Termination of Narrowing Revisited [1]

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Abstract

This paper describes several classes of term rewriting systems (TRS's) where narrowing has a finite search space and is still (strongly) complete as a mechanism for solving reachability goals. These classes do not assume confluence of the TRS. We also ascertain purely syntactic criteria that suffice to ensure the termination of narrowing and include several subclasses of popular TRS's such as right-linear TRS's, almost orthogonal TRS's, topmost TRS's, and left-flat TRS's. Our results improve and/or generalize previous criteria in the literature regarding narrowing termination.

1 Introduction

Narrowing is a generalization of term rewriting that allows free variables in terms (as in logic programming) and replaces pattern matching with syntactic unification in order to (non–deterministically) reduce these terms. Narrowing was originally introduced as a mechanism for solving equational unification problems (Fay, 1979) and then generalized to solve the more general problem of symbolic reachability (Meseguer and Thati, 2007). The narrowing mechanism has a number of important applications including automated proofs of termination (Arts and Zantema, 1996), execution of functional–logic programming languages (Dershowitz, 1995; Hanus, 1994; Reddy, 1985; Meseguer, 1992), verification of cryptographic protocols (Meseguer and Thati, 2007), and equational unification (Hullot, 1980), just to mention a few.

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Example 1 *Consider the following term rewriting system (TRS) defining the addition* add *on natural numbers built from* 0 *and* s*:*

$$\mathtt{add}(\mathtt{0}, y) \rightarrow y \qquad \text{(R1)}$$

$$\mathtt{add}(\mathtt{s}(x), y) \rightarrow \mathtt{s}(\mathtt{add}(x, y)) \qquad \text{(R2)}$$

There are infinitely many narrowing derivations issuing from the input expression $\mathtt{add}(w, \mathtt{s}(\mathtt{0}))$ *(at each step, the narrowing relation* $\rightsquigarrow$ *is labelled with the applied substitution and rule*[3]*, and the reduced subterm is underlined):*

$$\underline{\mathtt{add}(w, \mathtt{s}(\mathtt{0}))} \rightsquigarrow_{\{w \mapsto \mathtt{0}\},\text{(R1)}} \mathtt{s}(\mathtt{0})$$

$$\underline{\mathtt{add}(w, \mathtt{s}(\mathtt{0}))} \rightsquigarrow_{\{w \mapsto \mathtt{s}(x)\},\text{(R2)}} \mathtt{s}(\underline{\mathtt{add}(x, \mathtt{s}(\mathtt{0}))}) \rightsquigarrow_{\{x \mapsto \mathtt{0}\},\text{(R1)}} \mathtt{s}(\mathtt{s}(\mathtt{0}))$$

$$\underline{\mathtt{add}(w, \mathtt{s}(\mathtt{0}))} \rightsquigarrow_{\{w \mapsto \mathtt{s}(x)\},\text{(R2)}} \mathtt{s}(\underline{\mathtt{add}(x, \mathtt{s}(\mathtt{0}))}) \rightsquigarrow_{\{x \mapsto \mathtt{s}(x')\},\text{(R2)}} \mathtt{s}(\mathtt{s}(\underline{\mathtt{add}(x', \mathtt{s}(\mathtt{0}))}))$$

$$\rightsquigarrow_{\{x \mapsto \mathtt{0}\},\text{(R1)}} \mathtt{s}(\mathtt{s}(\mathtt{s}(\mathtt{0})))$$

$$\vdots$$

The following infinite narrowing derivation resulting from applying rule (R2) infinitely many times can also be proved

$$\underline{\mathtt{add}(w, \mathtt{s}(\mathtt{0}))} \rightsquigarrow_{\{w \mapsto \mathtt{s}(x)\},\text{(R2)}} \mathtt{s}(\underline{\mathtt{add}(x, \mathtt{s}(\mathtt{0}))}) \rightsquigarrow_{\{x \mapsto \mathtt{s}(x')\},\text{(R2)}} \mathtt{s}(\mathtt{s}(\underline{\mathtt{add}(x', \mathtt{s}(\mathtt{0}))})) \cdots$$

Due to nontermination, narrowing behaves as a semi-decision procedure for the problem of equational unification in a wide variety of equational theories. For instance, in the equational theory defined by the above rules (R1) and (R2), narrowing allows us to prove that the formula $\exists w \exists z$ s.t. $\mathtt{add}(w, \mathtt{s}(\mathtt{0})) = \mathtt{s}(\mathtt{s}(z))$ holds by computing the solution $\{w \mapsto \mathtt{s}(\mathtt{0}),\ z \mapsto 0\}$, whereas it cannot prove that the formula $\exists w$ s.t. $\mathtt{add}(w, \mathtt{s}(\mathtt{0})) = \mathtt{0}$ does not hold.

Under appropriate conditions, narrowing is complete as an equational unification algorithm as well as a procedure to solve *reachability* problems; that is, it is able to find "more general" solutions σ for the variables of terms s and t, such that $s\sigma$ rewrites to $t\sigma$ in $\mathcal{R}$ in a number of steps. For instance, narrowing computes the solution $\{w \mapsto \mathtt{s}(z)\}$ for the reachability problem $\exists w \exists z$ s.t. $\mathtt{add}(\mathtt{0}, w) \rightarrow^* \mathtt{s}(z)$.

In this paper, we are interested in identifying classes of TRS's where narrowing terminates and is still complete for solving reachability problems. Termination of narrowing is an important property for finitary equational unification (Dershowitz and Mitra, 1999; Fay, 1979; Hullot, 1980; Mitra and Dershowitz, 1996) and equational constraint solving (Alpuente et al., 1993, 1995a), as well as for developing semantics–based tools such as model checkers (Escobar and Meseguer, 2007), and program specializers or debuggers (Alpuente et al., 1998,

[3] Substitutions are restricted to the input variables.

2002) for functional logic programming languages whose operational principle is based on narrowing (Dershowitz, 1995; Hanus, 1994; Reddy, 1985; Meseguer, 1992). In this article, we do not consider extra artifacts to reduce or limit the narrowing space.

Basically, the only positive result in the literature concerning the termination of ordinary narrowing was proved by Christian (1992). It holds for every left-flat TRS $\mathcal{R}$ (each argument of the left–hand side of a rewrite rule is either a variable or a ground term) such that the rewrite rules are oriented by a termination ordering $>$: $\mathcal{R} \subseteq >$.

A faulty termination result for ordinary narrowing was published in (Hullot, 1980, Proposition 1) and is the starting point for our work. This result incorrectly stated that ordinary narrowing terminates in canonical TRS's if all *basic* narrowing derivations (narrowing derivations which do not reduce certain *blocked* positions) that issue from the right hand side of each rewrite rule terminate. Unfortunately, under the conditions established by Hullot, his proof only allows one to conclude the termination of basic narrowing, which was implicitly corrected in (Hullot, 1981). Results in the literature that take advantage of, or are built on top of, Hullot's termination result for narrowing are based on a false assumption and may need to be revised in light of the results presented in this article.

A detailed discussion of existing completeness and termination results for narrowing is given in Section 3.

1.1 Our contributions

The contributions of this paper are as follows:

(1) We fix Hullot's termination result for ordinary narrowing in canonical TRS's where all basic derivations issuing from the rhs's of the rules terminate. This is achieved by requiring the TRS to satisfy Réty's maximal commutation conditions, which allow the establishment of a correspondence between ordinary and basic narrowing derivations (Corollary 13). In the process we explicitly drop the superfluous requirement of canonicity from Hullot's result, as few cognoscenti tacitly do.

To our knowledge, this is the first termination result in the literature for ordinary narrowing which holds in (a subclass of) linear TRS's and is enunciated in Hullot's style without requiring canonicity.

(2) From Corollary 13, we distill a practical criterion for the termination of narrowing that has not been previously identified in the related literature and that does not yet require confluence of the TRS nor a termination

ordering. We achieve this by imposing that the TRS be linear and rnf-*based*, a novel class of TRS's that can be seen as a generalization of left-linear constructor systems and that satisfy Réty's normalization condition (Corollary 22).

A TRS is rnf-*based* if each argument occurring in the lhs of every rewrite rule is "unnarrowable", called *rigid normal form* (rnf), i.e., contains no subterm that unifies with the lhs of any rule. The class of rnf-based TRS's includes both constructor systems and almost orthogonal TRS's as a particular case.

(3) Then, we consider the class of TRS's where narrowing is strongly complete, as a procedure to solve reachability goals. This allows us to prove narrowing termination in a number of TRS's where right–linearity is not explicitly required (Corollary 32).

(4) Inspired by Christian's termination result (Christian, 1992), we are able to further improve our results and also get rid of left–linearity, by proving termination for a subclass of *left-plain* TRS's, a novel class where arguments of the lhs's can be either ground or rnf–patterns (Theorem 44).

(5) Finally, by using the known results for the strong reachability completeness of narrowing recently given by Meseguer and Thati (2007), we identify several purely syntactical, non–trivial classes of TRS's where narrowing has a finite search space and is still (strongly) complete as a procedure to solve reachability goals (Corollary 45).

From the above results, termination of several popular TRS's follow, including right–rnf TRS's which are either (i) almost orthogonal, (ii) constructor and either right–linear or confluent, (iii) topmost, and (iv) right–linear. These results are particularly practical since many interesting TRS's fit into one of these classes. Differently from Christian's criterion (Christian, 1992), our termination criteria do not resort to termination orderings, and are thus simpler to check.

A table summarizing the relevant results is included at the end of the paper.

1.2 Plan of the paper

Section 2 presents some preliminary notions and results. Section 3 summarizes the main completeness and termination results in the literature of narrowing. In Section 4, we clarify the main source of error in Hullot's termination result for canonical TRS's, and we correct it by using Réty's maximal commutation property (Réty, 1987). In Section 5, we show that canonicity is a superfluous requirement in Hullot's termination result, and then distill a practical criterion for narrowing termination which holds for TRS's that are linear and rnf–based. Section 6 introduces the class of reachability-complete TRS's, which allows us to get rid of right–linearity. Finally, Section 7 provides a strong narrowing termination criterion which holds in left-plain, right–rnf TRS's, provided they are also reachability–complete. Section 8 concludes. Proofs of the main technical results are given in Appendix.

2 Preliminaries

In this section, we briefly recall the essential notions and terminology of term rewriting (Dershowitz and Jouannaud, 1990; Ohlebusch, 2002; TeReSe, 2003). $\mathcal{V}$ denotes a countably infinite set of variables, and Σ denotes a set of function symbols, or signature, each of which has a fixed associated arity. Terms are viewed as labelled trees in the usual way, where $\mathcal{T}(\Sigma,\mathcal{V})$ and $\mathcal{T}(\Sigma)$ denote the non-ground term algebra and the ground algebra built on $\Sigma \cup \mathcal{V}$ and Σ, respectively. Positions are defined as sequences of positive natural numbers used to address subterms, with the empty sequence ϵ as the root (or top) position. Concatenation of positions p and q is denoted by $p.q$, and $p < q$ is the usual prefix ordering. The concatenation of a position p and a set of positions P is $p.P = \{p.q \mid q \in P\}$. Two positions p, q are disjoint, denoted by $p \parallel q$, if neither $p < q$, $p > q$, nor $p = q$. Given $S \subseteq \Sigma \cup \mathcal{V}$, $\mathcal{P}os_S(t)$ denotes the set of positions of a term t that are rooted by function symbols or variables in S. $\mathcal{P}os_{\{f\}}(t)$ with $f \in \Sigma \cup \mathcal{V}$ is simply denoted by $\mathcal{P}os_f(t)$, and $\mathcal{P}os_{\Sigma\cup\mathcal{V}}(t)$ is simply denoted by $\mathcal{P}os(t)$. $t|_p$ is the subterm at the position p of t. $t[s]_p$ is the term t with the subterm at the position p replaced with term s. Syntactic equality of terms is represented by $\equiv$. By $Var(s)$, we denote the set of variables occurring in the syntactic object s. By $\bar{x}$, we denote a tuple of pairwise distinct variables. A *fresh* variable is a variable that appears nowhere else. A *linear* term is one where every variable occurs only once.

