Matemáticas II: Segundo curso del Grado en Ingeniería Aeroespacial

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Ecuaciones diferenciales y transformadas de Laplace con aplicaciones
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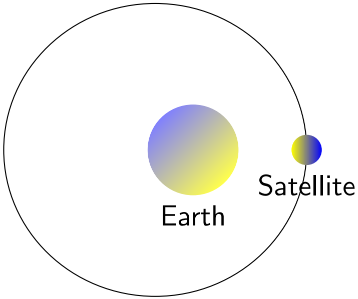
Chapter 3

Systems of differential equations

1. Introduction
2. Matrix method
   - Homogeneous linear system
   - Non-homogeneous linear system
3. Elimination method

Orbit of a satellite

A simplified model to compute the orbit of a satellite around the Earth assumes that the total mass of the Earth is allocated at the center of mass and only the gravitational force acts.



If the position of the satellite is in the plane and given by $\mathbf{r} = (x, y)$, the Newton's equations are given by

$$m\frac{d^2\mathbf{r}}{dt^2} = -G\,M\,m\frac{\mathbf{r}}{\|\mathbf{r}\|^3},$$

$G = 6{,}674 \cdot 10^{-20}\,\frac{Km^3}{Kg\cdot s^2}$: Universal gravitational constant
$M = 5{,}9722 \cdot 10^{-20} Kg$: Mass of the Earth
m: Mass of the satellite

$$\begin{cases} x' &=& v_x \\ v_x' &=& -GM\dfrac{x}{(x^2+y^2)^{3/2}} \\ y' &=& v_y \\ v_y' &=& -GM\dfrac{y}{(x^2+y^2)^{3/2}} \end{cases}$$

This is a **non-linear** system of ODEs, and we will only consider linear systems of ODEs.

Linear system

A linear system of DE that involves n scalar functions $y_1(x),\ldots,y_n(x)$, on the variable x has the general form

$$\begin{aligned} P_{11}(D)y_1+\cdots+P_{1n}(D)y_n &= f_1(x) \\ &\vdots \\ P_{n1}(D)y_1+\cdots+P_{nn}(D)y_n &= f_n(x) \end{aligned}$$

where $P_{ij}(D)$are polynomial functions on the operator $D\equiv\frac{d}{dx}$.

Example: Write the system in terms of polynomials of D.

$$\left\{ \begin{array}{l} x''+2y'-x'+3y=2t \\ y'''+x'-x-2y=\sin(t) \end{array} \right. \quad\Rightarrow\quad \left\{ \begin{array}{l} (D^2-D)x+(2D+3)y=2t \\ (D-1)x+(D^3-2)y=\sin(t) \end{array} \right.$$

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Normal form

$$
\begin{aligned}
Dy_1 &= a_{11}(x)y_1 \cdots + a_{1n}(x)y_n + f_1(x) \\
&\vdots \\
Dy_n &= a_{n1}(x)y_1 \cdots + a_{nn}(x)y_n + f_n(x)
\end{aligned}
$$

We only consider systems with constant coefficients and they can be written in **matrix form**

$$
D\left\{\begin{array}{c} y_1 \\ \vdots \\ y_n \end{array}\right\} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \left\{\begin{array}{c} y_1 \\ \vdots \\ y_n \end{array}\right\} + \left\{\begin{array}{c} f_1(x) \\ \vdots \\ f_n(x) \end{array}\right\}
$$

and in compact form

$$
\boxed{DY = AY + F(x)}
$$

Example: Let us consider the system of DEs

$$\left\{ \begin{array}{l} x' = 3x - 2y + t \\ y' = -x + 4y + 2t \end{array} \right.$$

$$D \left\{ \begin{array}{c} x \\ y \end{array} \right\} = \left(\begin{array}{cc} 3 & -2 \\ -1 & 4 \end{array} \right) \left\{ \begin{array}{c} x \\ y \end{array} \right\} + \left\{ \begin{array}{c} t \\ 2t \end{array} \right\}$$

or equivalently

$$D \left\{ \begin{array}{c} y_1 \\ y_2 \end{array} \right\} = \left(\begin{array}{cc} 3 & -2 \\ -1 & 4 \end{array} \right) \left\{ \begin{array}{c} y_1 \\ y_2 \end{array} \right\} + \left\{ \begin{array}{c} x \\ 2x \end{array} \right\}$$

