2$ . Therefore, we can expect that the matrix  $E(X)$  is ‘close to’  $E(L(A)) = A$ . But

$$E(X) = E \left[ \frac{1}{2}(L(A) - L(A)^T) \right] = (A \odot J(A)^T)^{(1/2)},$$

where we have denoted with the superscript ‘ $(1/2)$ ’ the ‘component-wise square root’ (i.e., if  $M \in M_{n,n}^+$ , then  $[M^{(1/2)}]_{i,j} = [M]_{i,j}^{1/2}$ ). Since  $X$  is skew-Hermitian, then  $E(X)$  is reciprocal.

Now, our purpose is to find a consistent matrix  $C$  ‘close’ to  $A$ , where  $A \in M_{n,n}^+$  is a non-consistent given matrix in  $M_{n,n}^+$ . To this end, we linearize this problem in view of Theorem 2.1. Recall the definition of the map given in (1).

**Problem 2.1.** *For a given non-consistent matrix in  $M_{n,n}^+$ , say  $A$ , find a matrix  $X \in \mathcal{L}_n$  (or a vector  $\mathbf{x} \in \mathbb{R}$ ) such that  $X$  (or  $\phi(\mathbf{x})$ ) is ‘close’ to  $L(A)$ .*

Since  $\mathcal{L}_n$  is a linear subspace of  $M_{n,n}$  the answer to Problem 2.1 is given by the next result (obtained from a standard result of linear algebra, see e.g., [22, Pg. 436]).

**Theorem 2.3.** *Let  $A \in M_{n,n}^+$ . There exists a unique matrix  $X \in \mathcal{L}_n$  such that*

$$\|L(A) - X\|_F \leq \|L(A) - Y\|_F, \quad \forall Y \in \mathcal{L}_n.$$

*This matrix  $X$  is the orthogonal projection of  $L(A)$  onto  $\mathcal{L}_n$ .*

Let  $X$  be the matrix given in Theorem 2.3. Since  $X \in \mathcal{L}_n$ , we have that  $E(X)$  is consistent. Since  $X$  approximates to  $L(A)$ , then  $E(X)$  approximates to  $E(L(A)) = A$ . To find the orthogonal projection of  $L(A)$  onto  $\mathcal{L}_n$  it is useful to find an orthogonal basis of  $\mathcal{L}_n$ . Recall the definition of the map given in (1) and that the dimension of  $\mathcal{L}_n$  is  $n - 1$ . Also, recall that in  $M_{n,n}$  we have defined the inner product  $\langle A, B \rangle = \text{trace}(A^T B)$  and  $\mathcal{L}_n$  inherits this inner product as a subspace of  $M_{n,n}$ .

From now on, we suppose that in  $\mathbb{R}^n$  we define the standard inner product (i.e.,  $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$  for  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ) whose induced norm is the Euclidean norm (i.e.,  $\|\mathbf{u}\|^2 = (\mathbf{u}^T \mathbf{u})^{1/2}$  for  $\mathbf{u} \in \mathbb{R}^n$ ).

**Theorem 2.4.** *Let  $\{\mathbf{y}_1, \dots, \mathbf{y}_{n-1}\}$  be an orthogonal basis of the orthogonal complement to  $\text{span}\{\mathbf{1}_n\}$ . Then  $\{\phi(\mathbf{y}_1), \dots, \phi(\mathbf{y}_{n-1})\}$  is an orthogonal basis in  $\mathcal{L}_n$ .*

