Matemáticas II: Segundo del Grado en Ingeniería Aeroespacial

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Ecuaciones diferenciales y transformadas de Laplace con aplicaciones
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Capítulo 4

Laplace Transform

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Introduction: Laplace Transform

Laplace Transform de $f(x)$

The **Laplace Transform** of a funtion $f(x)$ is a **new** function depending on a **new** variable, $F(s)$, given by

$$\mathcal{L}\left[f(x)\right] = F(s) = \int_0^{\infty} e^{-sx} f(x) dx.$$

This is an improper integral that must be understood as

$$F(s) = \lim_{b\to\infty} \int_0^{b} e^{-sx} f(x) dx,$$

so, the function $F(s)$ is only well defined for those values of s where the integral converges.

Examples. To compute the LT of:

a) $f(x) = 1$, **b**) $f(x) = x$, **c**) $f(x) = e^x$, **d**) $f(x) = \sin(x)$.

Introduction: Laplace Transform

Mathematica has a function that allows us to compute the LT of a function, f[x]

$$LaplaceTransform[f[x], x, s]$$

We can check easily that

$$\begin{array}{l} LaplaceTransform[1, x, s] = \frac{1}{s} \\ LaplaceTransform[x, x, s] = \frac{1}{s^2} \\ LaplaceTransform[Exp[x], x, s] = \frac{1}{s-1} \\ LaplaceTransform[Sin[x], x, s] = \frac{1}{s^2+1} \end{array}$$

Question: For which values of s the LT is well defined?

Introduction: Laplace Transform

If we write

$$F(s) = \int_0^N e^{-sx} f(x)dx + \int_N^\infty e^{-sx} f(x)dx$$

with N large enough, so the integrals converge

- $\displaystyle\int_N^\infty e^{-sx} f(x)dx$: $f(x)$ must not grow for $x \to \infty$ than the function faster than the inverse of e^{-sx} for some value of s.
- $\displaystyle\int_0^N e^{-sx} f(x)dx$: $f(x)$ must have no singularities that make the integral to be divergent. For example, if the function be piecewise continuous.

We say that $f(x)$ is of **exponential order** if
$\exists \quad \gamma, M, T \in \mathbb{R} \; / \; |f(x)| < Me^{\gamma x}, \quad \forall x > T$
We say that $f(x)$ is of **exponential order** γ_f with γ_f the infimum of the values of γ.

Introduction: Laplace Transform

- $x^n, \sin(\alpha x), \cos(\alpha x) : \ \gamma_f = 0$
- $e^{ax} : \ \gamma_f = a$
- $e^{-x^2} : \ \gamma_f = -\infty$
- $e^{x^2} : \ \gamma_f = \infty$ and we say it is not of exponential order.

In addition, it is easy to deduce from the definition that if the integral coverges for a given value of s_0, then it converges for $s > s_0$.

Theorem

If $f(x)$ is piecewise continuous and of exponential order γ_f, then, it exists $F(s) = \mathcal{L}[f(x)]$ for $s > \gamma_f$.

Introduction: Laplace Transform

Theorem

If $f(x)$ is piecewise continuous and $\int_0^\infty e^{-s_0 x} f(x)dx$ is convergent, then $\exists\ F(s) = \mathcal{L}[f(x)]$ for $s > s_0$ and satisfies that $\lim_{x\to\infty} F(s) = 0$.

Abscissa of convergence, s_f: the infimum of the values of s in which $F(s)$ is well defined. If $f(x)$ piecewise continuous and of exponential order γ_f, then $s_f \leq \gamma_f$.
Then, $F(s)$ is defined for $s > s_f$.

Since the limits of the integral in the LT are $[0, \infty[$ we will work with **causal functions**, i.e.

$$f(x) = \begin{cases} 0, & x < 0 \\ f(x), & x \geq 0 \end{cases}$$

For this purpose we introduce the unit step function (often called the **Heaviside function**) $u(x)$

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}, \qquad u(x-a) = \begin{cases} 0, & x < a \\ 1, & x \geq a \end{cases}$$

Then, when writing $f(x)$ we mean

$$u(x)f(x)$$

Introduction: Laplace Transform

The piecewise causal functions can be written using the Heaviside function.