A *substitution* is a mapping from the set of variables $\mathcal{V}$ into the set of terms $\mathcal{T}(\Sigma,\mathcal{V})$. A substitution is represented as $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ for variables $x_1, \ldots, x_n$ and terms $t_1, \ldots, t_n$. The application of substitution θ to term t is denoted by $t\theta$, using postfix notation. Composition of substitutions is denoted by juxtaposition, i.e., the substitution $\sigma\theta$ denotes $(\theta \circ \sigma)$. The *domain* of a substitution σ is $Dom(\sigma) = \{x \in \mathcal{V} \mid x\sigma \not\equiv x\}$, and $Rng(\sigma) = \{x\sigma \mid x \in Dom(\sigma)\}$ is its *range*. The set of variables in $Rng(\sigma)$ is denoted by $VRng(\sigma)$. The *empty substitution* is denoted by *id*, i.e., $Dom(id) = \emptyset$. A substitution θ is more (or equally) general than σ, denoted by $\theta \leq \sigma$, if there is a substitution γ such that $\sigma = \theta\gamma$. We write $\theta_{\restriction Var(s)}$ to denote the restriction of the substitution θ to the set of variables in s; by abusing notation, we often simply write $\theta_{\restriction s}$. Given a set of variables W, we write $\theta = \nu\ [W]$ for $\theta_{\restriction W} = \nu_{\restriction W}$, i.e., $\forall x \in W$,

$x\theta \equiv x\nu$. A *renaming* is a substitution σ for which there exists the inverse σ^{-1}, such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = id$. A *unifier* of terms s and t is a substitution ϑ such that $s\vartheta \equiv t\vartheta$. The *most general unifier* of terms s and t, denoted by $mgu(s,t)$, is a unifier θ such that for each other unifier θ', $\theta \leq \theta'$.

A *term rewriting system* $\mathcal{R}$ (TRS for short) is a pair (Σ, R), where R is a finite set of rewrite rules of the form $l \to r$ such that $l, r \in \mathcal{T}(\Sigma, \mathcal{V})$, $l \notin \mathcal{V}$, and $Var(r) \subseteq Var(l)$. We will often write just $\mathcal{R}$ instead of (Σ, R). For TRS $\mathcal{R}$, $l \to r \ll \mathcal{R}$ denotes that $l \to r$ is a new variant of a rule in $\mathcal{R}$ such that $l \to r$ contains only *fresh* variables, i.e., contains no variable previously met during any computation (standardized apart). A TRS $\mathcal{R}$ is called *conservative* (or regular) if, for every $l \to r \in \mathcal{R}$, $Var(l) = Var(r)$. A TRS $\mathcal{R}$ is called *left–linear* (respectively *right–linear*) if, for every $l \to r \in \mathcal{R}$, l (respectively r) is a linear term. A *linear* TRS is both left and right–linear.

Given a TRS $\mathcal{R} = (\Sigma, R)$, the signature Σ is often partitioned into two disjoint sets $\Sigma := \mathcal{C} \uplus \mathcal{D}$, where $\mathcal{D} := \{f \mid f(t_1, \ldots, t_n) \to r \in R\}$ and $\mathcal{C} := \Sigma \setminus \mathcal{D}$. Symbols in $\mathcal{C}$ are called *constructors*, and symbols in $\mathcal{D}$ are called *defined functions*. The elements of $\mathcal{T}(\mathcal{C}, \mathcal{V})$ are called *constructor terms*. A TRS is a *constructor system* (CS for short) if the left–hand sides of $\mathcal{R}$ are *patterns*, i.e., terms of the form $f(d_1, \ldots, d_k)$ where $f \in \mathcal{D}$ and $d_1, \ldots, d_k$ are constructor terms.

A rewrite step is the application of a rewrite rule to an expression. A term $s \in \mathcal{T}(\Sigma, \mathcal{V})$ *rewrites* to a term $t \in \mathcal{T}(\Sigma, \mathcal{V})$, denoted by $s \to_{\mathcal{R}} t$, if there exist $p \in \mathcal{P}os_{\Sigma}(s)$, $l \to r \ll \mathcal{R}$, and substitution σ such that $s|_p \equiv l\sigma$ and $t \equiv s[r\sigma]_p$. When no confusion can arise, we omit the subscript $\mathcal{R}$. A term s is a *normal form* w.r.t. the relation $\to_{\mathcal{R}}$ (or simply a normal form), if there is no term t such that $s \to_{\mathcal{R}} t$. This notion is lifted to substitutions as follows: a substitution σ is *normalized* if, for every $x \in \mathcal{V}$, $x\sigma$ is a normal form.

A TRS $\mathcal{R}$ is *terminating* (also called strongly normalizing or noetherian) if there are no infinite reduction sequences $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$. In other words, every reduction sequence eventually ends in a normal form. A TRS $\mathcal{R}$ is *confluent* if, whenever $t \to^*_{\mathcal{R}} s_1$ and $t \to^*_{\mathcal{R}} s_2$, there exists a term w s.t. $s_1 \to^*_{\mathcal{R}} w$ and $s_2 \to^*_{\mathcal{R}} w$. A confluent and terminating TRS is called canonical[4]. In canonical TRS's, each term has one (and only one) normal form. Two (possibly renamed) rules $l \to r$ and $l' \to r'$ *overlap* if there is $p \in \mathcal{P}os_{\Sigma}(l)$ and substitution σ such that $l|_p\sigma \equiv l'\sigma$. The pair $\langle l\sigma[r'\sigma]_p, r\sigma\rangle$ is called a *critical pair*; it is called an *overlay* if $p \equiv \epsilon$. A critical pair $\langle t, s\rangle$ is *trivial* if $t \equiv s$. A left-linear TRS without critical pairs is called *orthogonal*. A left-linear TRS whose critical pairs are trivial overlays is called *almost orthogonal*. Note that orthogonal TRS's are almost orthogonal and almost orthogonality implies confluence (TeReSe,

[4] Canonical TRS's are sometimes called *complete* (Knuth and Bendix, 1970; Hullot, 1980; Middeldorp and Hamoen, 1994).

2003).

A TRS $\mathcal{R}$ is called *topmost* if, for every term t, all rewritings on t are performed at the root position of t. Although topmost TRS's are not commonly used in term rewriting, they are relevant in programming languages. For instance, in Haskell (Peyton Jones, 2003) or Maude (Clavel et al., 2007), rewrite rules can be defined so that the type (or sort) information forces rewrites to happen only at the top of terms. In Maude, it is also possible to introduce freezing specifications that block rewrites at any proper subterm position. Actually, many concurrent systems of interest, including the vast majority of distributed algorithms, admit quite natural topmost specifications (Meseguer and Thati, 2007). In an unsorted setting like ours, topmost TRS's are only those that do not contain any function symbol whose arity is greater than 0 (that is, all rules have the form $a \to b$).

Narrowing is a symbolic computation mechanism that generalizes rewriting by replacing pattern matching with syntactic unification. W.l.o.g. we restrict ourselves to narrowing of terms; the extension of narrowing for (equational as well as reachability) goals is straightforward, see e.g. (Hölldobler, 1989; Meseguer and Thati, 2007). A term $s \in \mathcal{T}(\Sigma, \mathcal{V})$ narrows to $t \in \mathcal{T}(\Sigma, \mathcal{V})$, denoted by $s \leadsto_{\theta,\mathcal{R}} t$ if there exist $p \in \mathcal{P}os_{\Sigma}(s)$, $l \to r \ll \mathcal{R}$, and substitution θ such that $\theta = mgu(s|_p, l)$ and $t \equiv (s[r]_p)\theta$. When we want to emphasize the position p where a rewriting (respectively narrowing) step took place, we write $s \stackrel{p}{\to}_{\mathcal{R}} t$ (respectively $s \stackrel{p}{\leadsto}_{\theta,\mathcal{R}} t$). We may also write $s \stackrel{p}{\leadsto}_{\theta,l\to r} t$ when we also want to emphasize the applied rule. We denote the transitive and reflexive closure of $\to_{\mathcal{R}}$ (respectively $\leadsto_{\theta,\mathcal{R}}$) by $\to^*_{\mathcal{R}}$ (respectively $\leadsto^*_{\theta',\mathcal{R}}$).

3 Existing Termination and Completeness Results for Narrowing

Existing termination results for narrowing have been obtained as a by-product of other works that address the decidability of equational unification or the completeness of narrowing-based equational unification algorithms. To facilitate the understanding of our results, let us first summarize the existing completeness results for narrowing as a procedure to solve equational unification as well as reachability goals.

3.1 Existing completeness results for narrowing

Fay (1979) and Hullot (1980) demonstrated that narrowing is a complete method for solving equational unification goals $s_1 = t_1, \ldots, s_n = t_n$ in an equational theory defined by a canonical term rewriting system $\mathcal{R}$. In the equational setting, completeness means that, for every solution ρ to a given equational goal G (i.e., $\mathcal{R} \vdash s_i\rho = t_i\rho$, for all i s.t. $1 \leq i \leq n$), a more general solution η can be found by narrowing. Strictly speaking, the relative generality of substitution η w.r.t. ρ holds *modulo* $\mathcal{R}$ and is restricted to the variables of G, or more formally:

$$\eta \leq_{\mathcal{R}} \rho \; [Var(G)] \; (\textit{unification–completeness})$$

This means that there exists a substitution σ s.t., for all $x \in Var(G)$, the equation $x\rho = x\eta\sigma$ holds in $\mathcal{R}$, which can be proved by rewriting terms $x\rho$ and $x\eta\sigma$ in $\mathcal{R}$ to the same normal form, due to canonicity. The subindex $\mathcal{R}$ in $\leq_{\mathcal{R}}$ can be dropped only when we restrict our interest to *normalized* (or irreducible) substitutions, which is generally understood as a *weaker* result from both the semantic as well as the pragmatic point of view (Meseguer and Thati, 2007). If we drop the termination of $\mathcal{R}$ while keeping confluence, narrowing is (unification-) complete only w.r.t. normalizable solutions (Middeldorp and Hamoen, 1994).

In the extensive literature about narrowing, unification–completeness has been thoroughly investigated for a number of narrowing restrictions which are obtained by imposing specific narrowing *strategies*; see (Hanus, 1994) for a survey. In this work, we restrict our interest to *ordinary* (sometimes called full, unrestricted or simple) narrowing, as defined in Section 2. An investigation of completeness or termination for sophisticated narrowing strategies is beyond the scope of this paper.

From a practical point of view, equational unification problems can be seen as a special case of reachability problems. Namely, under canonicity of $\mathcal{R}$, solving a unification problem $\exists\bar{x}.\ s = t$ can be transformed into solving the corresponding reachability problem $\exists\bar{x}.\ (s \approx t) \to^* \mathtt{true}$ in the extended term rewriting system $\mathcal{R} \cup (x \approx x \to \mathtt{true})$ where both problems have the same solutions provided that $\approx$ is a fresh binary function symbol and $\mathtt{true}$ is a fresh constant (Meseguer and Thati, 2007; Middeldorp and Hamoen, 1994). The extension of $\mathcal{R}$ with the extra rule $(x \approx x \to \mathtt{true})$ allows treating equality $=$ as an ordinary function symbol $\approx$ and syntactic unification as a narrowing step, i.e., in the extended TRS, the "term" $s \approx t$ narrows to $\mathtt{true}$ with substitution σ iff σ is the most general unifier of s and t. Alternative formulations of narrowing-based equational unification procedures that do not extend $\mathcal{R}$ by this extra rewrite rule complement the narrowing calculus with an additional inference rule to cope with syntactic unification, e.g. (Hölldobler, 1989).

As stated above, the completeness of narrowing as a procedure to solve equational goals heavily depends on the condition that the rewrite rules are confluent. Actually, in the standard equational setting, confluence is the property which allows considering equations as rewrite rules (oriented from left to right). The equational theory axiomatized by $\{\texttt{f(a)} = \texttt{b}, \texttt{f(a)} = \texttt{c}\}$ is a trivial counter-example to unification–completeness when confluence does not hold. Here narrowing fails to prove the equation $\texttt{b} = \texttt{c}$ in the corresponding (oriented) TRS $\mathcal{R} = \{\texttt{f(a)} \to \texttt{b}, \texttt{f(a)} \to \texttt{c}\}$, whereas $\texttt{b} = \texttt{c}$ holds in the original equational theory.

In (Meseguer and Thati, 2007), reachability goals $s_1 \to^* t_1, \ldots, s_n \to^* t_n$ are investigated in non–confluent term rewriting systems in order to solve verification problems of cryptographic protocols. Many safety properties (i.e., properties of a system that are defined in terms of certain events not happening) can be characterized in terms of reachability problems. By finding all solutions to a reachability goal $s \to^* t$ (i.e., the substitutions σ such that $\mathcal{R} \vdash s\sigma \to^* t\sigma$), the subset of the states denoted by s that can reach a subset of the states denoted by t can be easily inferred. Hence, reachability problems extend narrowing capabilities to a wider spectrum that includes the analysis of concurrent systems. Similarly to the equational case, the procedure for solving reachability goals performs syntactic unification at the last step of the derivation; this way, trivial goals such as $x \to^* y$ (where there is no redex to narrow) do succeed in computing a more general solution. In the reachability context, confluence is no longer a reasonable (or needed) assumption and is thus done away with (e.g., concurrent systems are inherently non–deterministic).