*Proof.* Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be the standard basis in  $\mathbb{R}^n$ . Firstly, for  $1 \leq i, j \leq n$ , let us find  $\langle \phi(\mathbf{e}_i), \phi(\mathbf{e}_j) \rangle$ . For the sake of clarity, we shall compute  $\langle \phi(\mathbf{e}_1), \phi(\mathbf{e}_1) \rangle$  and  $\langle \phi(\mathbf{e}_1), \phi(\mathbf{e}_2) \rangle$ . Observe that

$$\phi(\mathbf{e}_1) = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 0 \end{pmatrix} = (\mathbf{e}_1 - \mathbf{1}_n \mid \mathbf{e}_1 \mid \dots \mid \mathbf{e}_1).$$

and analogously we have  $\phi(\mathbf{e}_2) = (\mathbf{e}_2 \mid \mathbf{e}_2 - \mathbf{1}_n \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_2)$ . Now we obtain

$$\langle \phi(\mathbf{e}_1), \phi(\mathbf{e}_1) \rangle = \text{trace} \left\{ \left( \begin{pmatrix} \mathbf{e}_1^T - \mathbf{1}_n^T \\ \mathbf{e}_1^T \\ \vdots \\ \mathbf{e}_1^T \end{pmatrix} (\mathbf{e}_1 - \mathbf{1}_n \mid \mathbf{e}_1 \mid \cdots \mid \mathbf{e}_1) \right) \right\} = 2n - 2$$

and

$$\langle \phi(\mathbf{e}_1), \phi(\mathbf{e}_2) \rangle = \text{trace} \left\{ \left( \begin{pmatrix} \mathbf{e}_1^T - \mathbf{1}_n^T \\ \mathbf{e}_1^T \\ \vdots \\ \mathbf{e}_1^T \end{pmatrix} (\mathbf{e}_2 \mid \mathbf{e}_2 - \mathbf{1}_n \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_2) \right) \right\} = -2.$$

By symmetry,

$$\langle \phi(\mathbf{e}_i), \phi(\mathbf{e}_j) \rangle = \begin{cases} 2n - 2 & \text{if } i = j, \\ -2 & \text{if } i \neq j. \end{cases} \quad (2)$$

Now, let  $\mathbf{v} = (v_1, \dots, v_n)^T \in \mathbb{R}^n$  and  $\mathbf{w} = (w_1, \dots, w_n)^T \in \mathbb{R}^n$ . Then

$$\langle \phi(\mathbf{v}), \phi(\mathbf{w}) \rangle = \left\langle \sum_{i=1}^n v_i \phi(\mathbf{e}_i), \sum_{j=1}^n w_j \phi(\mathbf{e}_j) \right\rangle = \sum_{i,j=1}^n v_i w_j \langle \phi(\mathbf{e}_i), \phi(\mathbf{e}_j) \rangle = \mathbf{v}^T \Phi \mathbf{w}, \quad (3)$$

where  $\Phi \in M_{n,n}$  is defined by  $[\Phi]_{i,j} = \langle \phi(\mathbf{e}_i), \phi(\mathbf{e}_j) \rangle$ . Observe that by (2) we have

$$\Phi = 2nI_n - 2U_n, \quad (4)$$

being  $U_n = (\mathbf{1}_n \mid \cdots \mid \mathbf{1}_n) \in M_{n,n}$ . Now, we are ready to prove this theorem. Pick any  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ . By using (3) and (4) we have

$$\langle \phi(\mathbf{y}_i), \phi(\mathbf{y}_j) \rangle = \mathbf{y}_i^T (2nI_n - 2U_n) \mathbf{y}_j = 2n\mathbf{y}_i^T \mathbf{y}_j - 2\mathbf{y}_i^T U_n \mathbf{y}_j. \quad (5)$$

Observe that  $\mathbf{y}_i^T \mathbf{y}_j = 0$ , since  $\{\mathbf{y}_1, \dots, \mathbf{y}_{n-1}\}$  is an orthogonal system by hypothesis, and

$$\mathbf{y}_i^T U_n \mathbf{y}_j = \mathbf{y}_i^T (\mathbf{1}_n \mid \cdots \mid \mathbf{1}_n) \mathbf{y}_j = (\mathbf{y}_i^T \mathbf{1}_n \mid \cdots \mid \mathbf{y}_i^T \mathbf{1}_n) \mathbf{y}_j = 0,$$

because  $\mathbf{y}_i \in (\text{span}\{\mathbf{1}_n\})^\perp$ . Thus (5) reduces to  $\langle \phi(\mathbf{y}_i), \phi(\mathbf{y}_j) \rangle = 0$ .  $\square$