Introduction

Given the nth-order linear DE

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x) \qquad (1)$$

if we introduce the following change of variables

$$y_1 = y, \quad y_2 = y' = Dy_1, \quad \ldots, \quad y_n = y^{(n-1)} = Dy_{n-1},$$

we can write it as a first order linear system of DEs

$$D\left\{\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{array}\right\} = \left(\begin{array}{cccc} 0 & 1 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & 0 \\ & & & 1 \\ -a_0 & & \cdots & -a_{n-1} \end{array}\right)\left\{\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{array}\right\} + \left\{\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{array}\right\}$$

Example: Transform second order linear the ODE

$$x'' + 3x' - 5x = \cos(t)$$

into a linear system of first order ODEs

$$\begin{cases} x' = v \\ v' = -3v + 5x + \cos(t) \end{cases}$$

$$D\begin{Bmatrix} x \\ v \end{Bmatrix} = \begin{pmatrix} 0 & 1 \\ 5 & -3 \end{pmatrix}\begin{Bmatrix} x \\ v \end{Bmatrix} + \begin{Bmatrix} 0 \\ \cos(t) \end{Bmatrix}$$

or equivalently

$$D\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{pmatrix} 0 & 1 \\ 5 & -3 \end{pmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \cos(x) \end{Bmatrix}$$

Introductory video (MIT)

https://www.youtube.com/watch?v=_Fo3Jq1blKk

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Several techniques are considered to solve this class of systems

- First, we consider matrix techniques
- Next, we show how to transform the system into a set of uncoupled equations depending on each dependent variable (but with higher derivatives).

Introduction

Example: Let us consider the system of DEs with solutions

$$\left\{ \begin{array}{ll} u' = -u & \Rightarrow \qquad u(t) = C_1 e^{-t} \\ v' = v & \Rightarrow \qquad v(t) = C_2 e^{t} \end{array} \right.$$

and apply the change of variables

$$\left\{ \begin{array}{l} x = u + v \\ y = -2u - 4v \end{array} \right. \Rightarrow D \left\{ \begin{array}{c} x \\ y \end{array} \right\} = \underbrace{\begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix}}_{A} \left\{ \begin{array}{c} x \\ y \end{array} \right\}$$

$$Sol. : \qquad \left\{ \begin{array}{c} x \\ y \end{array} \right\} = C_1 \left\{ \begin{array}{c} 1 \\ -2 \end{array} \right\} e^{-t} + C_2 \left\{ \begin{array}{c} 1 \\ -4 \end{array} \right\} e^{t}$$

Matrix techniques: The solution is related with the eigenvalues and eigenvectors of the matrix A.

Introductory videos

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https://www.youtube.com/watch?v=93QiCJ1XSPM

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Homogeneous linear system

$$DY = AY$$

Since the system is linear, if Y_1 and Y_2 are particular solutions of the system (i.e. $DY_i = AY_i,\ i = 1, 2$) then

$$Y_3 = \alpha Y_1 + \beta Y_2$$

with α, β arbitrary constants, is a solution of the system of DE too. If the set of vector functions $\{Y_1, \ldots, Y_n\}$ are particular solutions of the DE, $DY = AY$, and in addition, they are linearly independent (LI), then they form a **fundamental system of solutions**, and the matrix

$$\Phi = [Y_1 \ \cdots \ Y_n]$$

is a **fundamental matrix solution**.

Example: Given the system of DEs with solution

$$D\left\{\begin{array}{c} x \\ y \end{array}\right\} = \left(\begin{array}{cc} -3 & -1 \\ 8 & 3 \end{array}\right)\left\{\begin{array}{c} x \\ y \end{array}\right\}$$

$$Sol.: \qquad \left\{\begin{array}{c} x \\ y \end{array}\right\} = C_1\left\{\begin{array}{c} 1 \\ -2 \end{array}\right\}e^{-t} + C_2\left\{\begin{array}{c} 1 \\ -4 \end{array}\right\}e^{t}$$

Taking $C_1 = 1, C_2 = 0$ and $C_1 = 0, C_2 = 1$ we get the part. sols.

$$Y_1 = \left\{\begin{array}{c} 1 \\ -2 \end{array}\right\}e^{-t}, \qquad Y_2 = \left\{\begin{array}{c} 1 \\ -4 \end{array}\right\}e^{t}$$

and a **fundamental matrix solution** is

$$\Phi = \left(\begin{array}{cc} e^{-t} & e^{t} \\ -2e^{-t} & -4e^{t} \end{array}\right)$$

Theorem

If $\Phi = [Y_1 \ \cdots \ Y_n]$ is a fundamental matrix solution of the homogeneous linear system, $DY = AY$, of order n in $x \in (a, b)$, where A is continuous in (a, b), then the general solution is

$$Y = C_1 Y_1 + \ \cdots \ + C_n Y_n$$

or equivalently

$$Y = \Phi C = [Y_1 \ \cdots \ Y_n] \left\{ \begin{array}{c} C_1 \\ \vdots \\ C_n \end{array} \right\}$$

with $C_i, i = 1, \ldots, n$ being arbitrary constants.