The new completeness results for narrowing given in (Meseguer and Thati, 2007)[5] for solving reachability goals in (possibly) non–confluent TRS's are summarized as follows. Narrowing is *weakly* complete, i.e., complete w.r.t. *normalized* solutions: for every normalized solution ρ to a reachability goal G, a (syntactically) more general solution η is found by narrowing, in symbols:

$$\eta \leq \rho \; [\mathit{Var}(G)] \; (\textit{weak reachability–completeness})$$

Note that neither confluence nor termination of $\mathcal{R}$ are required.

In (Meseguer and Thati, 2007), *strong* reachability–completeness (i.e., completeness w.r.t. not necessarily normalized solutions, i.e. solutions that can be further rewritten by $\mathcal{R}$) is proved to hold only in the following two particular classes of TRS's: (i) topmost, and (ii) right–linear (provided that we additionally restrict ourselves to linear reachability goals $\wedge_{i=1}^{n}(s_i \to^* t_i)$, where each s_i is linear). Under these asumptions, for every solution ρ to a reachability goal G, a more general solution η (*modulo* $\mathcal{R}$) is computed by narrowing, i.e., $\eta \leq_{\mathcal{R}} \rho \; [\mathit{Var}(G)]$. In the reachability setting, where confluence cannot be assumed and thus equality in $\mathcal{R}$ cannot be decided by rewriting, the definition is translated as follows: there is a (syntactic) instance θ of the computed sub-

[5] The completeness results in (Meseguer and Thati, 2007) concern more general rewrite theories that consist of a set of rewrite rules $\mathcal{R}$ together with a set of equations E so that rewriting and narrowing in $\mathcal{R}$ are defined *modulo* E.

stitution η such that the (possibly not normalized) solution ρ reduces to θ. To be precise:

$$\rho_{\upharpoonright Var(G)} \rightarrow^*_{\mathcal{R}} \theta_{\upharpoonright Var(G)} \text{ and } \eta \leq \theta\ [Var(G)]\ (\textit{strong reachability–completeness})$$

Of course, unification–completeness trivially implies reachability–completeness, hence (strong) reachability–completeness of narrowing holds for canonical programs, whereas narrowing is not unification–complete in either right–linear or topmost TRS's (Middeldorp and Hamoen, 1994).

In the case of right–linear TRS's, linearity of the goal is a key requirement which cannot be dropped, as shown in the following example.

Example 2 (Meseguer and Thati, 2007) *Consider the TRS* $\mathcal{R} = \{\texttt{f(b,c)} \rightarrow \texttt{d}, \texttt{a} \rightarrow \texttt{b}, \texttt{a} \rightarrow \texttt{c}\}$*. The non–linear reachability goal* $\texttt{f(x,x)} \rightarrow^* \texttt{d}$ *has a solution* $\{x \mapsto \texttt{a}\}$*, whereas there is no narrowing derivation stemming from the term* `f(x,x)`*.*

This example shows that reachability–incompleteness of narrowing for general TRS's is mainly due to rewrites that must happen within non–normalized substitutions but are missed by the narrowing procedure, since narrowing steps do not apply to variable positions. In the standard equational setting, these "under the feet" rewritings are inconsequential, due to confluence.

3.2 Existing termination results for narrowing

In the literature, the termination of narrowing has received less attention than completeness. Actually, termination of narrowing is a much more difficult property to achieve than termination of standard term rewriting; see (Ohlebusch, 2002) for a survey on rewriting termination.

Termination results for narrowing calculi have been obtained as a by-product of other works that address the decidability of equational unification; a summary can be found in (Dershowitz and Mitra, 1999). Most of these results are truly restrictive and do not allow any recursively defined function. Most works introduce specially-tailored equational unification procedures based on the generally more expensive "top-down decomposition approach" outlined in (Martelli et al., 1986) (not considered in this paper). Narrowing–based procedures with a finite search space often incorporate a test to cut unproductive, infinitely failing derivations (Alpuente et al., 1995b; Chabin and Réty, 1991; Dershowitz and Sivakumar, 1988) or a kind of graph-based *memoization* technique (Antoy and Ariola, 1997; Escobar and Meseguer, 2007) to achieve, in some cases, a finite representation of an infinite narrowing space. There are popular[6] (syntactic) conditions that, together with termination and (often) confluence of $\mathcal{R}$, are required for the termination of these procedures. These include (Dershowitz and Mitra, 1999): *left–linearity* (no variable appears in the lhs of a rewrite rule more than once); right–hand side (rhs) groundness, *right–groundness* (rhs's of rewrite rules contain no variable); and *left–flatness* (each argument occurring at the lhs of a rewrite rule is either a variable —often called *shallow* (Comon et al., 1994)— or a ground term).

Unfortunately, the decidability of unification for a given equational theory does not imply the termination of ordinary narrowing in the corresponding TRS. For instance, unification is decidable in the equational theory associated to the function `add` of Example 1 above (see e.g. (Dershowitz and Mitra, 1999)) whereas narrowing does not terminate for the input equation $\mathtt{add}(w, \mathtt{s}(\mathtt{0})) = \mathtt{0}$ (as we have shown). Achieving termination without losing completeness is possible for this particular example by adding an extra "failure rule", which is able to detect a clash conflict between the irreducible symbols `0` and `s` in the derived equational goal $\mathtt{s}(\mathtt{add}(x, \mathtt{s}(\mathtt{0}))) = \mathtt{0}$. However, as the following example shows it is more difficult in general.

Example 3 *Consider the TRS consisting of the "shallow" oriented commutativity axiom for a binary symbol f: $\mathcal{R} = \{\mathtt{f(x,y)} \to \mathtt{f(y,x)}\}$. An extra artifact such as a "loop checker" would be needed to stop narrowing from the input equation $\mathtt{f(x,y)} = \mathtt{z}$ in $\mathcal{R}$, whereas the corresponding equational theory defined by $\mathcal{R}$ is not only decidable but actually finitary (Siekmann, 1989) (actually, the considered equational goal has exactly two solutions $\{z \mapsto \mathtt{f(x,y)}\}$ and $\{z \mapsto \mathtt{f(y,x)}\}$).*

Summarizing, the only positive result in the literature concerning the termination of ordinary narrowing was proved in (Christian, 1992) and holds for every left-flat TRS $\mathcal{R}$ that is compatible with a termination ordering $<$. Termination of narrowing does not hold for systems with flat right-hand sides (even if linearity is also imposed), as proved in (Mitra and Dershowitz, 1996).

In general, whenever the lhs of a rewrite rule is not flat, aliasing due to repeated variables can cause troublesome propagation of hazardous structure as shown by the following example.

Example 4 *(Christian, 1992) The non–flat rule $\mathtt{f(f(x))} \to \mathtt{x}$ is "safe" when used to narrow a linear term like $\mathtt{c(f(u),v)}$: it produces the term $\mathtt{c(x,v)}$, which cannot be further narrowed. However, the non–linear term $\mathtt{c(f(x),x)}$*

[6] These properties have been studied in the context of other rewriting-related properties and problems also, such as joinability, modularity of termination, and modularity of confluence.

can be narrowed indefinitely:

$$\texttt{c(}\underline{\texttt{f(x)}}\texttt{,x)} \leadsto_{\{x\mapsto \texttt{f(x')}\}} \texttt{c(x',}\underline{\texttt{f(x')}}\texttt{)} \leadsto_{\{x'\mapsto \texttt{f(x'')}\}} \texttt{c(}\underline{\texttt{f(x'')}}\texttt{,x'')} \cdots$$

A number of mistakes concerning completeness and termination proofs and results for narrowing (and some of its variants) have been pointed out in the related literature and summarized in (Middeldorp and Hamoen, 1994). In the following section, we focus on one of them, which is the starting point for our work.

3.3 A faulty result concerning termination of narrowing

Hullot (1980) introduced a restricted form of narrowing called *basic narrowing* (see the next section for details) which obtains a search space reduction by restricting narrowing steps to subterms that were not introduced by instantiation, while still being unification–complete for canonical TRS's.

For canonical TRS's, the seminal paper by Hullot (1980) establishes a faulty result for the termination of narrowing in (Hullot, 1980, Proposition 1). The result incorrectly stated that ordinary narrowing terminates in canonical TRS's when all *basic* narrowing derivations issued from the right hand side of each rewrite rule terminate. This result can be refuted by the following counterexample.

Example 5 *Consider again the TRS of Example 4, which is canonical and trivially satisfies the requirement that (basic) narrowing terminates for the rhs* `x`*. However, Example 4 above shows that an infinite narrowing derivation exists in* $\mathcal{R}$*.*

Actually, under the conditions established by Hullot's proof, nothing beyond the termination of basic narrowing can be concluded, as implicitly [7] corrected in Hullot's thesis (1981). Note that basic narrowing does "safely" handle the TRS $\{\texttt{f(f(x))} \to \texttt{x}\}$ of Example 4 and blocks the infinite narrowing derivation after the first step.

[7] The correct termination result which only guarantees the termination of basic narrowing under the same assumptions was established in (Hullot, 1981), and subsequently referred to in a number of works (Hölldobler, 1989; Middeldorp and Hamoen, 1994; Réty, 1987).

4 Repairing Hullot's termination result for Canonical TRS's

Here we formulate basic narrowing using the original definition, given by Hullot and subsequently used by (Réty, 1987; Middeldorp and Hamoen, 1994), which is based on restricting narrowing steps to a distinguished set of *basic* positions. Nevertheless, for the proofs given in Appendix B, we find more convenient to use an equivalent, easier formalization of (Hölldobler, 1989).

Given a narrowing derivation D: $t_0 \overset{p_1}{\rightsquigarrow}_{\theta_1,\mathcal{R}} t_1 \overset{p_2}{\rightsquigarrow}_{\theta_2,\mathcal{R}} \cdots \overset{p_n}{\rightsquigarrow}_{\theta_n,\mathcal{R}} t_n$, where $l_i \rightarrow r_i \in R$ is used at step i, we inductively define the *basic positions* of D as $B_0 = \mathcal{P}os_\Sigma(t_0)$ and $B_i = (B_{i-1} \setminus p_i.\mathcal{P}os(t_{i-1}|_{p_i})) \cup p_i.\mathcal{P}os_\Sigma(r_i)$. Informally, a basic occurrence is a non-variable occurrence of the original term or one that was introduced by the non-variable content of the rhs of an applied rule.

We define a *basic narrowing derivation* $s \rightsquigarrow^*_\theta t$ as $s_0 \overset{p_1}{\rightsquigarrow}_{\theta_1} s_1 \cdots s_{n-1} \overset{p_n}{\rightsquigarrow}_{\theta_n} s_n$ such that $s \equiv s_0$, $t \equiv s_n$, $\theta \equiv \theta_1 \cdots \theta_n$, and $p_i \in B_{i-1}$ for $1 \leq i \leq n$.

Example 6 *Consider the TRS* $\mathcal{R} = \{\mathtt{a} \rightarrow \mathtt{0}, \mathtt{f(x)} \rightarrow \mathtt{h(x)}\}$ *and input term* `f(a)`*. The following narrowing derivation is not basic* `f(a)` $\rightsquigarrow_{id,f(x)\rightarrow h(x)}$ `h(a)` $\rightsquigarrow_{id,a\rightarrow 0}$ `h(0)`*, since position* 1 *selected at the second narrowing step is not basic (the narrowing redex* `a` *was introduced by instantiation of the rhs* `h(x)` *of the second rule). A basic narrowing derivation is* `f(a)` $\rightsquigarrow_{id,a\rightarrow 0}$ `f(0)` $\rightsquigarrow_{id,f(x)\rightarrow h(x)}$ `h(0)`.

As mentioned above, Hullot (1981) proved two different results for *basic narrowing*:

(1) its unification–completeness for canonical TRS's, and
(2) its termination for canonical TRS's where all basic narrowing derivations issuing from the right–hand side of every rule terminate.

It is important to recall here that, in contrast to ordinary narrowing, unification–completeness of basic narrowing is lost when termination is dropped, even if we restrict ourselves to normalizable substitutions (Middeldorp and Hamoen, 1994). Unification–completeness of basic narrowing can be restored (for normalizable substitutions) by additionally requiring $\mathcal{R}$ to be right–linear (Middeldorp and Hamoen, 1994).

The termination of basic narrowing was established in Hullot's PhD thesis for canonical TRS's as follows.

Proposition 7 (Termination of B. Narrowing for Canonical TRS's) (Hullot, 1981, Proposition 7.1) *Let* $\mathcal{R}$ *be a canonical TRS. If for every* $l \rightarrow r \in \mathcal{R}$*, all basic narrowing derivations issuing from* r *terminate, then any* basic *narrowing derivation issuing from any term terminates.*

Hullot's condition on the rhs's of rewrite rules is essential for the termination of basic narrowing, as illustrated in the following example.