Observe that the proof of this theorem (see (5)) leads to

$$\|\phi(\mathbf{y}_i)\|_F^2 = 2n\|\mathbf{y}_i\|^2 \quad \forall i = 1, \dots, n-1.$$

Hence the answer to Problem 2.1 is given by the Fourier expansion (see e.g. [22, Pg. 299]) of  $L(A)$  onto the orthogonal system obtained in the former theorem. Precisely, we have the following result

**Theorem 2.5.** *Let  $A \in M_{n,n}^+$ . Then the following matrix*

$$X_A = \frac{1}{2n} \sum_{i=1}^{n-1} \frac{\text{trace}(L(A)^T \phi(\mathbf{y}_i))}{\|\mathbf{y}_i\|^2} \phi(\mathbf{y}_i)$$

*is the closest matrix in  $\mathcal{L}_n$  to  $L(A)$ , where  $\{\mathbf{y}_1, \dots, \mathbf{y}_{n-1}\}$  is an orthogonal basis of the orthogonal complement to  $\text{span}\{\mathbf{1}_n\}$ .*

**Note 1:** Since  $X_A$  is the closest matrix in  $\mathcal{L}_n$  to  $L(A)$ , it is expected that  $E(X_A)$  approximates to  $E(L(A)) = A$ .

The following results show that calculations involved in the Fourier expansion given in Theorem 2.5 are straightforward.

**Theorem 2.6.** *Let  $(Y_n)_{n=2}^\infty$  be the sequence of matrices defined as follows:*

$$Y_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad Y_{n+1} = \begin{pmatrix} Y_n & \mathbf{1}_n \\ 0 & -n \end{pmatrix}, \quad n \geq 2.$$

*Then for every  $n \geq 2$ , the columns of  $Y_n$  are orthogonal and belong to  $(\text{span}\{\mathbf{1}_n\})^\perp$ .*

*Proof.* The theorem is equivalent to saying that

$$Y_n^T Y_n \text{ is diagonal} \quad \text{and} \quad \mathbf{1}_n^T Y_n = 0 \tag{6}$$

for every  $n \geq 2$ . Let us prove (6) by induction on  $n$ . For  $n = 2$ , the theorem is obviously true. Assume that (6) holds for  $n$  and we will prove that (6) holds for  $n + 1$ .

$$\mathbf{1}_{n+1}^T Y_{n+1} = (\mathbf{1}_n^T \mid 1) \begin{pmatrix} Y_n & \mathbf{1}_n \\ 0 & -n \end{pmatrix} = (\mathbf{1}_n^T Y_n \mid \mathbf{1}_n^T \mathbf{1}_n - n) = 0.$$

Transposing the right identity of (6) leads to  $Y_n^T \mathbf{1}_n = 0$ , hence

$$Y_{n+1}^T Y_n = \begin{pmatrix} Y_n^T & 0 \\ \mathbf{1}_n^T & -n \end{pmatrix} \begin{pmatrix} Y_n & \mathbf{1}_n \\ 0 & -n \end{pmatrix} = \begin{pmatrix} Y_n^T Y_n & 0 \\ 0 & \mathbf{1}_n^T \mathbf{1}_n + n^2 \end{pmatrix}.$$

Thus, by one of the induction hypotheses,  $Y_{n+1}^T Y_{n+1}$  is diagonal. □

Below we write  $Y_2$ ,  $Y_3$ , and  $Y_4$ :

$$Y_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -2 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{pmatrix}.$$

**Note 2:** Observe that by Theorem 2.6, the  $n - 1$  columns of  $Y_n$  (each of these columns belongs to  $\mathbb{R}^n$ ) is an orthogonal basis of  $(\text{span}\{\mathbf{1}_n\})^\perp$ . Moreover, it is easy to prove that if  $\mathbf{y}_1, \dots, \mathbf{y}_n$  are the columns of  $Y_n$ , then  $\|\mathbf{y}_k\|^2 = k + k^2$  for  $1 \leq k \leq n - 1$ .

