Why can we not make a perfect map?

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Abstract

The purpose of this note is to present a simple proof (without using the Gauss egregium theorem) of the following fact: To make a lenght preserving projection of the Earth is impossible.

Try to wrap an orange without wrinkling the paper. Can you do this? Why? This same problem sprang up when geographers wanted to make a map of the Earth's surface with no length distortions.

In the 16th century, Mercator (a Flemish cartographer whose original name was Gerard Kremer), conceived a projection in which angles are preserved (such kind of projection is called *conformal*). On this type of map the direction remains true, making it particularly practical for navigation at sea. On Mercator's projection parallels have increased spacing in proportion to the secant of the latitude. See for example [5] about the Mercator projection and how Kremer developed it. The title of this work refers to the fact that Mercator invented it before the neccesary mathematics was invented. The equations for this projection involve a logarithm (Napier didn't invent logarithms until 1614). Deriving the equations require calculus (Newton and Leibniz, who discovered calculus, were not born until 50 years after Mercator died) and differential geometry (Gauss, who developed differential geometry, was born in 1777). See [1] about the mathematical aspects of Mercator's projection and other projections.

From this moment several kinds of projections have been designed in which angles are preserved. Another example of conformal mapping is the *stereographic projection* (see, for example [3] or [4], for a deeper study of map projections). And, what happens with the distances? It is possible to depict the Earth without any distortion?

In 19th century, Gauss solved definitively this question. The answer is no. The reason comes from the *Gauss egregium theorem* (see, for example [2]). This theorem says that the Gaussian curvature of a surface is invariant under local isometries. If a projection preserves distances, then the Gaussian curvature of the plane and the sphere must be equal, and this is false, since the Gaussian curvature of the sphere of radius R is $1/R^2$, and the Gaussian curvature of the plane is 0.

Obviously, to show this fact to student is impossible without any knowledge of differential geometry. We must use some difficult tools: definition of local isometry and Gaussian curvature, calculation of the Gaussian curvature for the plane and the sphere, and above all the Gauss egregium theorem. The purpose of this note is to give a simple proof of the following fact: To make a length preserving projection is impossible.

A map is a piece of paper where a part of the Earth's surface is depicted. We can model it as a mapping $\mathbf{r}(u, v) : D \to S$ differentiable enough , where D is an open set of $\mathbb{R}^2$ and S is the sphere whose equation is $x^2 + y^2 + z^2 = R^2$.

We can say that a curve $\mathbf{c}$ on the sphere is the image by $\mathbf{r}$ of a curve $\alpha(t) = (u(t), v(t))$ of the uv plane for $t \in [a, b]$. That is, $\mathbf{c}(t) = \mathbf{r}(\alpha(t))$. It is well-known that the length of a piecewise differentiable curve $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is $\int_a^b \|\mathbf{c}'(t)\| \, \mathrm{d}t$. By the chain's rule, it is easy to check that $\|\mathbf{c}'\| = \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}$; where $E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle$, $F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle$ and

$G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle$ (see, for example [2] for a deep study of the meaning of the coefficients E, F, G). So, the length of the curve $\mathbf{r}(\alpha(t)) = \mathbf{r} \circ \alpha$ is

$$L(\mathbf{r} \circ \alpha) = \int_a^b \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, \mathrm{d}t.$$

The map preserves the proportion of the distances if there is a constant $C > 0$ (the *scale* of the projection) such that $C \cdot L(\alpha) = L(\mathbf{r} \circ \alpha)$ for all curve $\alpha(t) = (u(t), v(t))$.