Example 8 (Chabin and Réty, 1991) *Consider the canonical TRS* $\mathcal{R} = \{\mathtt{h(f(y))} \to \mathtt{h(y)}\}$. *The following infinite basic narrowing derivation can be proved:*

$$\underline{\mathtt{h(x)}} \overset{id}{\leadsto}_{\{x\mapsto\mathtt{f(y)}\},\mathcal{R}} \underline{\mathtt{h(y)}} \overset{id}{\leadsto}_{\{y\mapsto\mathtt{f(y')}\},\mathcal{R}} \underline{\mathtt{h(y')}} \ldots$$

A termination result similar to Proposition 7 does not hold for ordinary narrowing, even when strengthen the condition by requiring termination of ordinary narrowing for the rhs's of the rules (instead of the less demanding condition of basic narrowing termination). The TRS of Example 4 would be an easy counter–example.

In the following, we ascertain the conditions which allow us to achieve the first positive termination result which holds for ordinary narrowing and is formulated in Hullot's style. This is done by considering a particular class of TRS's where there is a precise correspondence between basic narrowing and ordinary narrowing derivations. This class was first identified in a commutation result for narrowing sequences proved by Réty (Réty, 1987, June) (for the sake of self–containment, Réty's technical result is recalled in Appendix A).

Réty's commutation result is based on the condition that narrowing produces only normalized substitutions, as formalized in the following definition.

Definition 9 (Rety's normalization condition) (Réty, 1987) *A TRS* $\mathcal{R}$ *satisfies Rety's normalization condition if, for every term* s, *every substitution* θ *computed by an ordinary narrowing derivation issuing from* s *satisfies that* $\theta_{\upharpoonright Var(s)}$ *is normalized.*

A popular class of TRS's that satisfy the normalization condition is the class of left-linear constructor systems (Reddy, 1985), that only compute [8] constructor substitutions. Nevertheless, in Section 5.1 we are able to define a more general, syntactic characterization of TRS's satisfying this condition.

Together with the normalization condition, Réty's "maximal commutation property" of narrowing sequences requires two additional conditions: right–linearity, and either left–linearity or conservativeness (Réty, 1987). By requiring all these properties, we are able to achieve the desired narrowing termination result. The proof of this result is given in Appendix A.

Theorem 10 (Termination of Narrowing) *Let* $\mathcal{R}$ *be a right–linear TRS*

[8] This is desired in some functional logic languages (Hanus, 1994), since a broader class of solutions may contain unevaluated or undefined expressions.

which satisfies Réty's normalization condition and is either left–linear or conservative. If basic narrowing terminates in $\mathcal{R}$, then ordinary narrowing also terminates in $\mathcal{R}$.

Note that Example 4 satisfies all conditions required in Theorem 10, except for Réty's normalization condition. In the following section, we improve this result by explicitly getting rid of canonicity.

5 Getting rid of canonicity and characterizing Réty's normalization condition

Hullot's basic narrowing termination result for canonical TRS's recalled in Proposition 7 has been referred to in a number of works, e.g. (Hölldobler, 1989; Middeldorp and Hamoen, 1994; Réty, 1987). However, to the best of our knowledge, no one has explicitly pointed out that canonicity is not explicitly used in Hullot's proof. This seems to suggest that canonicity of $\mathcal{R}$ might be superfluous for Hullot's basic narrowing termination result and that is only required for deriving both termination and unification completeness of the basic narrowing mechanism in one go. By providing a new proof for Hullot's basic narrowing termination result, in this section we confirm this presumption and demonstrate that canonicity can be safely removed.

The following result establishes the termination of basic narrowing without the canonicity requirement. A proof of this result is given in Appendix B.

Theorem 11 (Termination of Basic Narrowing) *Let $\mathcal{R}$ be TRS. If for every $l \to r \in \mathcal{R}$, all basic narrowing derivations issuing from r terminate, then every basic narrowing derivation issuing from any term terminates.*

Note that the termination of basic narrowing in $\mathcal{R}$ does not imply that $\mathcal{R}$ is terminating.

Example 12 *Consider the following non-terminating and non-confluent TRS $\mathcal{R}$ borrowed from (Toyama, 1987), which satisfies Réty's normalization condition*[9] *:*

$$\texttt{f(b,c,x)} \to \texttt{f(x,x,x)} \qquad \texttt{a} \to \texttt{b} \qquad \texttt{a} \to \texttt{c}$$

By applying Theorem 11, there is no infinite basic narrowing derivation in $\mathcal{R}$.

The following Hullot–like termination result follows from Theorem 11.

[9] It satisfies the sufficient characterization given in Section 5.1.

Corollary 13 (Termination of Narrowing) *Let $\mathcal{R}$ be a right–linear TRS which satisfies Réty's normalization condition and is either left–linear or conservative. If for every $l \to r \in \mathcal{R}$ all basic narrowing derivations issuing from r terminate, then every narrowing derivation issuing from any term terminates.*

Proof. It follows immediately from Theorem 10 and Theorem 11. □

Example 14 *Consider the following linear TRS $\mathcal{R}$ satisfying*[10] *Réty's normalization condition.*

$$\mathtt{f(a,x)} \to \mathtt{a} \qquad \mathtt{f(f(b,x),a)} \to \mathtt{c(h(x))} \qquad \mathtt{h(c(x)))} \to \mathtt{x}$$

By applying Corollary 13, since all basic narrowing derivations issuing from the rhs's of the rules in $\mathcal{R}$ terminate, then narrowing terminates in $\mathcal{R}$.

Note that right-linearity is essential for Réty's maximum commutation property and hence cannot be dropped from Corollary 13, as shown in the following example.

Example 15 *Consider again the TRS of Example 12, which also satisfies Réty's normalization condition. However, note that it is not right–linear. Basic narrowing terminates in this TRS, as seen before, but an infinite ordinary narrowing sequence exists for input term* `f(a,a,a)`*, which is set off when we instantiate the rhs* `f(x,x,x)` *of the first rule using the non-normalized binding* $\{x \mapsto \mathtt{a}\}$*:*

$$\mathtt{f}(\underline{\mathtt{a}},\mathtt{a},\mathtt{a}) \to \mathtt{f}(\mathtt{b},\underline{\mathtt{a}},\mathtt{a}) \to \underline{\mathtt{f}(\mathtt{b},\mathtt{c},\mathtt{a})} \to \mathtt{f}(\underline{\mathtt{a}},\mathtt{a},\mathtt{a}) \to \mathtt{f}(\mathtt{b},\underline{\mathtt{a}},\mathtt{a}) \cdots .$$

Unfortunately, both Hullot's termination condition based on the rhs's of rewrite rules and Réty's normalization condition are not syntactical. Hullot's termination condition has been approximated in the related literature by the following syntactic criterion, assuming that $\mathcal{R}$ terminates: every non–ground rhs of a rewrite rule is a constructor term (Dershowitz et al., 1992; Prehofer, 1994). This generalizes the original characterization given by Hullot (Hullot, 1980), who required all non–ground rhs's to be variables. Note that these syntactic characterizations do not work under the conditions of Theorem 11 since termination is not explicitly required, and we would require also ground rhs's to be constructor terms (the rule $\mathtt{a} \to \mathtt{a}$ would be an easy counter—example).

With regard to Réty's normalization condition, we already mentioned a popular class of TRS's satisfying this property: left-linear constructor systems.

[10] It satisfies the sufficient characterization of TRS's satisfying Réty's normalization condition given in Section 5.1.

In the following section, we demonstrate that Réty's condition also holds in the more general class of left-linear, rnf–based TRS's. This leads to a practical approximation of the termination result for ordinary narrowing given in Corollary 13 which holds in (a subclass of) linear, rnf–based TRS's.

Moreover, by further exploring the notion of rigid normal form, in Sections 6 and 7, we will be also able to generalize the popular approximation of Hullot's termination condition based on the rhs's of the rules, and provide stronger (purely syntactical in some cases) termination results for ordinary narrowing in a class of systems where right–linearity as well as left–linearity are no longer required.

5.1 Rigid normal forms and rnf*-based TRS's*

Let us define the class of rnf–based TRS's by introducing the notion of *rigid normal form*[11] (rnf), which lifts the standard notion of (rewriting) normal form to narrowing.

Definition 16 (Rigid normal form) *A term s is a* rigid normal form *(rnf) if there is no term t, substitution θ, and position p such that $s \overset{p}{\leadsto}_{\theta,\mathcal{R}} t$.*

The notion of rnf is stronger than the standard notion of rewriting normal form but can still be easily decided by simply checking that no subterm of the considered term unifies with the lhs of any rule in $\mathcal{R}$. This notion extends to *rigidly normalized substitutions* in the obvious way.

We define the new class of rnf–based TRS's as follows.

Definition 17 (rnf–pattern) *A term $f(t_1, ..., t_n) \in \mathcal{T}(\Sigma, \mathcal{V})$ is a* rnf*–pattern if, for all i s.t. $1 \leq i \leq n$, t_i is a* rnf*.*

Definition 18 (rnf–based TRS) *Given a TRS $\mathcal{R}$, we call it* rnf-based *if the left–hand side of every rule in $\mathcal{R}$ is a* rnf*–pattern.*

Note that two popular classes of rnf–based, left-linear TRS's are: (i) left-linear constructor systems, and (ii) almost orthogonal TRS's, i.e., typical functional programs.

[11] Our rnf notion is more general than the *strongly $\leadsto$-irreducible* terms proposed in (Escobar et al., 2006) for topmost theories, where t is strongly $\leadsto$-irreducible if $t\sigma$ is a normal form for every normalized substitution σ. Consider, e.g. the non-confluent, non-topmost TRS $\mathcal{R} = \{\texttt{f(a)} \to \texttt{b}, \texttt{a} \to \texttt{b}\}$. The term $\texttt{f(x)}$ is strongly $\leadsto$-irreducible, since non-normalized substitutions such as $\{\texttt{x} \mapsto \texttt{a}\}$ are not considered within the definition. However, it is not a rigid normal form.

Proposition 19 *Almost orthogonal TRS's are* rnf*-based.*

Proof. By definition of almost orthogonal TRS, every critical pair is an overlay, i.e., two lhs's overlap only at the root position. Therefore, the lhs of every rewrite rule is a rnf-pattern. □

The following result is instrumental and shows that rnf's are closed under substitution.

Lemma 20 *For every rigidly normalized substitution θ, if t is a rigid normal form, then $t\theta$ is also a rigid normal form.*

Proof. By contradiction. Let us assume that $t\theta$ is not a rigid normal form, i.e., there is a term s, substitution σ, rule R, and $p \in \mathcal{P}os(t\theta)$ such that $t\theta \overset{p}{\leadsto}_{\sigma,R} s$. Actually, since θ is rigidly normalized, then $p \in \mathcal{P}os_{\Sigma}(t)$. Therefore, we have that $t\theta|_p$ and l unify with unifier σ, whereas by hypothesis $t|_p$ and l do not unify, which leads to contradiction. □

From Lemma 20, it follows that, in rnf–based left–linear TRS's, all substitutions computed by narrowing are rigidly normalized, hence also normalized.

Theorem 21 (Rigid normalization) *Let $\mathcal{R}$ be a* rnf*–based, left–linear TRS. Every substitution θ computed by an ordinary narrowing derivation issuing from the term t satisfies that $\theta_{\restriction Var(t)}$ is rigidly normalized.*

Proof. Consider a narrowing sequence

$$t \equiv t_0 \overset{p_1}{\leadsto}_{\theta_1,l_1\to r_1} t_1 \cdots t_{n-1} \overset{p_n}{\leadsto}_{\theta_n,l_n\to r_n} t_n \equiv s$$

At each narrowing step $t \overset{p}{\leadsto}_{\theta,l\to r} s$, the substitution $\theta_{\restriction Var(t)}$ is rigidly normalized, since l is linear and every subterm of l is a rnf. We proceed by induction on n. The base case $n = 0$ is trivial. For the case when $n > 0$, by induction hypothesis we have that $\vartheta \equiv (\theta_1 \cdots \theta_{n-1})_{\restriction Var(t)}$ is rigidly normalized, i.e., for each binding $x \mapsto w \in \vartheta$, we have that w is a rigid normal form. Now, by Lemma 20, we have that $w\theta_n$ is also a rigid normal form, and the conclusion follows. □

From Theorem 21 and Corollary 13, the following practical criterion for termination of narrowing in rnf–based, linear TRS's easily follows.