**Note 3:** The map  $\phi : \mathbb{R}^n \rightarrow M_{n,n}$  defined in (1) accepts a matrix representation. It is simple to prove that

$$\phi(\mathbf{v}) = \mathbf{v} \mathbf{1}_n^T - \mathbf{1}_n \mathbf{v}^T.$$

Formulas of theorems 2.5 and 2.6 are extremely simple, the number of operations really reduced, and its implementation in Matlab is completely straightforward as the following Matlab codes show

```
function y = y(n)
% This function calculates matrices Y of theorem 2.6
y = zeros(n,n-1);
for k=1:n-1
```

```

        y(1:k,k)=ones(k,1);
        y(k+1,k)=-k;
    end

function matrix =matrix(A)
% This function calculates the sought consistent matrix of theorem 2.5
B = log(A);
[n m] = size(A);
Y = y(n);
X = zeros(size(A));
for i = 1:n-1
    phiy = Y(:,i)*ones(1,n)-ones(n,1)*Y(:,i)';
    factor = trace(B'*phiy)/(i+i^2);
    X = X + factor*phiy;
end
X = X/(2*n);
matrix = exp(X);

```

### 3 Application to leakage policy in water supply

In this section, a comparison between active leakage control (ALC) and passive leakage control (PLC) in water supply is considered. The developed technique is applied in the decision-making process.

ALC involves taking actions in distribution systems or individual district metered areas to identify and repair leaks that have not been reported. PLC boils down to just repairing reported or evident leaks [23]. The main objective is the minimization of water loss by means of suitable leakage control. The criteria used to decide on the alternatives are manifold, but decision makers should be concerned with the tangible and quantitative factors, such as cost in engineering selection problems; as well as the intangible and qualitative factors, such as environmental and social impacts [24]. We consider the following criteria:

- C<sub>1</sub>: planning development cost and its implementation;
- C<sub>2</sub>: damage to properties and other service networks;
- C<sub>3</sub>: effects (cost or compensations) of supply disruptions;
- C<sub>4</sub>: inconveniences caused by closed or restricted streets;
- C<sub>5</sub>: water extractions (benefits for aquifers, wetlands or rivers);
- C<sub>6</sub>: construction of tanks and reservoirs (environmental and recreational impacts);
- C<sub>7</sub>: CO<sub>2</sub> emissions.

Upon evaluation and following the nine-point Saaty scale, the matrix  $A$  in Table 1 is obtained, which reflects the opinions of a panel of experts of a water company in Valencia (Spain) about the relative importance among the seven criteria. Only the  $n(n-1)/2$  entries above the main diagonal of  $A$  are provided, the lower triangular part being completed by reciprocity.

|                | C <sub>1</sub> | C <sub>2</sub> | C <sub>3</sub> | C <sub>4</sub> | C <sub>5</sub> | C <sub>6</sub> | C <sub>7</sub> |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| C <sub>1</sub> | 1              | 1/3            | 1/5            | 1              | 1/4            | 2              | 3              |
| C <sub>2</sub> | 3              | 1              | 1/2            | 2              | 1/3            | 3              | 3              |
| C <sub>3</sub> | 5              | 2              | 1              | 4              | 5              | 6              | 5              |
| C <sub>4</sub> | 1              | 1/2            | 1/4            | 1              | 1/4            | 1              | 2              |
| C <sub>5</sub> | 4              | 3              | 1/5            | 4              | 1              | 3              | 1              |
| C <sub>6</sub> | 1/2            | 1/3            | 1/6            | 1              | 1/3            | 1              | 1/3            |
| C <sub>7</sub> | 1/3            | 1/3            | 1/5            | 1/2            | 1              | 3              | 1              |

