Consistent completion of incomplete judgments in decision making using AHP

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What is the Analytic Hierarchy Process?

Decision making is becoming increasingly complex due to the large number of alternatives and multiple conflicting goals.

The Analytic Hierarchy Process (AHP) has been accepted as a leading multiattribute decision-aiding model.

Decisions arise in every type of application including

- Planning
- Economy
- Resource allocation
- Medical management
- Emergency situations
- ...
What is the Analytic Hierarchy Process?

Some recent uses:

- Deciding how best to reduce the impact of global climate change (Fondazione Eni Enrico Mattei)
- Quantifying the overall quality of software systems (Microsoft Corporation)
- Selecting university faculty (Bloomsburg University of Pennsylvania)
- Deciding where to locate offshore manufacturing plants (University of Cambridge)
- Assessing risk in operating cross-country petroleum pipelines (American Society of Civil Engineers)
- Deciding how best to manage U.S. watersheds (U.S. Department of Agriculture)

*Extracted from Wikipedia*

- Water management (leaks, distribution system, ...) developed by the group Fluing (part of the group are the authors of this talk).
What is the Analyty Hierarchy Process?

Example: (Reciprocal matrices)

- Journal of Superb Mathematics
- International Bulletin of Excellent Theorems
- Annals of the Best Mathematical Results

\[
\begin{pmatrix}
\text{JSM} & \text{IBET} \\
\text{JSM} & \text{ABMR} \\
\text{IBET} & \text{ABMR}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 6 & 5 \\
1/6 & 1 & 3 \\
1/5 & 1/3 & 1
\end{pmatrix}
\]
What is the **Analyty Hierarchy Process**?

**Definition**

A matrix \( A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_{n,n} \) is **reciprocal** when \( a_{ij} > 0 \) and \( a_{ij} = 1/a_{ji} \) for all \( i, j \in \{1, \ldots, n\} \).
What is the Analyty Hierarchy Process?

Example: (Consistent matrices)

\[
\begin{pmatrix}
JSM & IBET & ABMR \\
JSM & 1 & 6 & 5 \\
IBET & 1/6 & 1 & 3 \\
ABMR & 1/5 & 1/3 & 1 \\
\end{pmatrix}
\]

- I prefer JSM 6 times over IBET.
- I prefer IBET 3 times over ABMR.
- I prefer JSM 5 times over ABMR.

This is not “rational”. I had to prefer JSM \(6 \times 3\) times over ABMR!

\[6 \times 3 \neq 5\]
What is the Analytic Hierarchy Process?

**Definition**
A matrix $\mathbf{A} = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}_{n,n}$ is consistent when $a_{ij} > 0$ and $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k \in \{1, \ldots, n\}$.

**Theorem**
Let $\mathbf{A} \in \mathbb{R}_{n,n}$ be positive. If $\mathbf{A}$ is consistent, then $\mathbf{A}$ is reciprocal.

However, if $\mathbf{A}$ is reciprocal, it may happen that $\mathbf{A}$ is not consistent.

$$\mathbf{A} = \begin{bmatrix} 1 & 6 & 5 \\ 1/6 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix}.$$
What is the Analytical Hierarchy Process?

**Theorem**

If $A \in \mathbb{R}_{n,n}$ is consistent, then exists $\mathbf{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$ such that $v_i > 0$ and $A =$

$$
\begin{bmatrix}
\frac{1}{v_1} & \cdots & \frac{1}{v_n} \\
\vdots & \ddots & \vdots \\
\frac{1}{v_n} & \cdots & \frac{1}{v_1}
\end{bmatrix}.
$$

Therefore, the rank of any consistent matrix is 1.
What is the Analytic Hierarchy Process?

**Theorem**

If \( A \in \mathbb{R}_{n,n} \) is consistent, then exists \( \mathbf{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n \) such that

\[ v_i > 0 \text{ and } A = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} 1/v_1 & \cdots & 1/v_n \end{bmatrix}. \]

This vector \( \mathbf{v} \) is important:

**Theorem**

If \( A \in \mathbb{R}_{n,n} \) is consistent, then \((n, \mathbf{v})\) is an eigenpair.

**Proof:**

\[ A\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} 1/v_1 & \cdots & 1/v_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = n\mathbf{v}. \]

This vector \( \mathbf{v} \) (normalised) tells us how to rank the alternatives. It is the priority vector.
What is the Analytic Hierarchy Process?