Corollary 22 *Let $\mathcal{R}$ be a linear,* rnf*–based TRS. If for every $l \to r \in \mathcal{R}$, all basic narrowing derivations issuing from r terminate, then every narrowing derivation issuing from any term terminates.*

6 Getting rid of right–linearity

Our narrowing termination results in Section 5.1 rely on Réty's commutation result (Réty, 1987, June), which requires right–linearity and either left–linearity or conservativeness. In this section, we provide new termination results that are not based on Réty's commutation property, and thus get rid of linearity in some cases.

The notions of *root-stable rigid normal form* (rs−rnf) and *stable rigid normal form* (srnf) are the key for achieving termination when right–linearity is dropped.

6.1 Stable and Root-stable Rigid normal forms

Let us highlight the insufficiency of considering rigid normal forms for ensuring the narrowing termination when right–linearity of $\mathcal{R}$ is not imposed. Basically, the problem lies in the fact that rigid normal forms are not stable under instantiation by non-normalized substitutions, as illustrated in the following example.

Example 23 *Consider again the left–linear and* rnf*–based TRS $\mathcal{R}$ of Example 12, which is non–confluent and not right–linear. The term* `f(x,x,x)` *in the rhs of the first rule is a rigid normal form since it does not unify with lhs* `f(b,c,x)`*; hence, it cannot be narrowed. However, the instance* `f(a,a,a)` *is no longer a* rnf *since it can be rewritten (in two steps) to* `f(b,c,a)`*, which can then be rewritten (hence narrowed) at the top position by using the first rule of $\mathcal{R}$.*

Let us introduce the notion of root-stable rigid normal form, which lifts to narrowing the standard notion of root-stable (or head) normal form. Then, a suitable definition of "stable rigid normal form t" is provided which ensures that every subterm s of t is conveniently "protected", in the sense that no instantiation can enable a "non-topmost" rewriting sequence such that then the resulting term can be narrowed at the top.

Definition 24 (stable and root-stable rigid normal forms) *A term s is a* root-stable rigid normal form *(*rs−rnf*) if either s is a variable or there are no substitutions θ and θ' and terms s' and s'' s.t. $s\theta \stackrel{>\epsilon *}{\to}_{\mathcal{R}} s' \stackrel{\epsilon}{\leadsto}_{\theta'} s''$. A term t is a* stable rigid normal form *(*srnf*) if every subterm of t is a root-stable rigid normal form.*

The above notions extend to *root-stable rigidly normalized substitutions* and *stable rigidly normalized substitutions* in the natural way.

Note that the notion of stable rigid normal form is stronger than the notion of rigid normal form. Example 23 above shows that the inverse does not hold. By definition, non-variable stable rigid normal forms are stable under instantiation, even under non-normalized substitutions. Also, constructor terms as well as ground normal forms are trivial cases of stable rigid normal forms. Therefore, the approximation of Hullot's basic narrowing termination condition based on checking that the rhs's of the rules are constructor terms is subsumed by the more general right–srnf condition.

Definition 25 (Right–rnf TRS) *A TRS is called* right–rnf *if the right-hand side of every rule in $\mathcal{R}$ is a* rnf*.*

The notion of right–srnf TRS can be defined similarly. The following interesting property holds.

Proposition 26 *Every right–*srnf *TRS is terminating.*

Proof. (Sketch) We apply the dependency pairs technique (Arts and Giesl, 2000) for proving termination of rewriting. Since by definition a right–srnf TRS $\mathcal{R}$ can have no chains, then $\mathcal{R}$ terminates by (Arts and Giesl, 2000, Thm. 6). □

Note that the right–srnf condition required in Proposition 26 cannot be weakened to right–rnf. The TRS of Example 12 is an easy counterexample.

In order to provide a general termination result for right–srnf TRS's, we need the following notion.

Definition 27 (Stable rigid normalization condition (SRNC)**)** *A TRS $\mathcal{R}$ satisfies the* stable rigid normalization condition *if, for every term s, every substitution θ computed by an ordinary narrowing derivation issuing from s satisfies that $\theta_{\upharpoonright Var(s)}$ is stable rigidly normalized.*

By requiring the SRNC (instead of Réty's maximal commutation condition), we are able to provide the following termination result for narrowing. The proof is in Appendix C.

Theorem 28 (Termination of narrowing under the SRNC**)** *Let $\mathcal{R}$ be a right–*srnf *TRS that satisfies the stable rigid normalization condition. Every narrowing derivation issuing from any term terminates.*

Even if the above result may seem of little interest in the context of functional (logic) programming, since it precludes recursion, we would like to highlight its interest for proving the termination of narrowing–based procedures that are used in the context of bottom–up program analysis and abstract diagnosis. The key ingredient for the analyses is often a suitable, collecting program semantics that is also expressed as a set of rules. And it happens that those rules are often right–srnf. (Alpuente et al., 2003).

The following example demonstrates that stable rigid normal forms cannot be replaced by rough rigid normal forms in Theorem 28.

Example 29 *Consider again the left–linear and* rnf*–based TRS of Example 12, where we showed that the term* `f(x,x,x)` *in the rhs of the first rule is a* rnf*. However, it is not a* srnf *and actually narrowing does not terminate for the input term* `f(a,a,a)`*, as shown in Example 12.*

In the following section, we characterize the class of TRS's where all rigid normal forms are stable thus guaranteeing that the new structure that is introduced through ordinary narrowing steps by instantiation cannot burst an infinite derivation. This is the final ingredient we need in order to derive a purely syntactical characterization of narrowing termination which does not require the right–linearity of $\mathcal{R}$.

6.2 Reachability-complete TRS's

Let us introduce a new class of TRS's (which we call reachability–complete TRS's) where narrowing is strongly reachability–complete. This is inspired by the commonly used terminology which, recalling the unification-completeness of narrowing for canonical TRS's, uses the name "complete TRS" as an alternative terminology to refer to this particular class (Knuth and Bendix, 1970; Hullot, 1980; Middeldorp and Hamoen, 1994).

Definition 30 (Reachability–complete TRS) *A TRS $\mathcal{R}$ is reachability–complete iff the narrowing procedure is strongly reachability–complete for $\mathcal{R}$.*

The following interesting result holds for reachability–complete TRS's.

Proposition 31 *Let $\mathcal{R}$ be a reachability–complete TRS. If s is a rigid normal form, then s is also a stable rigid normal form.*

Proof. By contradiction. Assume that s is a rigid normal form and there is a position p in s such that $s|_p$ is not a root-stable rigid normal form. Then, there are two substitutions ρ and ρ' and terms t and t' such that $s|_p\rho \stackrel{>\epsilon *}{\to}_{\mathcal{R}} t \stackrel{\epsilon}{\leadsto}_{\rho'} t'$. Let $s' = s[t']_p$. Since $\mathcal{R}$ is reachability–complete, for the reachability goal $s \to^* s'$ narrowing computes a solution η more general than $\rho\rho'$ s.t. $s \leadsto^*_{\eta} s''$, with $s'' \leq s'$. Hence, s is not a rigid normal form, which contradicts the initial assumption. □

Proposition 31 reveals that reachability-completeness can be understood as the property that shelters rnf's with a suitable form of stability which suffices to prevent non-normalized bindings from introducing the possibility of initiating an infinite narrowing derivation. Actually, under reachability-completeness we are able to weaken stable rigid normal forms down to the purely syntactic notion of rigid normal form, which is easier to check.

As a corollary of Theorem 28, by using Proposition 31, we achieve the following termination result for reachability–complete TRS's. Note that reachability–complete TRS's that satisfy Réty's normalization condition also satisfy SRNC.

Corollary 32 *Let $\mathcal{R}$ be a reachability–complete, right–*rnf *TRS which satisfies Réty's normalization condition. Every narrowing derivation issuing from any term terminates.*

In the above result, reachability–completeness allows us to get rid of right–linearity, e.g. in TRS's that are confluent or topmost (Meseguer and Thati, 2007). Unfortunately, this is not the case for left–linearity, which is still required in the sufficient criteria for Réty's normalization condition.

Inspired by Christian's narrowing termination result for left–flat TRS's (Christian, 1992), in the last section we further refine our termination results by also getting rid of left–linearity, and syntactically characterize a very wide class of TRS's where narrowing terminates, while still being complete as a procedure for solving reachability goals.

7 Getting rid of left–linearity

In (Christian, 1992), termination of narrowing was proved for left–flat TRS's (i.e., each argument occurring in the lhs of a rewrite rule is either a variable or a ground term), provided the rewrite rules are also compatible with a termination ordering $<$. A termination ordering $<$ is a well–founded ordering on ground terms such that, if $s < t$, then $s\sigma < t\sigma$ for any substitution σ; see (Dershowitz, 1987) for a survey on termination orderings. Christian formalized a stability ("harmlessness") criterion for narrowing as an extension $<_{\mathcal{L}}$ of $<$ as follows: $s <_{\mathcal{L}} t$ whenever the number of distinct variables in s is either (i) less than the number in t; or (ii) equal to the number in t, and s and t are identical everywhere, except at some position p such that $s|_p < t|_p$. Then he demonstrated that, whenever any term t narrows to t', then $t' <_{\mathcal{L}} t$, which ensures termination of narrowing.

Informally, the reason why left–flat rules "behave well" is that they do not introduce *new* variables in the term: each narrowing step either reduces the number of distinct variables, or produces a smaller term under the $<$ well–founded ordering.

Example 33 *Consider the following non–flat TRS* `f(f(x))` $\rightarrow$ `f(x)` *which can be oriented with the following termination ordering:* $t > s$ *iff* $|t\sigma| > |s\sigma|$ *for every substitution* σ*, where* $|t|$ *denotes the size of* t*. However, this rule raises the infinite narrowing sequence*

$$\underline{\texttt{f(x)}} \leadsto_{\{\texttt{x}\mapsto\texttt{f(x')}\}} \underline{\texttt{f(x')}} \leadsto_{\{\texttt{x'}\mapsto\texttt{f(x'')}\}} \underline{\texttt{f(x'')}} \leadsto_{\{\texttt{x''}\mapsto\texttt{f(x''')}\}} \cdots$$

Note that the ultimate source of narrowing non-termination in this TRS is the introduction of "fresh variables" `x'`, `x''`*, which causes the terms* `f(x')`, `f(x'')`*,... to enter at some point in the derivation, whereas* `f(x')` $\not<_{\mathcal{L}}$ `f(x)`*.*

In order to combine and generalize the termination results that hold for TRS's which are either left–flat (Christian, 1992) or `rnf`-based (Section 6), we extend the stable rigid normalization condition (SRNC) as follows. Informally, the key idea is to ensure that the substitutions applied in narrowing steps cannot introduce any new term that is not a `rs`$-$`rnf` and may only replicate in the worst case (strict) subterms of existing ones.

Definition 34 (Quasi stable rigidly normalized substitution) *Given a TRS* $\mathcal{R}$*, a term* s*, a substitution* θ *is* quasi stable rigidly normalized w.r.t. s and $\mathcal{R}$ *if, for each variable* $x \in Var(s)$ *that appears in* s *more than once,* $x\theta$ *is either (i) a ground term, (ii) a stable rigid normal form, or (iii) there exists a position* $p \in \mathcal{P}os_{\Sigma}(s)$ *such that* $x\theta \equiv (s\theta)|_p$*.*

Note that every substitution is quasi stable rigidly normalized w.r.t. a linear term, for any TRS.

Example 35 *Consider the TRS* R *of Example 4, and the term* $s =$ `c(c(x,f(x)),f(y))`*. Assume* `a` *is a new constant in the signature of* R*. The following substitutions are QSRNC w.r.t.* s *and* R*:* $\{$`x` $\rightarrow$ `a`$\}$*, by (i);* $\{$`x` $\rightarrow$ `c(z,z)`$\}$*, by (ii);* $\{$`x` $\rightarrow$ `f(y)`$\}$*, by (iii). Note that* $\{$`x` $\rightarrow$ `f(z)`$\}$ *is not QSRNC w.r.t.* s *and* R*.*

The following result is trivial due to linearity.

Corollary 36 *In a right-linear TRS* $\mathcal{R}$*, every substitution computed by narrowing for a linear term* s *is quasi stable rigidly normalized w.r.t.* s *and* $\mathcal{R}$*.*

Definition 37 (Quasi stable rigid normalization condition (QSRNC)**)** *A TRS* $\mathcal{R}$ *satisfies the* quasi stable rigid normalization condition *if, for every term* s*, every substitution* θ *computed by an ordinary narrowing derivation issuing from* s *satisfies that* $\theta_{\restriction Var(s)}$ *is quasi stable rigidly normalized w.r.t.* s *and* $\mathcal{R}$*.*

Note that SRNC implies QSRNC. Now we are ready to provide our most general result for narrowing termination. The proof of the following theorem is given in Appendix D.