Table 1: Matrix of criteria,  $A$

Let us note that this matrix is inconsistent. For example,  $a_{34}a_{45} = 4(1/4) \neq a_{35} = 5$ . The Perron eigenvalue is  $\lambda_{\max} \simeq 7.9$ . According to [1], the consistency index is  $CI = (\lambda_{\max} - 7)/6 \simeq 0.148$ , and the consistency ratio, obtained by comparing  $CI$  with Saaty's random consistency index value is  $CR \simeq 10.95\%$ , which shows that even when almost acceptable, the matrix consistency is unacceptable. Also, the Monsuur's consistency threshold for  $\lambda_{\max}$  being smaller than 7.87 is not satisfied. Thus,  $A$  lacks a minimum of consistency. Following [11], additional efforts to lower the consistency ratio will improve, on average, the reliability of the analysis. With the proposed linearization process the inconsistency ratio is reduced to 0.

By using the proposed linearization approach, the new matrix given in Table 2 is obtained. This matrix uses the same Saaty's nine point scale, with the only difference of allowing intermediate values in the calculations. The entries of this matrix, being the result of a numerical process, logically do not strictly follow the integer semantics to the Saaty's nine-point scale. Nevertheless, both matrices share the same verbal scale and enable us to find a reliable vector of priorities [11].

|                | C <sub>1</sub> | C <sub>2</sub> | C <sub>3</sub> | C <sub>4</sub> | C <sub>5</sub> | C <sub>6</sub> | C <sub>7</sub> |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| C <sub>1</sub> | 1              | 0.526          | 0.154          | 0.794          | 0.471          | 1.738          | 1.17           |
| C <sub>2</sub> | 1.902          | 1              | 0.293          | 1.51           | 0.896          | 3.306          | 2.225          |
| C <sub>3</sub> | 6.487          | 3.411          | 1              | 5.149          | 3.055          | 11.28          | 7.59           |
| C <sub>4</sub> | 1.26           | 0.662          | 0.194          | 1              | 0.593          | 2.19           | 1.474          |
| C <sub>5</sub> | 2.123          | 1.116          | 0.327          | 1.685          | 1              | 3.691          | 2.484          |
| C <sub>6</sub> | 0.575          | 0.302          | 0.089          | 0.457          | .271           | 1              | 0.673          |
| C <sub>7</sub> | 0.855          | 0.449          | 0.132          | 0.678          | 0.403          | 1.486          | 1              |

Table 2: Consistent matrix closest to  $A$

For this consistent matrix the normalized Perron eigenvector, the priority vector (which is any of its columns normalized so that its components add 1), is

$$Z = (0.070, 0.134, 0.457, 0.089, 0.149, 0.041, 0.060)^T.$$

In this case, the largest value corresponds to  $C_3$ : effects (cost or compensations) of supply disruptions. The smallest value is attributed to criterion  $C_6$ : constructing tanks and reservoirs (environmental and recreational impacts).

We have to note that if the consistency of the original matrix had been considered accept-



able then the priority vector would have been

$$\mathbf{w} = (0.082, 0.147, 0.381, 0.072, 0.196, 0.046, 0.076)^T.$$

As expected, the qualitative response given by both vectors seems to be equivalent. In fact, by performing sensitivity analysis using standard perturbation methods (see next paragraph), it can be easily seen that the largest partial derivatives of  $\lambda_{\max}$  with respect to the entries  $a_{ij}$  of matrix  $A$ , with  $i < j$ , correspond to  $a_{35}$ ,  $a_{17}$ , and  $a_{36}$ . After going back to the experts and asking them if the new corresponding values for these entries in the consistent matrix (Table 2) are acceptable, the answer was clearly positive. As a consequence, this consistent matrix was considered to represent the know-how of the experts.