Suppose that $\mathbf{r} : D \to S$ preserves the proportion of the distances. Let $\mathbf{p}$ be an arbitrary point of $\mathbf{r}(D)$. There exists $(u_0, v_0) \in D$ such that $\mathbf{r}(u_0, v_0) = \mathbf{p}$. Consider the segment from (u_0, v_0) to $(u_0 + h, v_0 + k)$, whose parametrization is $\alpha(t) = (u_0 + th, v_0 + tk)$ for $t \in [0, 1]$. Pick $h, k \geq 0$ enough small so that $\alpha([0, 1]) \subset D$. Obviously the length of α is $\sqrt{h^2 + k^2}$ and the length of $\mathbf{r} \circ \alpha$ is $\int_0^1 \sqrt{Eh^2 + 2Fhk + Gk^2}\, \mathrm{d}t$. By hipothesis

$$\int_0^1 \sqrt{Eh^2 + 2Fhk + Gk^2}\, \mathrm{d}t = C\sqrt{h^2 + k^2}, \qquad \forall\ h, k. \tag{1}$$

If $h = 0$, then $k \int_0^1 \sqrt{G}\, \mathrm{d}t = Ck$. By the integral mean-value theorem, there exists $\xi \in [0, 1]$ (it depends on k) such that $G(\mathbf{r}(\alpha(\xi))) = C^2$. If $k \to 0+$, by continuity, we get $G(\mathbf{p}) = C^2$. Using a similar argument (make $k = 0$ in (1)), we get $E(\mathbf{p}) = C^2$. Now, if in (1) we make $h = k$, then $\int_0^1 \sqrt{E + 2F + G}\, \mathrm{d}t = C\sqrt{2}$. Again, by the integral mean-value theorem, we get $F(\mathbf{p}) = 0$. Since $\mathbf{p}$ is arbitrary, we conclude that $E = G$ is a constant and $F = 0$.

Evidently, it is trivial to check that if $E = G$ is a constant and $F = 0$, then the map $\mathbf{r}$ preserves the proportion of the distances. So we have found a characterization in terms of E, F, G of a projection that preserves the proportion of the distances. This is a corollary of a general fact of differential geometry of surfaces (see, for example, [2]).

Now suppose that $\mathbf{r} : D \to S$ is a projection that preserves the proportion of the distances. So, there exists $c \in \mathbb{R}$ such that

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = c, \qquad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0, \qquad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = c. \tag{2}$$

If we take the derivaties in (2) relative to u and v we get that $\mathbf{r}_{uu}, \mathbf{r}_{uv}$ and $\mathbf{r}_{vv}$ are orthogonal to $\mathbf{r}_u$ and to $\mathbf{r}_v$. Hence $\mathbf{r}_{uu}, \mathbf{r}_{uv}$ and $\mathbf{r}_{vv}$ are parallel to a normal vector of the sphere $x^2+y^2+z^2 = R^2$. So, there are functions α, β and γ such that $\mathbf{r}_{uu} = \alpha\mathbf{r}$, $\mathbf{r}_{uv} = \beta\mathbf{r}$ and $\mathbf{r}_{vv} = \gamma\mathbf{r}$. Now we get

$$\begin{aligned}\mathbf{0} &= (\mathbf{r}_{uu})_v - (\mathbf{r}_{uv})_u = (\alpha\mathbf{r})_v - (\beta\mathbf{r})_u \\ &= \alpha_v\mathbf{r} + \alpha\mathbf{r}_v - \beta_u\mathbf{r} - \beta\mathbf{r}_u = (\alpha_v - \beta_u)\mathbf{r} + \alpha\mathbf{r}_v - \beta\mathbf{r}_u.\end{aligned}$$

Since $\{\mathbf{r}, \mathbf{r}_u, \mathbf{r}_v\}$ is a basis of $\mathbb{R}^3$, we get $\alpha = \beta = 0$. From $0 = (\mathbf{r}_{vv})_u - (\mathbf{r}_{vu})_v$, analogously, we get $\gamma = 0$. So $\mathbf{r}_{uu} = \mathbf{r}_{uv} = \mathbf{r}_{vv} = \mathbf{0}$. Now it is easy to infer that there exists vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{r}(u,v) = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$. This is a contradiction, since this last equation is the equation of a plane. Do not forget that we are depicting the Earth, not a “flat Earth”!

References

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