Theorem 38 (Termination of narrowing under QSRNC**)** *Let $\mathcal{R}$ be a right–*srnf *TRS that satisfies the quasi stable rigid normalization condition. Every narrowing derivation issuing from any term terminates.*

Theorem 38 and Corollary 36 provide the following result.

Corollary 39 (Termination of Narrowing for right–linear TRS's) *Let $\mathcal{R}$ be a right–linear, right–*srnf *TRS. Every narrowing derivation in $\mathcal{R}$ issuing from any linear term terminates.*

Now we are ready to introduce the notion of *left–plain* TRS's as a natural generalization, with regard to narrowing termination, of both left–flat as well as rnf-based TRS's. Note that the case of a variable argument is considered in the definition below, since variables are rigid normal forms.

Definition 40 (Left–plain TRS) *A TRS $\mathcal{R}$ is called* left–plain *if every non-ground strict subterm of the left–hand side of every rule of $\mathcal{R}$ is a rigid normal form.*

Example 41 *The following TRS defining a specialized version of the* `xor` *operator used in many security protocols (Comon-Lundh, 2004; Cortier et al., 2006) is left–plain. The symbol* `h` *is constructor; it might represent e.g. the hash of a message.*

$$\texttt{x + x} \to \texttt{0} \qquad \texttt{x + 0} \to \texttt{x} \qquad \texttt{(0 + 0) + h(x)} \to \texttt{h(x)}$$

Note that the third rule is neither left–flat nor rnf*-based.*

Example 42 *The rule* `0 + (0 + x)` $\to$ `x` *is not left–plain, since the non–ground subterm* `0 + x` *is not a* rnf*. Indeed, the following infinite narrowing derivation can be proved*

$$\texttt{c(}\underline{\texttt{0 + x}}\texttt{,x)} \leadsto_{\{x \mapsto \texttt{0 + x'}\}} \texttt{c(x',}\underline{\texttt{0 + x'}}\texttt{)} \leadsto_{\{x' \mapsto \texttt{0 + x''}\}} \texttt{c(}\underline{\texttt{0 + x''}}\texttt{,x'')} \cdots$$

By using Proposition 31, we are able to demonstrate the QSRNC property for left–plain, reachability–complete TRS's.

Lemma 43 *Every left-plain, reachability-complete TRS satisfies the quasi stable rigid normalization condition.*

Now, by using Lemma 43, the following result directly follows as a specialization of Theorem 38 for left-plain TRS's.

Corollary 44 (Termination of Narrowing for left-plain TRS's) *Let $\mathcal{R}$ be a left–plain, reachability complete, right–*rnf *TRS. Every narrowing derivation issuing from any term terminates.*

Note that the above result is very handy as it can be applied to TRS's which are neither purely left–flat or rnf-based, as illustrated in Example 41.

Finally, by using the known results for the strong reachability-completeness of narrowing given by Meseguer and Thati (2007), we are able to particularize Corollary 44 to a number of purely syntactical, non–trivial classes of TRS's where narrowing has a finite search space and is still (strongly) complete as a procedure to solve reachability goals. The following result also subsumes Corollary 39.

Corollary 45 (Termination of Narrowing for right-rnf TRS's) *Let $\mathcal{R}$ be a right-*rnf *TRS which is either*

(1) right-linear,
(2) confluent and left–plain, or
(3) topmost.

Then, every narrowing derivation issuing from any term terminates. In the case of (1)*, the termination (proved in Corollary 39) only holds for linear input terms.*

Example 46 *Let us consider the following rule defining the exponentiation function used as a primitive operation for key exchange in the Diffie-Hellman key agreement protocol (Comon-Lundh, 2004; Cortier et al., 2006), where symbols* `*` *and* `g` *are constructors*[12] *.*

$$\texttt{exp(exp(g,y),z)} \to \texttt{exp(g, y * z)}$$

This rule satisfies both criteria 1 and 2 of Corollary 45, hence we conclude that narrowing derivations w.r.t. this rule terminate.

The criteria given in Corollary 45 are particularly practical, since many interesting TRS's fit in one of the above classes. For instance, termination of the following TRS's follows from Corollary 45 straightforwardly (other examples are given in Table 1):

- almost orthogonal, right-rnf TRS's (including right-rnf orthogonal TRS's as a particular case);
- constructor, confluent, and right-rnf TRS's;
- right-linear, right-rnf TRS's (only for linear input terms).

[12] `*` is commonly defined as a (built-in) associative commutative operator with identity element `1`.

Restrictions on $\mathcal{R}$	Reference
LF + cT	**(Christian, 1992, Lemma 2)**
RL + (LL or Co) + NC + bnT	**Thm. 10 (Hullot's result generalized)**
RL + (LL or Co) + NC + R–bnT	**Cor. 13 (Hullot's result repaired)**
R-rnf + L + rnf–B	Corollary 22
	*e.g. R-*rnf *+ L + CS*
RL + R-rnf (+linear term)	**Corollary 39**
LP + RC + R-rnf	**Corollary 44**
R-rnf + LP + C	Corollary 45
	*e.g. R-*rnf *+ (either aO or CS + C)*
R-rnf + Tp	Corollary 45
RL + (LL or Co) + NC + St	**Thm. 10, by (Nieuwenhuis, 1996)**

Legend

C	confluent	LL	left–linear	RL	right–linear
Tp	topmost	Co	conservative	CS	constructor system
R-rnf	right–rnf	rnf–B	rnf–based	LP	left–plain
LF	left–flat	L	linear	aO	almost Orthogonal
bnT	basic narrowing terminates			NC	Rety's normalization condition
R–bnT	all basic narrowing derivations starting from rule rhs's terminate				
St	standard theories saturated by basic paramodulation				
cT	compatible with a termination ordering				

Fig. 1. Criteria for Narrowing termination

Note that the TRS in Example 1 satisfies all the above requirements, except for the condition to be right-rnf.

We would like to note that our results are not comparable to those of (Christian, 1992), i.e., we do not claim to subsume Christian's results. As a counterexample, it suffices to consider any left–flat TRS that is compatible with a termination ordering but is neither right–rnf nor reachability–complete. Obviously, (Christian, 1992) does not subsume our results either, since Christian's criterion cannot deal with TRS's that are not left–flat.

The main advantage of our approach w.r.t. (Christian, 1992) is that our criteria are truly syntactic and do not rely on termination orderings. As an additional advantage, note that some of our results are based (and hence preserve) the strong reachability–completeness of $\mathcal{R}$, besides ensuring the narrowing termination, which is not guaranteed by Christian's result.

8 Conclusion

We conclude by summarizing in Figure 1 all known results (including the ones presented in this paper) for termination of ordinary narrowing. We would like to point out that, even if functional programs may unlikely fulfil the right–

rnf condition required for some of our results, it might be still very useful as a criterion for proving the termination of narrowing–based procedures that use an extensional, rule-based presentation of the program semantics (where the rhs's of the equations in the semantics are rnf's, e.g. values), rather than termination of the program itself.

It is challenging to identify more general classes of TRS's where narrowing terminates. However, this seems difficult without losing the ability to test (almost purely) syntactic properties of individual rewrite rules. Let us emphasize that all the results in this paper apply to proving termination of sophisticated narrowing strategies such as innermost or lazy narrowing (Hanus, 1994), where narrowing steps are restricted to a suitable subset of the term positions. Obviously, more general classes of TRS's may exist where a particular narrowing strategy terminates.

Theorem 10 provides a powerful criterion for proving narrowing termination in TRS's or theories where basic narrowing terminates, often called BNT-theories; see e.g. (Schmidt-Schauß, 1988). We consider that this criterion is quite versatile and lays the ground for further research in the area. Recently, we have studied in (Alpuente et al., 2008) the modularity of basic narrowing termination, showing that it is modular for several classses of unions of TRS's. Under the conditions for Theorem 10, the modularity results of (Alpuente et al., 2008) also apply to ordinary narrowing. On the other hand, Nieuwenhuis (1996) demonstrated that, for some kinds of theories closed under some basic inference rules, equational unification can be proved terminating by again applying these inference rules. This entails termination of basic narrowing e.g. in shallow theories (where all variables in the axiomatization are shallow) that are saturated under a rule which subsumes basic narrowing, called basic paramodulation. A similar result holds in standard theories, which extend shallow theories by only requiring shallowness to the variables that appear on both sides of the equations. We consider standard theories to be an interesting topic for further research on narrowing termination.

Part of the inspiration for this work goes back to 1991, when María Alpuente developed her PhD thesis under the supervision of Giorgio Levi regarding CLP(H/E), an instance of the constraint logic programming scheme CLP(X) (Jaffar and Lassez, 1987) which used an incremental constraint solver based on narrowing to semi–decide the solvability of equational constraints (Alpuente and Falaschi, 1991; Alpuente et al., 1992, 1993, 1995a). Termination of the narrower was an important problem in CLP(H/E), which led to the development of static analysis techniques to finitely approximate the unsatisfiability of a set of equations with respect to a given equational theory (Alpuente et al., 1995b).

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APPENDIX

A Proof of Theorem 10

Our restoration of Hullot's result is based on Réty's commutation properties for basic narrowing, which rely on the following notion of *antecedent* of a position in a rewrite sequence (Réty, 1987, June).

Definition 47 (Antecedent of a position) (Réty, 1987, June) *Let* $t \xrightarrow{p}_{l \to r} t'$ *be a rewriting step,* $v \in \mathcal{P}os(t)$*, and* $v' \in \mathcal{P}os(t')$*. We say position* v *is an* antecedent *of* v' *iff*

(1) $v \parallel p$*, i.e.,* v *is incomparable to* p*, and* $v \equiv v'$*, or*
(2) there is a variable $x \in Var(r)$*,* $u' \in \mathcal{P}os_x(r)$*, and* $u \in \mathcal{P}os_x(l)$ *s.t.* $v' \equiv p.u'.w$ *and* $v \equiv p.u.w$*.*

This notion extends to a rewrite sequence by transitive closure of the rewriting relation in the usual way.

With the notations of the previous definition, we have:

(1) $t|_v \equiv t'|_{v'}$,
(2) v' may have no antecedent if $v' = p.u'$ with $u' \in \mathcal{P}os_\Sigma(r)$, or if $v' < p$,
(3) v' may have several antecedents if l is not linear.

Therefore, the notion of antecedent is (nearly) dual to the standard notion of *descendants* of a position in a rewrite sequence (TeReSe, 2003). The main difference is that, given a rewriting step $t \xrightarrow{p}_{l \to r} t'$ and a position q such that $q \leq p$, then q is not an antecedent of any position in t' whereas the same position q in t' is commonly considered the descendant of q in t. Therefore, there are positions that do not have an antecedent in any previous term in the rewriting sequence.

Definition 48 (Terminal antecedents) (Réty, 1987, June) *Let* $\mathcal{D}$ *be a rewrite sequence* $t_0 \to_\mathcal{R} t_1 ... \to_\mathcal{R} t_n$*, and* $q_n \in \mathcal{P}os(t_n)$*. Given an antecedent* $q_i \in \mathcal{P}os(t_i)$ *of* q_n*, we say that* q_i *is* terminal *in* $\mathcal{D}$ *iff either* $i = 0$ *or* q_i *has no antecedent in* t_{i-1}*.*

The notion of antecedent can be extended to narrowing as follows:

Definition 49 (Narrowing antecedent of a position) (Réty, 1987, June) *Let* $t \leadsto^*_{\sigma,\mathcal{R}} t'$*,* $v \in \mathcal{P}os(t)$*, and* $v' \in \mathcal{P}os(t')$*. We say* v *is a (terminal) antecedent of* t *iff* v *is a (terminal) antecedent of* v' *in the rewrite sequence* $t\sigma \to^*_\mathcal{R} t'$*.*

The following proposition holds.

Proposition 50 (Réty, 1987) *Given a narrowing sequence* $t_0 \overset{p_1}{\leadsto}_{\sigma_1, l_1 \to r_1} t_1 \cdots t_{n-1} \overset{p_n}{\leadsto}_{\sigma_n, l_n \to r_n} t_n$*. If* $q_i \in \mathcal{P}os(t_i)$ *is an antecedent of* $q_n \in \mathcal{P}os(t_n)$*, then* $(t_i(\sigma_{i+1}..\sigma_n))|_{q_i} \equiv t_n|_{q_n}$*.*

As we mentioned, when $\mathcal{R}$ is not left-linear, a given position may have several antecedents in a previous term in the derivation, and may also have antecedents in different previous terms which are not antecedents from one another. Therefore, a position may have terminal antecedents in different previous terms of the sequence.

Also note that, whenever an expression is introduced by instantiation, and subsequently propagated along the narrowing derivation, its terminal antecedents are all in the initial input term of the sequence, and occur exactly at the positions of the input term which become instantiated. This is due to the absence of extra variables in rhs's.