The matrix of partial derivatives of  $\lambda_{\max}$  with respect to the entries of  $A$ , with  $i < j$ , can be obtained (see [25, Section 3, (3.22)]) by taking the entries with  $i < j$  of

$$\left( \frac{\partial \lambda_{\max}}{\partial a_{ij}} \right) = \mathbf{w} \mathbf{v}^T - A \odot \mathbf{v} \mathbf{w}^T.$$

where  $\mathbf{v}$  is the left eigenvector of  $A$  corresponding to  $\lambda_{\max}$ , normalized so that  $\mathbf{v}^T \mathbf{w} = 1$ .

For the necessary feedback with the expert, taking into account that the expert is asked to specify his judgments via a qualitative scale [26], entries bigger than 1 in Table 2 must be replaced with the closest figures with semantic values, completing the whole matrix with reciprocity. In this process we suggest restricting the rounding to integer or to integer plus one half values (starting with 1.5 and ending with 8.5). In any of these cases the consistency ratio always remain far from unacceptable values, as can be shown using sensitivity analysis (applied herein to the consistent matrix). In this specific case, we have the values in Table 3 for the different matrices. We also include the scale-independent test for acceptable consistency given by Monsuur [10], stating for  $n = 7$  a maximum value for  $\lambda_{\max}$  of 7.87.

| Matrix            | Original | Consistent | Consistent-( $n$ ) | Consistent-( $n.5$ ) |
|-------------------|----------|------------|--------------------|----------------------|
| $\lambda_{\max}$  | 7.89     | 7          | 7.05               | 7.01                 |
| Consistency ratio | 10.95 %  | 0 %        | 0.67%              | 0.15%                |

Table 3: Consistency values

We address now the last step in AHP, as a multi-criteria decision making method. It consists in deriving, firstly, priorities for the alternatives with respect to the different criteria; and, secondly, aggregating priorities by multiplying each priority of an alternative by the priority of its corresponding criterion, by using additive aggregation, we add through all the criteria to obtain the overall priority of that alternative [1, 3, 27]. Accordingly, the next step is to obtain vectors of priorities for our two alternatives, namely ALC and PLC, for each criterion. These vectors will reflect the weight or relative importance of each alternative for each criterion. Calculation of these priority vectors is straightforward since the seven matrices are  $2 \times 2$ . In fact, it is easy to prove that positive, reciprocal  $2 \times 2$  matrices are always consistent. As a result, any column of such matrices is a principal eigenvector (corresponding to  $\lambda_{\max} = 2$ ). Consequently, normalization of any of these columns directly gives the sought priority vectors. The seven priority vectors are given in Table 4 for any of the alternative comparison matrices. In each matrix entry (1, 2) corresponds to the attributed importance of ALC over PLC for the displayed criterion.

|          |      |         |      |         |      |          |      |
|----------|------|---------|------|---------|------|----------|------|
| Planning | p.v. | Damage  | p.v. | Supply  | p.v. | Closed   | p.v. |
| 1    1/8 | 0.11 | 1    5  | 0.83 | 1    5  | 0.83 | 1    1/3 | 0.25 |
| 8    1   | 0.89 | 1/5   1 | 0.17 | 1/5   1 | 0.17 | 3    1   | 0.75 |

  

|            |      |           |      |                 |      |
|------------|------|-----------|------|-----------------|------|
| Extraction | p.v. | Reservoir | p.v. | CO <sub>2</sub> | p.v. |
| 1    4     | 0.80 | 1    1/5  | 0.17 | 1    6          | 0.86 |
| 1/4   1    | 0.20 | 5    1    | 0.83 | 1/6   1         | 0.14 |

Table 4: Matrices of alternative comparisons according to established criteria and their corresponding priority vectors (p.v.)

Finally, a score is computed for an alternative by multiplying its priority value by the priority of any criterion and summing through all the criteria. This score is shown in the coordinates of vector

$$W = \begin{pmatrix} 0.11 & 0.83 & 0.83 & 0.25 & 0.80 & 0.17 & 0.86 \\ 0.89 & 0.17 & 0.17 & 0.75 & 0.2 & 0.83 & 0.14 \end{pmatrix} \begin{pmatrix} 0.07 \\ 0.13 \\ 0.46 \\ 0.09 \\ 0.15 \\ 0.04 \\ 0.06 \end{pmatrix} = \begin{pmatrix} 0.70 \\ 0.30 \end{pmatrix}.$$

The largest coordinate of  $W$  will be associated with ‘the best alternative’ and the lowest with ‘the worst alternative’ [4].