If $A \in \mathbb{R}_{n,n}$, it is defined the spectral radius of $A$ as

$$\rho(A) = \{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

**Theorem**

*If $A \in \mathbb{R}_{n,n}$ is reciprocal, then*

- $n \leq \rho(A)$.
- $n = \rho(A)$ if and only if $A$ is consistent.

If $A$ is a reciprocal matrix “near” to be consistent, the eigenvector corresponding to $\rho(A)$ is called the priority vector. **Note:** $\rho(A)$ is an eigenvalue of $A$ with simple geometric multiplicity and a corresponding eigenvector can be taken positive (Perron Theorem).
What is the Analyty Hierarchy Process?

Is A near to be consistent?

Yes

Obtain the priority vector of A

Yes

Is A OK for the experts?

No

The experts modify A

No

Improve the consistency of A
Statement of the problem

Sometimes, the appearing comparison matrices can be incomplete.

- Some experts may not be completely familiar with one or more of the elements.
- It is possible that there exist conflicts of interests.
- Perhaps, some experts don’t want to express their opinion.
- Some data may be lost.
- ...
Sometimes, the appearing reciprocal matrices can be incomplete.

\[
A = \begin{bmatrix}
1 & 2 & 3 & \ast \\
1/2 & 1 & 3 & 4 \\
1/3 & 1/3 & 1 & \ast \\
\ast & 1/4 & \ast & 1
\end{bmatrix}
\]

Since \( \text{rk}(A) \geq 3 \), then \( A \) can’t be completed to be consistent.

**Problem**

*If \( A \) is an incomplete reciprocal matrix, find the “most consistent” completion of \( A \).*
Statement of the problem

\[ A = \begin{bmatrix}
1 & 2 & 3 & \star \\
1/2 & 1 & 3 & 4 \\
1/3 & 1/3 & 1 & \star \\
\star & 1/4 & \star & 1 \\
\end{bmatrix} \]
We need a distance in the set of reciprocal matrices!
Statement of the problem

Example

\[
A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 8 \\ 1/8 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 9 \\ 1/9 & 1 \end{bmatrix}
\]

- Matrix \(A_1\) reflects the fact that the two criteria are equivalent, while \(B_1\) reflects that the second criterion is twice as important as the second criterion.
- The importance of the criteria in \(A_2\) and \(B_2\) are very close.
- Thus, in an intuitive point of view, the distance between \(A_1\) and \(B_1\) must be much greater than the distance between \(A_2\) and \(B_2\).

\[
\|A_1 - B_1\|_F \simeq 1.118, \quad \|A_2 - B_2\|_F \simeq 1.001.
\]
Statement of the problem

We define the following distance in the set of $n \times n$ reciprocal matrices

**Definition**

$$d(A, B) = \|L(A) - L(B)\|_F,$$

where $L(A)$ is a $n \times n$ matrix whose $(i, j)$ entry is $\log(a_{ij})$.

**Example**

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 8 \\ 1/8 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 9 \\ 1/9 & 1 \end{bmatrix}$$

$$d(A_1, B_1) \simeq 0.9803, \quad d(A_2, B_2) \simeq 0.1666.$$ 

Also we need the mapping $E : M_{n,n} \to M_{n,n}$ given by $E(A)_{ij} = \exp(A_{ij})$. 

The structure of reciprocal completions

**Example**

Let \( A = \begin{bmatrix} 1 & \ast & 2 \\ \ast & 1 & \ast \\ 1/2 & \ast & 1 \end{bmatrix} \). If \( \hat{A} \) is any reciprocal completion of \( A \), then

\[
L(\hat{A}) = \begin{bmatrix} 0 & \lambda & \log 2 \\ -\lambda & 0 & \mu \\ -\log 2 & -\mu & 0 \end{bmatrix} = B_0 + \lambda \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}
\]

\[
e_1e_2^T - e_2e_1^T + e_3e_2^T - e_2e_3^T
\]

**Fact:** Let \( A \in M_{n,n}^+ \) be an incomplete reciprocal matrix. If \( \hat{A} \) is any reciprocal completion of \( A \), then there exist a skew-Hermitian matrix \( B_0 \in M_{n,n} \) and \( \lambda_1, \ldots, \lambda_k \in \mathbb{R} \) such that

\[
L(\hat{A}) = B_0 + \sum_{r=1}^{k} \lambda_r \begin{pmatrix} e_{i_r}e_{j_r}^T - e_{j_r}e_{i_r}^T \end{pmatrix}.
\]
The structure of reciprocal completions