The following commutation property is the key of our proof. For $\vartheta \equiv \vartheta_1 \cdots \vartheta_k$, we use $t \overset{u^1,\ldots,u^k}{\leadsto}_{\vartheta, l \to r} s$ as a shorthand to denote the narrowing sequence $t \overset{u^1}{\leadsto}_{\vartheta_1, l \to r} s_1 \cdots \overset{u^k}{\leadsto}_{\vartheta_k, l \to r} s$.

Proposition 51 (Maximum commutation) (Réty, 1987, June) *Let* $\mathcal{R}$ *be a right–linear TRS, which is also either left–linear or conservative. Consider a narrowing sequence*

$$t_0 \overset{p_1}{\leadsto}_{\sigma_1, l_1 \to r_1} t_1 \cdots t_{n-1} \overset{p_n}{\leadsto}_{\sigma_n, l_n \to r_n} t_n$$

such that $\sigma_1 \cdots \sigma_n$*, restricted to* $Var(t_0)$*, is normalized. Then, there exists a* commuted *narrowing derivation*

$$\begin{array}{l} t_0 \overset{u_1^1,\ldots,u_1^{k_1}}{\leadsto}_{\theta_1, l_n \to r_n} \overset{p_1}{\leadsto}_{\sigma'_1, l_1 \to r_1} t'_1 \\ \vdots \\ t'_{n-2} \overset{u_{n-1}^1,\ldots,u_{n-1}^{k_{n-1}}}{\leadsto}_{\theta_{n-1}, l_n \to r_n} \overset{p_{n-1}}{\leadsto}_{\sigma'_{n-1}, l_{n-1} \to r_{n-1}} t'_{n-1} \\ t'_{n-1} \overset{p_n}{\leadsto}_{\theta_n, l_n \to r_n} t_n \end{array}$$

such that $\theta_1\sigma'_1 \cdots \theta_{n-1}\sigma'_{n-1}\theta_n \equiv \sigma_1 \cdots \sigma_n[Var(t_0)]$*, where* $u_1^1, \ldots, u_i^{k_i}$ *are the terminal antecedents of position* p_n *in term* t_i*.*

The following commutation result for ordinary narrowing derivations easily follows.

Proposition 52 *Let* $\mathcal{R}$ *be a TRS that satisfies Rety's normalization condition as well as the conditions for Rety's maximum commutation property* *(i.e. right–linearity, and either left–linearity or conservativeness). For every narrowing sequence* $t_0 \overset{p_1}{\leadsto}_{\theta_1,\mathcal{R}} t_1 \cdots \overset{p_n}{\leadsto}_{\theta_n,\mathcal{R}} t_n$*, there is a commuted basic narrowing sequence* $t_0 \overset{q_1}{\leadsto}_{\sigma_1,\mathcal{R}} t'_1 \cdots \overset{q_m}{\leadsto}_{\sigma_m,\mathcal{R}} t'_m$ *such that* $t'_m \equiv t_n$*, and* $\theta_1 \cdots \theta_n \equiv \sigma_1 \cdots \sigma_m[\mathit{Var}(t_0)]$.

Proof. By successive applications of Proposition 51.

Given the narrowing sequence $t_0 \overset{p_1}{\leadsto}_{\theta_1,\mathcal{R}} t_1 \cdots \overset{p_n}{\leadsto}_{\theta_n,\mathcal{R}} t_n$, assume that p_i is the first non-basic position selected in the derivation. By Proposition 51, we can commute the derivation so that the step i is performed on the terminal antecedent positions of p_i. Those terminal antecedents occur at basic positions, since redexes are never introduced in a basic narrowing derivation by instantiation due to the Rety's normalization condition. Note that the procedure that repeatedly applies Proposition 51 to the derivation which results from the previous commutation is finite since the number of non-basic steps to commute is reduced at each application. □

Now we are able to prove the desired termination result for ordinary narrowing.

Theorem 10 (Termination of Narrowing) *Let* $\mathcal{R}$ *be a right–linear TRS which satisfies Réty's normalization condition and is either left–linear or conservative. If basic narrowing terminates in* $\mathcal{R}$*, then ordinary narrowing also terminates in* $\mathcal{R}$*.*

Proof. By contradiction. Assume that there exists an infinite narrowing derivation $\mathcal{D}$ issuing from a given term t. Then, we can obtain infinitely many finite subsequences (prefixes) of $\mathcal{D}$. By Proposition 52, each of these finite subsequences has a corresponding, commuted basic narrowing derivation issuing from t. Hence, there are infinitely many basic narrowing derivations issuing from the very same term t, each of which is: (i) finite (by definition), and (ii) a prefix of the subsequent one (by Proposition 52), which yields to contradiction. □

B Proof of Theorem 11

To prove Theorem 11, we find it useful to use the alternative definition of basic narrowing given in (Hölldobler, 1989). In this formulation, elements of the derivation are split into a *skeleton* and an *environment* part. The environment part keeps track of the accumulated substitutions so that, at each step, substitutions are composed in the environment part, but are not applied to the expressions in the skeleton part, as opposed to ordinary narrowing. Due to this representation, the basic occurrences in $t\theta$ are all in t, whereas the non-basic occurrences are all in the codomain of θ. This ensures that no narrowing step will reduce any expression brought by a substitution computed in a previous step. Given a term $s \in \mathcal{T}(\Sigma, \mathcal{V})$ and a substitution σ, a basic narrowing step is defined by $\langle s, \sigma \rangle \leadsto_{\theta,\mathcal{R}} \langle t, \sigma' \rangle$ if there exist $p \in \mathcal{P}os_\Sigma(s)$, $l \to r \ll \mathcal{R}$, and substitution θ such that $\theta \equiv mgu(s|_p\sigma, l)$, $t \equiv (s[r]_p)$, and $\sigma' \equiv \sigma\theta$.

We say that two idempotent substitutions θ_1 and θ_2 are *compatible* if their correponding bindings "unify", that is, there is θ s.t. $x\theta_1\theta \equiv x\theta_2\theta$, for all $x \in Dom(\theta_1) \cup Dom(\theta_2)$.

Lemma 53 *Let $\mathcal{R}$ be a TRS, t be a term, and σ be a substitution. Let n be the length of the longest basic narrowing derivation for $\langle t, \sigma \rangle$ in $\mathcal{R}$. Then, for every substitution ϑ, n is an upper bound for the length of the basic narrowing derivations issuing from $\langle t, \sigma\vartheta \rangle$ in $\mathcal{R}$.*

Proof. By induction on n.

The case when $n = 0$ is straightforward, since no basic narrowing step issuing from $\langle t, \sigma\vartheta \rangle$ can be proved for any ϑ, either.

Consider now the case when $n > 0$. If there is no basic narrowing sequence such that the substitution θ computed in the first step $\langle t, \sigma \rangle \leadsto_{\theta,\mathcal{R}} \langle t', \sigma\theta \rangle$ is compatible with ϑ, then there is no basic narrowing sequence issuing from $\langle t, \sigma\vartheta \rangle$, and the conclusion follows. Assume that $\langle t, \sigma \rangle \leadsto_{\theta,\mathcal{R}} \langle t', \sigma\theta \rangle$ is the first step of a basic narrowing derivation for $\langle t, \sigma \rangle$ such that θ is compatible with ϑ. Since ϑ and θ are compatible, the narrowing step $\langle t, (\sigma\vartheta) \rangle \leadsto_{\theta',\mathcal{R}} \langle t', (\sigma\vartheta)\theta' \rangle$ can be proven, and $(\sigma\vartheta)\theta'$ is compatible with $\sigma\theta$. By hypothesis, the lengths of the derivations issuing from $\langle t', \sigma\theta \rangle$ are bounded by $n - 1$, hence so are the lengths of the derivations issuing from $\langle t', (\sigma\vartheta)\theta' \rangle$, which concludes the proof. □

Theorem 11 (Termination of Basic Narrowing) *Let $\mathcal{R}$ be a TRS. If for every $l \to r \in \mathcal{R}$, all basic narrowing derivations issuing from r terminate, then every basic narrowing derivation issuing from any term terminates.*

Proof. We prove the slightly more general result that, for every term t and substitution σ, every basic narrowing derivation issuing from $\langle t, \sigma \rangle$ terminates. We proceed by structural induction on the term t.

- The case when t is a variable is straightforward.
- Let $t \equiv f(t_1, \ldots, t_m)$, $m \geq 0$, and consider any basic narrowing derivation $\mathcal{D}$: $\langle t, \sigma \rangle \overset{p_1}{\leadsto}_{\theta_1,\mathcal{R}} \langle t_2, \sigma_2 \rangle \overset{p_2}{\leadsto}_{\theta_2,\mathcal{R}} \cdots$ stemming from $\langle t, \sigma \rangle$. We distinguish two cases: either none of the positions p_j for $j > 0$ is ϵ, or there is $k > 0$ such that the k–th narrowing step in $\mathcal{D}$ takes place at the root position of t_k. In the first case, by the induction hypothesis the derivation terminates, since every basic narrowing derivation issuing from $\langle t_i, \sigma \rangle$ terminates, for $i \in \{1, \ldots, m\}$. In the second case, $\langle t_k, \sigma_k \rangle \overset{\epsilon}{\leadsto}_{\theta_k,\{l \to r\}} \langle r, \sigma_{k+1} \rangle$. Since all

basic narrowing derivations issuing from r terminate, then by Lemma 53 the derivation terminates. Thus, the conclusion follows. □

C Proof of Theorem 28

The proof of Theorem 28 is subsumed by the more general result proved below in Theorem 38, since SRNC implies QSRNC.

D Proof of Theorem 38

We first prove the following auxiliary result.

Lemma 43 *Every left-plain, reachability-complete TRS satisfies the quasi stable rigid normalization condition.*

Proof. By reachability-completeness, we can safely consider rigid normal forms instead of stable rigid normal forms. On the other hand, since the composition of two rigidly normalized substitutions is also rigidly normalized, we can safely consider the substitutions computed at each narrowing step.

Let us consider a term t and the narrowing step $t \overset{p}{\rightsquigarrow}_{\sigma,l\to r} t'$. We prove the result by induction on the number of bindings in σ. If $\sigma = id$, the conclusion follows straightforwardly. Let $x \mapsto u \in \sigma$ and suppose that u does not satisfy any of the conditions (i), (ii), and (iii) of Definition 34, i.e., u is not ground, is not a rigid normal form, and is not a non–variable subterm of $t\sigma$. By definition, there is at least one position $p' \in \mathcal{P}os(l) \cap \mathcal{P}os(t|_p)$ s.t. $t|_{p.p'} = x$ and $t\sigma|_{p.p'} = l\sigma|_{p'} = u$. Let us consider an arbitrary such p'. We distinguish the cases when $l|_{p'}$ is a variable or not. If $l|_{p'} = y \in \mathcal{V}$, then $y \mapsto u \in \sigma$ and y must be a repeated variable in l, since u is not a variable (it is not a rigid normal form) and σ is the most general unifier. Therefore, there is a position $p'' \in \mathcal{P}os_\Sigma(t|_p)$ s.t. $t\sigma|_{p.p''} = u$. But this contradicts condition (iii) of Definition 34. If $l|_{p'} \notin \mathcal{V}$, then $l|_{p'}$ itself is not ground and is not a rigid normal form, since x cannot appear in $l|_{p'}$ and, by induction hypothesis, $\sigma \setminus \{x \mapsto u\}$ satisfies conditions (i), (ii), and (iii) of Definition 34. However, this contradicts condition (ii) of Definition 34, and the conclusion follows. □

In order to prove the main result in this section, let us introduce the following measure functions. We use the following notation: a term that is not a root-stable rigid normal form is called a non-$\mathsf{rs{-}rnf}$. Given a multiset M and an element u, we denote the number of occurrences of u in M by $M(u)$.

Definition 54 *Let $\mathcal{R}$ be a TRS and t be a term. We define $D^*_{\mathcal{R}}(t)$ (resp. $D_{\mathcal{R}}(t)$) to be the multiset of subterms (resp. non-ground subterms) of t that are not root-stable rigid normal forms.*

We drop the subindex $\mathcal{R}$ in $D_{\mathcal{R}}(t)$ and $D^*_{\mathcal{R}}(t)$ when it is clear from the context.