As a consequence, in this specific problem an ALC policy should clearly be preferred over PLC. According to the considerations already made, the effects in terms of cost or compensations due to supply disruptions play a leading role in the decision. This decision is partially influenced by consideration of damage to properties and other service networks, as well as benefits for aquifers, wetlands, or rivers resulting from less water extraction. The interesting aspect regarding the application of AHP is indeed the inclusion of social costs in decision-making. In a similar way, environmental costs, and all the externalities and the usual costs for leakage management can also be included.

## 4 Conclusions

AHP is a very well established technique for decision-making and enables the evaluation of complex multi-criteria problems through a hierarchical representation of the problem, including objectives, criteria, and alternatives. Although pairwise comparisons performed in AHP have been seen as an effective way for eliciting qualitative data, a major drawback is that judgments are rarely consistent when dealing with intangibles - no matter how hard one tries - unless they are forced in some artificial manner. In this paper, we show that when starting with an inconsistent matrix, consistency can be achieved in a direct (as opposed to iterative) and straightforward manner following the described process of linearization.

For the studied problem, corresponding to the conclusions of a panel of experts in a water company in Spain and compiled after a comprehensive discussion in a workshop organized

for the company's personnel by the second author, we have shown that the alternative of active leakage control clearly outperforms the classical passive leakage control. The main factor influencing this fact comes from the consideration of costs or compensations for supply disruptions. It must be emphasized at this point that many water supply companies are liable for maintaining quality standards, and supply disruptions often result in numerous complaints from customers. Moreover, if disruptions become a major problem then political responsibilities may be felt, since many water companies are municipal or mixed private-public entities. These aspects represent a different kind of 'toll' that managers of water companies and politicians are very reluctant to pay.

The obtained results have been applied to of a complex problem in engineering: the selection of a suitable policy to manage a water supply network and avoid water losses - a worrying and crucial issue in the management of a scarce resource. The results show that the inclusion of social and environmental costs clearly points in the direction of ALC as the best alternative in leakage control. In this specific case, the economic aspects are clearly left behind by a rise in other social and environmental factors - which are more subjective objectives. We must also note that legislation in many countries has been modified and new laws have been enforced to encourage cost recovery and environmental and social responsibility. Moreover, vast sums are invested around the world encouraging responsible consumption and raising awareness about the need to care for natural resources.

For these kinds of decisions to be valid, they must be obtained from consistent matrices. Since, in general, consistency is poor, especially when many criteria are simultaneously considered, methods for improving consistency are necessary. Trial and error methods are clearly devised as very inefficient. Iterative techniques based on optimization may provide a good solution. Nevertheless, in this paper, consistency is achieved by using a closed formula. Of course, the obtained consistent matrix must be validated through suitable sensitivity analyses and feedback from the decision-maker.

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## References

- [1] T.L. Saaty, Relative Measurement and Its Generalization in Decision Making. Why Pair-wise Comparisons are Central in Mathematics for the Measurement of Intangible Factors. The Analytic Hierarchy/Network Process, *Revista de la Real Academia de Ciencias Serie A: Matemáticas* 102(2) (2008) 251-318.
- [2] Y. Dong, Y. Xu, H. Li, M. Dai, A comparative study of the numerical scales and the prioritization methods in AHP, *European Journal of Operational Research* 186 (2008) 229-242.
- [3] T.L. Saaty, A scaling method for priorities in hierarchical structures, *Journal of Mathematical Psychology* 15 (1977) 234-281.