- For example

$$f(x) = \begin{cases} 0, & x < 0 \\ f_1(x), & 0 \leq x < a \\ f_2(x), & a \leq x < \infty \end{cases}$$

  can be written as

$$\begin{aligned} f(x) &= u(x)f_1(x) - u(x-a)f_1(x) + u(x-a)f_2(x) \\ &= \big(u(x) - u(x-a)\big)f_1(x) + u(x-a)f_2(x) \end{aligned}$$

- The function

$$\big(u(x-a) - u(x-b)\big)f(x)$$

  can be considered as the function that turn on and turn off the function $f(x)$ on the interval $x \in [a, b]$

Applications

One of the most relevant applications of the LT is to solve linear DEs that can describe an electrical circuit, a mechanical system, etc.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = e(t)$$

with initial conditions

$$y^{(n-1)}(0) = y_0^{(n-1)}, \ldots, \; y'(0) = y_0', \quad y(0) = y_0.$$

The coefficients a_i depend on how the circuit is built, the mechanical system, etc.

$e(t)$ is the entrance of the system: the voltage that is applied to the circuit, the external forces applied to the mechanical system, etc.

$y(t)$ is the response of the system.

Applications

The LT is useful if one is interested to study the solution of a system with many different initial conditions or different entrance functions (for example, different external forces). In addition, $e(t)$ can be a piecewise continuous function, a periodic and piecewise continuous function, a unit impulse signal, etc.

The LT are also useful to solve involved equations like some integro-differential equations. For example

$$y'' + \int_0^x e^{2(x-t)} y(t) dt = e^x$$

The procedure to solve the problem is:

1- To compute the LT of the whole equation.

2- To solve the equation for the LT, $Y(s) = \mathcal{L}[y(x)]$.

3- To find the solution $y(x)$ whose LT is $Y(s)$.

(**ILL**: It corresponds to $y(x) = \mathcal{L}^{-1}[Y(s)]$)

Properties of the LT

We now show how to compute the LT for large number of functions by taking into account a number of propertie tha we list as Theorems. These properties can be easily be proven by just applying the definition of the LT and using the properties of the integrals.

Theorem (Lineality)

If there exists $\mathcal{L}[f(x)],\ \mathcal{L}[g(x)]$ *and* $\alpha, \beta \in \mathbb{R}$*, then*

$$\mathcal{L}[\alpha f(x) + \beta g(x)] = \alpha \mathcal{L}[f(x)] + \beta \mathcal{L}[g(x)]$$

Find: $\mathcal{L}[\cosh(ax)],\ \mathcal{L}[\sinh(ax)]$.

Theorem (LT of a function's derivative)

Let f a function of exponential order γ_f, continuous in $]0,\infty[$ and f' being its derivative for those values of x where it exists. If f' a piecewise continuous and $\exists \lim_{x\to 0^+} f(x) = f(0^+) \in \mathbb{R}$, then

$$\mathcal{L}[f'(x)] = s\,\mathcal{L}[f(x)] - f(0^+), \qquad s > \gamma_f.$$

Find, using this property $\mathcal{L}[\cos(bx)], \mathcal{L}[\sin(bx)]$

Theorem

If $f, f', \ldots, f^{(n-1)} \in \mathbf{C}(]0,\infty[)$ are of exponential order, $f^{(n)}$ piecewise continuous and

$$\lim_{x\to 0^+} f^{(i)}(x) = f^{(i)}(0^+) \in \mathbb{R}, \qquad i = 0, 1, \ldots, n-1,$$

then

$$\mathcal{L}[f^{(n)}(x)] = s^n\,\mathcal{L}[f(x)] - s^{n-1} f(0^+) - \cdots - s f^{(n-2)}(0^+) - f^{(n-1)}(0^+).$$