Example

Let $A = \begin{bmatrix} 1 & * & 2 \\ * & 1 & * \\ 1/2 & * & 1 \end{bmatrix}$. If $\hat{A}$ is any reciprocal completion of $A$, then

$$L(\hat{A}) = \begin{bmatrix} 0 & \lambda & \log 2 \\ -\lambda & 0 & \mu \\ -\log 2 & -\mu & 0 \end{bmatrix} = B_0 + \lambda \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} e_1 e_2^T - e_2 e_1^T + e_2 e_3^T - e_3 e_2^T$$

Fact: Let $A \in M_{n,n}^+$ be an incomplete reciprocal matrix. If $\hat{A}$ is any reciprocal completion of $A$, then there exist a skew-Hermitian matrix $B_0 \in M_{n,n}$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$L(\hat{A}) = B_0 + \sum_{r=1}^{k} \lambda_r \left( e_{i_r} e_{j_r}^T - e_{j_r} e_{i_r}^T \right).$$

$$\left\{ L(\hat{A}) : \hat{A} \text{ is a reciprocal completion of } A \right\} \text{ is a linear manifold}$$
The problem

Theorem

Let \( \mathbf{A} \in M_{n,n} \) be a reciprocal incomplete matrix. If \( \hat{\mathbf{A}} \) is any reciprocal completion of \( \mathbf{A} \), then exists \( \mathbf{B}_0 \) a skew-Hermitian matrix and \( \lambda_1, \cdots, \lambda_k \in \mathbb{R} \) such that 

\[
L(\hat{\mathbf{A}}) = \mathbf{B}_0 + \sum_{r=1}^{k} \lambda_r \left( e_{ir} e_{jr}^T - e_{jr} e_{ir}^T \right).
\]

Problem

Given an incomplete reciprocal matrix \( \mathbf{A} \), find a reciprocal completion \( \hat{\mathbf{A}} \) and a consistent matrix \( \mathbf{X} \) such that

\[
d(\hat{\mathbf{A}}, \mathbf{X}) \leq d(\hat{\mathbf{B}}, \mathbf{Y})
\]

for any reciprocal completion \( \hat{\mathbf{B}} \) of \( \mathbf{A} \) and any consistent matrix \( \mathbf{Y} \).
The problem

Theorem

Let \( \mathbf{A} \in M_{n,n} \) be a reciprocal incomplete matrix. If \( \hat{\mathbf{A}} \) is any reciprocal completion of \( \mathbf{A} \), then exists \( \mathbf{B}_0 \) a skew-Hermitian matrix and \( \lambda_1, \cdots, \lambda_k \in \mathbb{R} \) such that

\[
L(\hat{\mathbf{A}}) = \mathbf{B}_0 + \sum_{r=1}^{k} \lambda_r \left( \mathbf{e}_r \mathbf{e}_r^T - \mathbf{e}_r \mathbf{e}_r^T \right).
\]

Problem

Given an incomplete reciprocal matrix \( \mathbf{A} \), find a reciprocal completion \( \hat{\mathbf{A}} \) and a consistent matrix \( \mathbf{X} \) such that

\[
\| L(\hat{\mathbf{A}}) - L(\mathbf{X}) \|_F \leq \| L(\hat{\mathbf{B}}) - L(\mathbf{Y}) \|_F
\]

for any reciprocal completion \( \hat{\mathbf{B}} \) of \( \mathbf{A} \) and any consistent matrix \( \mathbf{Y} \).

Theorem

The set \( \{ L(\mathbf{X}) : \mathbf{X} \text{ is consistent} \} \) is a linear subspace \((n - 1)\)-dimensional.
How can we solve this problem?

This problem can be solved by using the following result:

**Theorem**

Let $U, V$ be linear subspaces of an Euclidean vector space $E$, $p, q \in E$, and $u \in U, v \in V$. The following affirmations are equivalent.

(i) $\|p + u - (q + v)\| \leq \|p + u' - (q + v')\|$ for all $(u', v') \in U \times V$.

(ii) $p + u - (q + v) \in U^\perp \cap V^\perp$.