- [4] B. Srdjevic, Linking analytic hierarchy process and social choice methods to support group decision-making in water management, *Decision Support Systems* 42 (2007) 2261-2273.
- [5] M.S. Ozdemir, Validity and inconsistency in the analytic hierarchy process, *Applied Mathematics and Computation* 161 (2005) 707-720.
- [6] E.U. Choo, W.C. Wedley, A common framework for deriving preference values from pairwise comparison matrices, *Computers & Operations Research* 31 (2004) 893-908.
- [7] G. Crawford, C. Williams, A note on the analysis of subjective judgement matrices. *Journal of Mathematical Psychology*, 29 (1985) 387-405.
- [8] T.L. Saaty, G. Hu, Ranking by the eigenvector versus other methods in the analytic hierarchy process, *Applied Mathematical Letters* 11 (4) (1998) 121-125.
- [9] J. Aznar, F. Guijarro, Nuevos métodos de valoración. Modelos Multicriterio. Universidad Politécnica de Valencia. Spain, 2008.
- [10] H. Monsuur, An intrinsic consistency threshold for reciprocal matrices. *European Journal of Operational Research* 96 (1996) 387-391.
- [11] J. S. Finan, W. J. Hurley, The analytic hierarchy process: Does adjusting a pairwise comparison matrix to improve the consistency ratio help?, *Computers & Operations Research*, 24 (1997) 749-755.
- [12] T.L. Saaty, Decision-making with the AHP: Why is the principal eigenvector necessary. *European Journal of Operational Research* 145 (2003) 85-91.
- [13] J. Benítez, X. Delgado-Galván, J. Izquierdo, R. Pérez-García, Consistent Matrices and Consistency Improvement in Decision-making Processes, *Proceedings of The Seventh International Conference on Engineering Computational Technology*, 2010.
- [14] G. Crawford, C. Williams, A note on the analysis of subjective judgement matrices, *Journal of Mathematical Psychology* 29 (4) (1985) 387-405.
- [15] L. Mikhailov, A fuzzy programming method for deriving priorities in the analytic hierarchy process, *Journal of the Operational Research Society* 51 (3) (2000) 341-349.
- [16] C.-C. Lin, An enhanced goal programming method for generating priority vectors, *Journal of the Operational Research Society* 57 (12) (2006) 1491-1496.
- [17] B. Srdjevic, Combining different prioritization methods in the analytic hierarchy process synthesis, *Computers and Operations Research* 32 (7) (2005) 1897-1919.
- [18] T.L. Saaty, L.G. Vagas, Comparison of eigenvalue, logarithmic least squares and least squares methods in estimating ratios, *Mathematical Modelling* 5 (5) (1984) 309-324.
- [19] J. Barzilai, Deriving weights from pairwise comparison matrices, *Journal of the Operational Research Society* 48 (12) (1997) 1226-1232.
- [20] N. Bryson, A goal programming method for generating priorities vectors, *Journal of the Operational Research Society* 46 (5) (1995) 641-648.

- [21] A.T.W. Chu, R.E. Kalaba, K. Springarn, A comparison of two methods for determining the weights of belonging to fuzzy sets, *Journal of Optimization Theory and Applications* 27 (4) (1979) 531-541.
- [22] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM.
- [23] M. Farley, S. Trow, *Losses in Water Distribution Networks. A practitioner's guide to assessment, monitoring and control*, IWA publishing, UK, 2003.
- [24] X. Delgado-Galván, R. Pérez-García, J. Izquierdo, J. Mora-Rodríguez, Analytic Hierarchy Process for Assessing Externalities in Water Leakage Management, *Mathematical and Computer Modeling*, 52 (2010) 1194-1202.
- [25] G.W. Stewart, *Matrix Algorithms, Vol II*, SIAM, 2001.
- [26] R. Ramanathan, U. Ramanahan, A qualitative perspective to deriving weights from pairwise comparison matrices, *Omega* 38 (2010) 228-232.
- [27] J. Aczel, T. Saaty, Procedures for synthesising ratio judgements. *Journal of Mathematical Psychology*, 27 (1983) 93-102.