From the result of the LT of the function's derivative it can be easily proven that

Theorem (The LT of integrals)

If $\mathcal{L}[\varphi(x)] = F(s)$*,*

$$\mathcal{L}\left[\int_0^x \varphi(t)dt\right] = \frac{F(s)}{s}.$$

It is also straightful to prove (making use of these theorems) that

$$\mathcal{L}[x^n] = \frac{n!}{s^{n+1}}, \qquad n = 0, 1, 2, \ldots$$

This allows us to find the LT of all polynomials.
Example: Find $\mathcal{L}[x^3 - 3x^2 + 5]$

To find $\mathcal{L}[x^n]$ for n a real number such that $n > -1$, we review the definition of the **Gama function**

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx, \qquad n > 0$$

that satisfies $\Gamma(1) = 1$ and the recursion

$$\Gamma(n) = (n-1)\Gamma(n-1), \qquad n > 1$$

so $\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}$. For other values of n we need to use the recursion relation as well as a procedure to evaluate it on a given unit interval (it can be tabulatted on such interval). It is easy to prove that

$$\mathcal{L}[x^n] = \frac{\Gamma(n+1)}{s^{n+1}}, \qquad n > -1$$

Example: To find $\mathcal{L}\left[\sqrt[4]{x} - \frac{2}{\sqrt[5]{x}}\right]$.

Theorem (Change of scaling)

Given $h > 0$, *if*

$$\mathcal{L}[f(x)] = F(s),\ s > s_f \ \Rightarrow \ \mathcal{L}[f(hx)] = \frac{1}{h} F\left(\frac{s}{h}\right),\ s > hs_f$$

Theorem (First shift property)

Si $a \geq 0$ *y*

$$\mathcal{L}[f(x)] = F(s),\ s > s_f \ \Rightarrow \ \mathcal{L}[u(x-a)f(x-a)] = e^{-as} F(s),\ s > s_f$$

Theorem (Second shift property)

If $a \in \mathbb{R}$ *and*

$$\mathcal{L}[f(x)] = F(s),\ s > s_f \ \Rightarrow \ \mathcal{L}[e^{ax} f(x)] = F(s-a),\ s > a + s_f$$

Theorem (Multiplication by x)

If $\mathcal{L}[f(x)] = F(s)$ *then*

$$\mathcal{L}[x^n f(x)] = (-1)^n F^{(n)}(s), \; n \in \mathbb{N}$$

Theorem (Division by x)

If $\mathcal{L}[f(x)] = F(s)$ *and it exists*

$$\lim_{x \to 0^+} \frac{f(x)}{x} \in \mathbb{R} \; \Rightarrow \; \mathcal{L}\left[\frac{f(x)}{x}\right] = \int_s^\infty F(s) ds$$

Definition (Convolution of functions)

Given two functions $f(x)$ and $g(x)$, we define the convolution of the functions, denoted by $f * g$, as a new function given by

$$(f * g)(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

For causal functions then $f * g$ is causal and

$$(f * g)(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \int_0^x f(t)g(x-t)dt & \text{if} \quad x \geq 0 \end{cases}$$

Theorem (The convolution)

If $F(s)$ and $G(s)$ are the LT of the functions $f(x)$ and $g(x)$, respectively, then

$$\mathcal{L}\left[f(x) * g(x)\right] = F(s)G(s)$$

Theorem (Periodic functions)

If $f(x)$ is a periodic function with period, T, ($f(x+T)=f(x)$), and it exists $\mathcal{L}[f(x)]$ then

$$\mathcal{L}[f(x)] = \frac{1}{1-e^{-Ts}} \int_0^T e^{-sx} f(x)\,dx, \qquad s>0.$$

Example: Find the LT of the $2-$periodic causal function such that $f(x)=x,\ 0\le x<1,\ f(x)=2-x,\ 1<x\le 2$.