Let’s recall that the Frobenius norm satisfies

$$\|X\|_F^2 = \text{trace}(XX^T)$$

and $\text{trace}(XY^T)$ is an inner product in the space of $n \times n$ real matrices.
Main result

If $A \in M_{n,n}^+$ is a reciprocal incomplete matrix and $\hat{A}$ is any completion of $A$, then $L(A) = \sum_{i<j} \rho_{ij}(e_i e_j^T - e_j e_i^T) + \sum_{r=1}^{k} \lambda_r \left(e_{ir} e_{jr}^T - e_{jr} e_{rr}^T\right)$.

Theorem

If $\lambda = [\lambda_1 \cdots \lambda_k]^T$ and $\mu = [\mu_1 \cdots \mu_{n-1}]^T$ satisfy

$$\lambda = \Delta^T Y \mu, \quad Y^T \left(I_n - \frac{1}{n} \Delta \Delta^T \right) Y \mu = \frac{1}{n} \sum_{i<j} \rho_{ij} (e_j - e_i),$$

where $\Delta = [e_{j_1} - e_{j_1} \cdots e_{j_k} - e_{j_k}]$, $Y \in M_{n,n-1}$ is any matrix whose $n-1$ columns span $\{e_j\}^\perp$, then

$$E \left(B_0 + \sum_{r=1}^{k} \lambda_r \left(e_{ir} e_{jr}^T - e_{jr} e_{rr}^T\right)\right)$$

is the closest reciprocal completion of $A$ to the set of $n \times n$ consistent matrices.
The main objective is to minimize water loss through an appropriate leakage control policy.
To illustrate the application of the methodology developed, we consider a set of five criteria:

- $C_1$: Cost of development planning and implementation;
- $C_2$: budget and credit;
- $C_3$: payback;
- $C_4$: social costs;
- $C_5$: environmental costs.
For this problem, the next table presents the views of the actor involved in accordance with the Saaty scale.
On this occasion the process was undergone by an employee who is part of the team overseeing the renovation of the drinking water network in the city of Valencia (Spain).
The employee limited his judgments to those elements in which he was fully acquainted. The asterisks indicate judgments not provided.

<table>
<thead>
<tr>
<th></th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
<th>C₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₁</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>9</td>
<td>*</td>
</tr>
<tr>
<td>C₂</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>C₃</td>
<td>1/2</td>
<td>1/3</td>
<td>1</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>C₄</td>
<td>1/9</td>
<td>1/9</td>
<td>1/7</td>
<td>1</td>
<td>*</td>
</tr>
<tr>
<td>C₅</td>
<td>*</td>
<td>1/7</td>
<td>1/5</td>
<td>*</td>
<td>1</td>
</tr>
</tbody>
</table>
After applying the described process, the following values are obtained

\[
\lambda_1 = 2.07, \quad \lambda_2 = -0.24.
\]

With these values the entire matrix in the next table is built.

<table>
<thead>
<tr>
<th></th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
<th>C₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₁</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>9</td>
<td>7.92</td>
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<tr>
<td>C₂</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>C₃</td>
<td>1/2</td>
<td>1/3</td>
<td>1</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>C₄</td>
<td>1/9</td>
<td>1/9</td>
<td>1/7</td>
<td>1</td>
<td>0.79</td>
</tr>
<tr>
<td>C₅</td>
<td>0.13</td>
<td>1/7</td>
<td>1/5</td>
<td>1.27</td>
<td>1</td>
</tr>
</tbody>
</table>

The priority vector for this consistently completed matrix is

\[
Z = [0.351 \ 0.380 \ 0.189 \ 0.035 \ 0.045]^T,
\]

showing a clear dominance of the economic criteria.
Conclusions

- This mechanism of consistent completion is clearly explicit and involves only a few simple matrix calculations.
- It can be simply and directly implemented in any computing environment that includes matrix functions.
- Let us note that the size of the involved matrices in AHP, generally, is not very high.
Conclusions and further work

Conclusions

- This mechanism of consistent completion is clearly explicit and involves only a few simple matrix calculations.
- It can be simply and directly implemented in any computing environment that includes matrix functions.
- Let us note that the size of the involved matrices in AHP, generally, is not very high.

Further research

- Characterise the uniqueness of the solution.
Many thanks for your kind attention.

These slides are available at

http://personales.upv.es/jbenitez/investigacion.html