Matrix method: homogeneous system

Given n particular solutions of the DE, $\{Y_1, \ldots, Y_n\}$ we have to check if they are LI or not. To this purpose we build the Wronskian $W(Y_1, \ldots, Y_n; x)$ given by

$$W(Y_1, \ldots, Y_n; x) = \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix} \quad \text{with} \quad Y_j = \left\{ \begin{array}{c} y_{1j} \\ \vdots \\ y_{nj} \end{array} \right\}$$

Theorem

The set of solutions of the linear system of DE, $\{Y_1, \ldots, Y_n\}$, are LI **iff** (if and only if)

$$W(Y_1, \ldots, Y_n; x) \neq 0, \qquad \forall x$$

Matrix method: homogeneous system

We look for N solutions of the system of DE. In the same way as for the high order linear systems, we look for exponential solutions of the form

$$Y = \left\{ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_n \end{array} \right\} e^{\lambda x}.$$

Question: which conditions must satisfy λ and the constant vector $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)^T$ to be a solution of the equation? If we compute the derivative of the vector function Y

$$DY = \lambda Y$$

and since the equation is $DY = AY$, by equating the right hand side of both equations we have

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \left\{ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_n \end{array} \right\} e^{\lambda x} = \lambda \left\{ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_n \end{array} \right\} e^{\lambda x}$$

Matrix method: homogeneous system

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \left\{ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_n \end{array} \right\} e^{\lambda x} = \lambda \left\{ \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_n \end{array} \right\} e^{\lambda x}$$

and, since $e^{\lambda x} \neq 0$, we can eliminate it leading into an eigenvalue problem, where λ is the eigenvalue of A and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)^T$ is its associated eigenvector.

Theorem

If A is a constant matrix and $\sigma(A)$ contains n different (real or complex) numbers, $\lambda_1, \ldots, \lambda_n$, a fundamental system of solutions for the DE, $DY = AY$ is given by $\{\bar{\alpha}_1 e^{\lambda_1 x}, \ldots, \bar{\alpha}_n e^{\lambda_n x}\}$, where $\bar{\alpha}_i$ is a (non zero) eigenvector associated to the eigenvalue $\lambda_i, \; i = 1, \ldots, n$.

If $\lambda_1 = \delta_1 + i\delta_2$ is a complex eigenvalue, with associated complex eigenvector, $\alpha_1 = \gamma_1 + i\gamma_2$ with $\gamma_1, \gamma_2 \in \mathbb{R}^n$ then its conjugate, $\lambda_2 = \delta_1 - i\delta_2$, is an eigenvalue too with eigenvector $\alpha_2 = \gamma_1 - i\gamma_2$. The combination of these two solutions must provide real solutions because A is a real matrix. The linear combination

$$C_1(\gamma_1 + i\gamma_2)e^{(\delta_1 + i\delta_2)x} + C_2(\gamma_1 - i\gamma_2)e^{(\delta_1 - i\delta_2)x}$$

requires that $C_2 = C_1^*$ to give a real solution.

To simplify, consider the sol. from the first eig.

$$(\gamma_1 + i\gamma_2)e^{(\delta_1 + i\delta_2)x} = (\gamma_1 + i\gamma_2)\big(\cos(\delta_2 x) + i\sin(\delta_2 x)\big)e^{\delta_1 x}$$
$$= \underbrace{(\cos(\delta_2 x)\gamma_1 - \sin(\delta_2 x)\gamma_2)e^{\delta_1 x}}_{Y_1} + i\underbrace{(\sin(\delta_2 x)\gamma_1 + \cos(\delta_2 x)\gamma_2)e^{\delta_1 x}}_{Y_2}$$

The real and the imaginary parts correspond to two independent solutions.

If $\lambda_1 = \delta_1 + i\delta_2$ is a complex eigenvalue, with associated complex eigenvector, $\alpha_1 = \gamma_1 + i\gamma_2$ with $\gamma_1, \gamma_2 \in \mathbb{R}^n$ then its conjugate, $\lambda_2 = \delta_1 - i\delta_2$, is an eigenvalue too with eigenvector $\alpha_2 = \gamma_1 - i\gamma_2$. The combination of these two solutions must provide real solutions because A is a real matrix. The linear combination

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$$\begin{aligned}(\gamma_1 + i\gamma_2)e^{(\delta_1+i\delta_2)x} &= (\gamma_1 + i\gamma_2)\big(\cos(\delta_2 x) + i\sin(\delta_2 x)\big)e^{\delta_1 x} \\ &= \underbrace{(\cos(\delta_2 x)\gamma_1 - \sin(\delta_2 x)\gamma_2)e^{\delta_1 x}}_{Y_1} + i\underbrace{(\sin(\delta_2 x)\gamma_1 + \cos(\delta_2 x)\gamma_2)e^{\delta_1 x}}_{Y_2}\end{aligned}$$

The real and the imaginary parts correspond to two independent solutions.