Inverse Laplace Transform (ILT)

Definition (ILT)

Given the function $F(s)$, we say that a function $f(x)$ is the **inverse Laplace transform** *of $F(s)$ if $\mathcal{L}[f(x)] = F(s)$. It is denoted by $f(x) = \mathcal{L}^{-1}[F(s)]$*

The table at the end of the book, reading it from right to left, corresponds to a table of ILT. Most functions $F(s)$ in this course will have the form

$$F(s) = \frac{P(s)}{Q(s)}, \qquad \text{o} \qquad F(s) = e^{-as}\frac{P(s)}{Q(s)}$$

with $P(s), Q(s)$ polynomials and Q of a higher degree than P.

Inverse Laplace Transform

Theorem (Linearity)

If $c_1, c_2 \in \mathbb{R}$ *and* $F_1(s), F_2(s)$ *have ILT, then*

$$\mathcal{L}^{-1}\left[c_1 F_1(s) + c_2 F_2(s)\right] = c_1 \mathcal{L}^{-1}\left[F_1(s)\right] + c_2 \mathcal{L}^{-1}\left[F_2(s)\right]$$

Let us first consider the case

$$F(s) = \frac{P(s)}{Q(s)}$$

We can decompose $F(s)$ into simple fractions and to apply the linearity property. Then, it suffices to study each class of simple fractions that are obtained from general decompositions, i.e.

$$\frac{A}{s-a}, \quad \frac{B}{(s-a)^n}, \quad \frac{Cs+D}{(s-\alpha)^2+\beta^2}, \quad \frac{Es+F}{((s-\alpha)^2+\beta^2)^n}$$

$$\mathcal{L}^{-1}\left[\frac{A}{s-a}\right] = A\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = Ae^{ax}$$

$$\mathcal{L}^{-1}\left[\frac{B}{(s-a)^n}\right] = Be^{ax}\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = Be^{ax}\frac{x^{n-1}}{(n-1)!}$$

Taking into account that

$$\mathcal{L}\left[Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)\right] = \frac{A(s-\alpha)+B\beta}{(s-\alpha)^2+\beta^2}$$

then

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{Cs+D}{(s-\alpha)^2+\beta^2}\right] &= \mathcal{L}^{-1}\left[\frac{C(s-\alpha)+C\alpha+D}{(s-\alpha)^2+\beta^2}\right] \\ &= Ce^{\alpha x}\cos(\beta x) + \frac{C\alpha+D}{\beta}e^{\alpha x}\sin(\beta x)\end{aligned}$$

The ILT

$$\mathcal{L}^{-1}\left[\frac{Es+F}{((s-\alpha)^2+\beta^2)^n}\right]$$

can be solved recursively by convolution. For example, if

$$f(x)=\mathcal{L}^{-1}\left[\frac{Es+F}{(s-\alpha)^2+\beta^2}\right], \quad g(x)=\mathcal{L}^{-1}\left[\frac{1}{(s-\alpha)^2+\beta^2}\right]$$

then

$$\mathcal{L}^{-1}\left[\frac{Es+F}{((s-\alpha)^2+\beta^2)^2}\right]=\mathcal{L}^{-1}\left[\frac{Es+F}{(s-\alpha)^2+\beta^2}\ \frac{1}{(s-\alpha)^2+\beta^2}\right]=f*g$$

Heaviside method

We study the method only for the case of simple roots, i.e.

$$F(s) = \frac{P(s)}{Q(s)}$$

with Q of degree q and P of degree $p < q$ and, in addition, Q has q distinct simple roots, i.e.