Example 55 *Assume any TRS $\mathcal{R}$ such that any term rooted by symbol f is not a root-stable rigid normal form w.r.t. $\mathcal{R}$, whereas terms rooted by symbols a or s are root-stable rigid normal forms. Then,*

(1) for $t_1 = f(a, a)$, we have $D(t_1) = \emptyset$ and $D^*(t_1) = \{f(a, a)\}$,
(2) for $t_2 = f(s(x), f(a, a))$, we have $D(t_2) = \{f(s(x), f(a, a))\}$ and $D^*(t_2) = \{f(s(x), f(a, a)), f(a, a)\}$,
(3) for $t_3 = f(f(x, y), a)$, we have $D(t_3) = D^*(t_3) = \{f(f(x, y), a), f(x, y)\}$,
(4) for $t_4 = f(f(x, y), f(x, y))$, we have $D(t_4) = D^*(t_4) = \{f(f(x, y), f(x, y)), f(x, y), f(x, y)\}$, and
(5) for $t_5 = f(f(x, y), f(x', y'))$, we have $D(t_5) = D^*(t_5) = \{f(f(x, y), f(x', y')), f(x, y), f(x', y')\}$.

Let us now define an ordering $\rhd_\theta$ on terms. The main idea behind the definition is to capture that whenever t narrows to t' with substitution θ, all non-$\mathsf{rs}{-}\mathsf{rnf}$ terms in t' are just descendants of (possibly instantiated) strict subterms of non-$\mathsf{rs}{-}\mathsf{rnf}$ terms of t.

Definition 56 *Let t, s be two terms and θ a substitution. We say $t \rhd_\theta s$ if there is a position $p \in \mathcal{P}os_\Sigma(t)$ such that $s \equiv t\theta|_p$ and either $p > \epsilon$ or $\theta \neq id$. We write $t \blacktriangleright_\theta s$ whenever $t \rhd_\theta s$ and s is a strict subterm of t (i.e., $p > \epsilon$).*

We recall the definition of a multiset ordering.

Definition 57 (Multiset ordering) (Baader and Nipkow, 1998) *Let $(M, \succ)$ be a partial ordering. The multi-set extension of $\succ$ to multi-sets over M is defined by $S_1 \succ_{mul} S_2 \Leftrightarrow S_1 \not\equiv S_2$ and $\forall m \in M, S_2(m) > S_1(m) \Rightarrow \exists m' \in M : (m' \succ m, S_1(m') > S_2(m'))$.*

Since a term might be instantiated further and further, the orderings $\rhd_\theta$ and $\blacktriangleright_\theta$ are not well-founded, hence neither of their multiset extensions $(\rhd_\theta)_{mul}$ and $(\blacktriangleright_\theta)_{mul}$ are well-founded. Nevertheless, we can prove that there are no infinite decreasing sequences generated by narrowing steps. Informally, the idea is that no new non-$\mathsf{rs}{-}\mathsf{rnf}$ terms are introduced by narrowing and it may replicate in the worst case (strict) subterms of existing ones.

Definition 58 (Non-additive) *We say a decreasing sequence S_0 $(\rhd_{\theta_1})_{mul}$ S_1 $(\rhd_{\theta_2})_{mul}$ $\cdots$ $(\rhd_{\theta_n})_{mul}$ S_n of term multisets is* non-additive *if no new terms are introduced at any step of the sequence, i.e., for every $i > 0$ and term t in S_i such that $S_i(t) > 0$, there is a term t' in S_{i-1} such that $S_{i-1}(t') > 0$ and*

$t \equiv t'\theta_i$.

Definition 59 (Monotonically decreasing) *We say a non-additive decreasing sequence $S_0\ (\rhd_{\theta_1})_{mul}\ S_1\ (\rhd_{\theta_2})_{mul}\ \cdots\ (\rhd_{\theta_n})_{mul}\ S_n$ of term multisets is* monotonically decreasing *if replication of a term t implies consumption of a term u lying strictly above t, i.e., for every $i > 0$ and terms t in S_i and t' in S_{i-1} such that $t \equiv t'\theta_i$, $S_{i-1}(t') > 0$, and $S_i(t) > S_{i-1}(t')$, there are terms u in S_i and u' in S_{i-1} such that $u \equiv u'\theta_i$, $S_{i-1}(u') > S_i(u)$, and $u' \blacktriangleright_{id} t'$.*

Lemma 60 *Every monotonically decreasing sequence of term multisets is finite.*

Proof. By contradiction. Let us assume an infinite monotonically decreasing sequence

$$S_0\ (\rhd_{\theta_1})_{mul}\ S_1\ (\rhd_{\theta_2})_{mul}\ \cdots\ (\rhd_{\theta_n})_{mul}\ S_n \cdots$$

Since it is non-additive, there must be a term u_0 in the original multiset S_0 that is replicated infinitely many times, i.e., for all i there is u_i in S_i such that, for some p, $u_i \equiv u_0\theta_1 \cdots \theta_i|_p$ and $S_i(u_i) \geq S_0(u_0)$. However, this leads to a contradiction since the sequence is monotonically decreasing and u_0 is finite. □

We prove that that the conditions of the previous result do hold for the class of TRS's considered in this appendix.

Proposition 61 *Let $\mathcal{R}$ be a right–*srnf *TRS that satisfies the quasi stable rigid normalization condition. For each narrowing sequence $t_0 \overset{p_1}{\leadsto}_{\theta_1,l_1\to r_1} t_1 \cdots t_{n-1} \overset{p_n}{\leadsto}_{\theta_n,l_n\to r_n} t_n \cdots$ the sequence $D(t_0)\ (\rhd_{\theta_1})_{mul}\ D(t_1)\ (\rhd_{\theta_2})_{mul}\ D(t_2) \cdots$ of term multisets is monotonically-decreasing.*

Proof. The proof that the sequence is non-additive is obtained by considering that new non-ground, non-rs−rnf terms are never introduced by narrowing steps, since (i) $\mathcal{R}$ is right–srnf, and (ii) the computed substitutions are QSRNC and thus any eventual new non-rs−rnf brought by instantiation is ground.

The proof that the sequence is monotonically decreasing is obtained by considering that any new non-rs−rnf term u of t_i is ground, and any non-rs−rnf subterm u of t_{i-1} that has more occurrences in t_i than in t_{i-1} satisfies $t_{i-1}|_{p_i} \blacktriangleright_{id} u$. □

Now, we provide two auxiliary results for proving Theorem 38: (i) for the case when a narrowing step produces a stable rigidly normalized substitution, and (ii) for the case when a narrowing step produces a quasi stable rigidly normalized substitution. Intuitively, when a term t narrows to t', we take into account the number of variables of t and t' and the number of non-rs−rnf subterms in t and t', and show that at least one of these numbers decreases.

We first prove that whenever a term t narrows to t' by computing a stable rigidly normalized computed substitution θ, $D^*(t)\ (\rhd_\theta)_{mul}\ D^*(t')$.

Lemma 62 *Let $\mathcal{R}$ be a right–*srnf *TRS. For every narrowing step $t \overset{p}{\leadsto}_{\theta,l\to r} t'$ such that θ is a stable rigidly normalized substitution, $D^*(t)\ (\rhd_\theta)_{mul}\ D^*(t')$.*

Proof. By Definition 57, let us assume that there exists a term u such that $D^*(t')(u) > D^*(t)(u)$; otherwise it is trivial. We have to prove that there is a subterm w of t s.t. $w \rhd_\theta u$ and $D^*(t)(w) > D^*(t')(w)$. We consider the cases when $D^*(t)(u) = 0$ and $D^*(t)(u) > 0$ separately.

If $D^*(t)(u) = 0$, then u does not appear in t because u is an instantiated version of a subterm u' of t. That is, since θ is a stable rigidly normalized substitution and r is a srnf, there is a subterm u' of t such that $u \equiv u'\theta$ and $\theta_{\restriction Var(u')} \not\equiv id$. Therefore, $u' \rhd_\theta u$, $D^*(t)(u') > D^*(t')(u') = 0$, and the conclusion follows.

If $D^*(t)(u) > 0$, then the extra occurrences of u in t' have been introduced by propagation of the applied substitution due to the possible non-linearity of r (the possible non-linearity of l did not have any effect because θ is stable rigidly normalized), which implies that u is a strict subterm of $t|_p$. However, we have that $D^*(t)(t|_p) > D^*(t')(t|_p)$ (at least in one unit since $t|_p$ has been narrowed) and $t|_p \rhd_\theta u$, since u is a subterm of $t|_p$. Therefore, the conclusion follows. □

The previous result can be easily extended to $D(t)$ instead of $D^*(t)$ when we consider narrowing steps on non-ground terms.

Corollary 63 *Let $\mathcal{R}$ be a right–*srnf *TRS. For every narrowing step $t \overset{p}{\leadsto}_{\theta,l\to r} t'$ such that $t|_p$ is non-ground and θ is a stable rigidly normalized substitution, $D(t)\ (\rhd_\theta)_{mul}\ D(t')$.*

Now we are ready to extend the previous results to the case when the computed substitutions are not stable rigidly normalized.

Lemma 64 *Let $\mathcal{R}$ be a right–*srnf *TRS. For every narrowing step $t \overset{p}{\leadsto}_{\theta,l\to r} t'$ such that $t|_p$ is non-ground and θ is a quasi stable rigidly normalized substitution w.r.t. t, $D(t)\ (\rhd_\theta)_{mul}\ D(t')$.*

Proof. By Definition 57, let us assume that there exists a non–ground term u such that $D(t')(u) > D(t)(u)$; otherwise it is trivial. We have to prove that there is a subterm w of t s.t. $w \rhd_\theta u$ and $D(t)(w) > D(t')(w)$. We consider the cases when $D(t)(u) = 0$ and $D(t)(u) > 0$ separately.

If $D(t)(u) = 0$, then, since θ is a quasi stable rigidly normalized substitution w.r.t. t and r is a srnf, there is a subterm u' of t such that $u \equiv u'\theta$ and

$\theta_{\lceil Var(u')} \not\equiv id$. Therefore, $u' \rhd_\theta u$, $D(t)(u') > D(t')(u') = 0$, and the conclusion follows.

If $D(t)(u) > 0$, then the extra occurrences of u in t' have been introduced by propagation of the applied substitution, due to the possible non-linearity of either l or r. In both cases, u is a strict subterm of $t|_p$, and since $t|_p$ is non-ground and r is a srnf, $D(t)(t|_p) > D(t')(t|_p)$ (at least in one unit), $t|_p \blacktriangleright_\theta u$, and the conclusion follows. □

Let us finally demonstrate our main result in this section.

Theorem 38 (Termination of narrowing under QSRNC**)** *Let $\mathcal{R}$ be a right–*srnf *TRS that satisfies the quasi stable rigid normalization condition. Every narrowing derivation issuing from any term terminates.*

Proof. Given a narrowing sequence

$$\mathcal{D} = t_0 \overset{p_1}{\rightsquigarrow}_{\theta_1, l_1 \to r_1} t_1 \cdots t_{n-1} \overset{p_n}{\rightsquigarrow}_{\theta_n, l_n \to r_n} t_n \cdots$$

we define an order based on pairs $\langle D(t_i), D^*(t_i)\rangle$ and ordered by $\langle M_1, M_2\rangle \succ_\theta \langle M'_1, M'_2\rangle$ if $M_1\ (\rhd_\theta)_{mul}\ M'_1$ or $M_1 = M'_1$ and $M_2\ (\rhd_\theta)_{mul}\ M_2$. Note that the order is noetherian due to Proposition 61 and Lemma 60. Then, we prove termination of narrowing by noetherian induction on $\langle D(t_n), D^*(t_n)\rangle$ and $\succ_{\theta_n}$.

(1) (Base case) $\langle D(t_n), D^*(t_n)\rangle = \langle\emptyset, \emptyset\rangle$, which implies that there are no narrowable subterms in t_n, and the claim follows trivially.
(2) (Induction case) We have $\langle D(t_n), D^*(t_n)\rangle \neq \langle\emptyset, \emptyset\rangle$, and consider the subsequent narrowing step

$$t_n \overset{p_{n+1}}{\rightsquigarrow}_{\theta_{n+1}, l_{n+1} \to r_{n+1}} t_{n+1}$$

We consider the following three cases separately,
(a) if $t_n|_{p_{n+1}}$ is a ground term, then $D(t_n) = D(t_{n+1})$ and θ_{n+1} is a stable rigidly normalized substitution. Then by Lemma 62, $D^*(t_n)$ $(\rhd_{\theta_{n+1}})_{mul}$ $D^*(t_{n+1})$;
(b) if $t_n|_{p_{n+1}}$ is a non-ground term and θ_{n+1} is a stable rigidly normalized substitution, then by Corollary 63, $D(t_n)\ (\rhd_{\theta_{n+1}})_{mul}\ D(t_{n+1})$;
(c) if $t_n|_{p_{n+1}}$ is a non-ground term and θ_{n+1} is a quasi stable rigidly normalized substitution w.r.t. t_n, then by Lemma 64, $D(t_n)\ (\rhd_{\theta_{n+1}})_{mul}$ $D(t_{n+1})$.

In the three cases, the result follows by induction hypothesis. □