- If an eigenvalue λ_i has multiplicity m and we find m LI eigenvectors, $\bar{\alpha}_{i1},\ldots,\bar{\alpha}_{im}$ (this is the case when the matrix A is diagonalizable) then $\bar{\alpha}_{i1}e^{\lambda_i x},\ldots,\bar{\alpha}_{im}e^{\lambda_i x}$ are m LI solutions.
- If the matrix A is not diagonalizable, then the number of eigenvectors is smaller that the multiplicity of the eigenvalue. In this case, we will guess with the following vector functions

$$\left\{ \begin{array}{c} \alpha_1 x+\beta_1 \\ \vdots \\ \alpha_n x+\beta_n \end{array} \right\} e^{\lambda_i x}, \qquad \left\{ \begin{array}{c} \gamma_1 x^2+\delta_1 x+\varepsilon_1 \\ \vdots \\ \gamma_n x^2+\delta_n x+\varepsilon_n \end{array} \right\} e^{\lambda_i x}, \quad \text{etc.}$$

Matrix method: non-homogeneous system

Given the non-homogeneous system

$$DY = AY + F$$

the general solution is given by the sum of the homogeneous solution, $Y_H = C_1 Y_1 + \cdots + C_n Y_n$, and a particular solution of the non-homogeneous DE, Y_p, i.e. $Y = Y_H + Y_p$. Observe that Y is a solution of the non-homogeneous DE and contains n independent constants

$$\begin{aligned} DY &= D(Y_H + Y_p) = C_1 DY_1 + \cdots + C_n DY_n + DY_p \\ &= C_1 AY_1 + \cdots + C_n AY_n + AY_p + F \\ &= AY + F \end{aligned}$$

Matrix method: non-homogeneous system

Then, it only remains to look for a particular solution of the non-homogeneous DE. To this purpose we consider the method of **variation of constants**

$$Y_p = c_1(x)Y_1 + \cdots + c_n(x)Y_n.$$

We impose the functions $c_i(x)$ to satisfy similar conditions as in the previous chapter for high order linear DE.

Theorem

Let $DY = AY + F$ a non-homogeneous linear system and $\{Y_1, \ldots, Y_n\}$ a fundamental system of solutions of the associated homogeneous DE. Then, $Y_p = c_1(x)Y_1 + \cdots + c_n(x)Y_n$ is a particular solution of the non-homogeneous system if

$$c_1' Y_1 + \cdots + c_n' Y_n = F$$

Elimination method and Cramer's rule

As already mentioned, a linear system of DE can be written as

$$\begin{aligned} P_{11}(D)y_1 + \cdots + P_{1n}(D)y_n &= f_1(x) \\ &\vdots \\ P_{n1}(D)y_1 + \cdots + P_{nn}(D)y_n &= f_n(x) \end{aligned}$$

where $P_{ij}(D)$ are polynomial functions depending on the operator $D \equiv \frac{d}{dx}$.
We can see it as a linear system of equations.

- If we solve the system considering to $P_{ij}(D)$ as numbers (but taking into account that they are operators acting on functions) we obtain uncoupled DE, but of higher orders

$$\begin{aligned} R_1(D)y_1 &= h_1(x) \\ &\vdots \\ R_n(D)y_n &= h_n(x) \end{aligned}$$

- The number of independent constant is given by the degree of the polynomial we obtain from the following determinant

$$\Delta = \begin{vmatrix} P_{11}(D) & \cdots & P_{1n}(D) \\ \vdots & \ddots & \vdots \\ P_{n1}(D) & \cdots & P_{nn}(D) \end{vmatrix}$$

- We can solve the system following the **Cramer's rule**, but one has to take into account that the operators must act on functions. For example, for $n = 2$

$$y_1 = \frac{1}{P_{11}P_{22} - P_{12}P_{21}} \begin{vmatrix} f_1 & P_{12}(D) \\ f_2 & P_{22}(D) \end{vmatrix}$$

must be read as

$$\Big(P_{11}(D)P_{22}(D) - P_{12}(D)P_{21}(D)\Big) y_1 = P_{22}(D)f_1 - P_{12}(D)f_2$$