$$Q(s) = \alpha(s - a_1) \cdots (s - a_q)$$

We define $Q_{a_i}(s) = Q(s)/(s - a_i)$, i.e. the polynomial Q where we have eliminated the factor $s - a_i$. Then, we have

$$\frac{P(s)}{\alpha(s - a_1) \cdots (s - a_q)} = \frac{A_1}{(s - a_1)} + \cdots + \frac{A_q}{(s - a_q)}$$

with

$$A_i = \frac{P(a_i)}{Q_{a_i}(a_i)}$$

In some problems we find some instantaneous impulse or forces and this leads to functions, $F(s)$, that contain a constant term. In those cases it is convenient to introduce the **Dirac delta function** (also known as the **unit impulse symbol**)
Given $\epsilon > 0$, we define the function

$$\delta_\epsilon(x - x_0) = \begin{cases} 0, & x < x_0 \\ 1/\epsilon, & x_0 \le x < x_0 + \epsilon \\ 0, & x \ge x_0 + \epsilon \end{cases}$$

being a rectangle of unit area

$$\int_{-\infty}^{\infty} \delta_\epsilon(x - x_0) dx = 1$$

It can be seen as the impulse caused by a constant force of magnitude $\frac{1}{\epsilon}$ that applies for the interval ϵ, i.e.

$$\delta_\epsilon(x - x_0) = \frac{1}{\epsilon}\big(u(x - x_0) - u(x - (x_0 + \epsilon))\big)$$

The **Dirac delta** or **unit impulse symbol**) is defined in x_0 as the limit in which $\epsilon \to 0$, i.e.

$$\delta(x - x_0) = \lim_{\epsilon \to 0} \delta_\epsilon(x - x_0)$$

with the properties

$$a)\ \delta(x - x_0) = 0,\ x \neq x_0, \quad b)\ \int_{-\infty}^{\infty} \delta(x - x_0) dx = 1$$

We take $x_0 = 0$ and evaluate

$$\begin{aligned} \mathcal{L}[\delta(x)] &= \lim_{\epsilon \to 0} \mathcal{L}[\delta_\epsilon(x)] = \lim_{\epsilon \to 0} \mathcal{L}[\frac{1}{\epsilon}\big(u(x) - u(x - \epsilon)\big)] \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\frac{1 - e^{-s\epsilon}}{s}\right) = \lim_{\epsilon \to 0} \frac{se^{-s\epsilon}}{s} = 1 \end{aligned}$$

so

$$\mathcal{L}^{-1}[A] = A\delta(x)$$

To solve linear ODEs. Transfer function

Let us see how to solve a linear EDO using LT

$$y''' - 6y'' + 12y' - 8y = t\,e^{2t}$$
$$y(0^+) = 1, \quad y'(0^+) = 0, \quad y''(0^+) = -2.$$

If we compute the LT of the whole equation

$$\mathcal{L}[y''' - 6y'' + 12y' - 8y] = \mathcal{L}[t\,e^{2t}]$$

and we apply the linearity property as well as the LT of the function's derivative we easily find that

$$Y(s) = \mathcal{L}[y(t)] = \frac{s^2 - 6s + 10}{(s-2)^3} + \frac{1}{(s-2)^5}$$

and then, we obtain the solution by evluating the ILT

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2 - 6s + 10}{(s-2)^3} + \frac{1}{(s-2)^5}\right]$$

To solve linear ODEs. Transfer function

Let us consider the IVP

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = e(t)$$

with initial conditions

$$y^{(n-1)}(0^+) = y_0^{(n-1)}, \ldots, \; y'(0^+) = y_0', \quad y(0^+) = y_0.$$

We have the following:

Theorem

The response $y(t)$ of the system is

$$y(t) = y_{I_0}(t) + y_H(t)$$

the sum of the response $y_{I_0}(t)$ to $e(t)$ under nul initial conditions, and the response, $y_H(t)$ to the nul entrance with the given initial conditions.

Definition (weight function and transfer function)

Weight function : *it is the response,* $y(t) = h(t)$*, to the unit impulse with null initial conditions.*
Transfer function: *it is the LT of* $h(t)$*.*
Indicial admittance: *it is the response,* $y(t) = a(t)$ *of the system to the unit step function with null initial conditions, i.e.* $a(t) = h(t) * u(t)$

Example: To find tha response of a system to the input $e(t) = \cos(2t)$, if we know that the response to the null input is $y_H = e^{-t}$, and its indicial admitance is $a(t) = \sin(t)$. To find the response to $e(t) = \cos(3t)$ with null initial conditions.

Equations with variable coefficients

Most equations with variable coefficients cannot be solved using the techniques previously studied. However, some of these equations can be solved using the LT. We study some equations in which it appear terms of the form : $t^r y^{(n)}(t)$, where we know that

$$\mathcal{L}[t^r y^{(n)}(t)] = (-1)^r \frac{d^r}{ds^r} \mathcal{L}[y^{(n)}(t)]$$

Example: To solve the IVP

$$ty'' + (1-2t)y' - 2y = 0, \qquad y(0^+) = 1, \ y'(0^+) = 2.$$

Sol.: $\mathcal{L}[ty''] + \mathcal{L}[y'] - 2\mathcal{L}[ty'] - 2\mathcal{L}[y] = 0.$

$$(-s^2 + 2s)\frac{dY}{ds} - sY = 0 \ \Rightarrow \ Y = \frac{C}{s-2} \ \Rightarrow \ y = Ce^{2t}$$

$$y(0^+) = 1 \ \Rightarrow \ C = 1 \ \Rightarrow \ y(t) = e^{2t}$$

Integro-differential equations

We now show how to use the LT to solve some equations that involve derivatives as well as integrals of the unknown function. We use the following properties

$$\mathcal{L}\left[\int_0^x y(t)dt\right] = \frac{Y(s)}{s}, \quad \mathcal{L}\left[\int_0^x f(t)g(x-t)dt\right] = F(s)G(s)$$

Example: To find $m(t)$ where

$$m(t) = \int_0^t f(x)dx + \int_0^t m(t-x)f(x)dx$$

with $f(t) = \alpha e^{-\alpha t}, \ \alpha = 1.6 \cdot 10^{-8}$
Sol.: We evaluate the LT of the whole equation

$$M(s) = \frac{F(s)}{s} + M(s)F(s) \ \Rightarrow \ M(s) = \frac{F(s)}{s(1-F(s))} = \frac{\alpha}{s^2}$$

so

$$m(t) = \mathcal{L}^{-1}[M(s)] = \alpha\, t.$$

Systems of DEs

We can also use the LT to solve linear systems od DEs with given initial conditions.

Example: To find, by applying the TL, the solution to the IVP

$$\begin{aligned}\frac{dx}{dt}+x &= y+e^t, \qquad x(0^+)=1\\ \frac{dy}{dt}+y &= y+e^t, \qquad y(0^+)=1\end{aligned}$$

Sol.: We evaluate the LT of both equations and denote $X(s)=\mathcal{L}[x(t)],\ Y(s)=\mathcal{L}[y(t)]$, leading to the following algebraic system of equations

$$\begin{array}{c}(sX-x(0^+))+X=Y+\frac{1}{s-1}\\ (sY-y(0^+))+Y=X+\frac{1}{s-1}\end{array}$$

with solution

$$X(s)=\frac{1}{s-1},\quad Y(s)=\frac{1}{s-1} \quad\Rightarrow\quad x(t)=e^t,\ \ y(t)=e^t$$

Application: Improper Integrals

The LT can also be used to evaluate some improper integrals.
Example: To evaluate $\int_0^{+\infty} x^2 \cos x \, e^{-3x} dx$.
Sol.:

$$\begin{aligned}\int_0^{+\infty} x^2 \cos x e^{-sx} dx &= \mathcal{L}\left[x^2 \cos x\right] = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\left[\cos x\right] \\ &= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{2s^3-6s}{(s^2+1)^3} = F(s).\end{aligned}$$

Since the abscissa of convergence of $f(x) = x^2 \cos x$ is $s_f = 0$, the integral can be evaluated as follows

$$\int_0^{+\infty} x^2 \cos x \, e^{-3x} dx = F(3) = \frac{2\cdot 3^3 - 6\cdot 3}{(3^2+1)^3} = \frac{9}{250}.$$