1.- Considera la ED: $y'' - 2xy' - 2y = 2x\, e^{x^2}$

Comprueba que $y_0 = e^{x^2}$ es sol. de la hom. y emplea el mét. de red. de orden

$$\left.\begin{aligned} y_0 &= e^{x^2} \\ y_0' &= 2x\, e^{x^2} \\ y_0'' &= (2+4x^2)\, e^{x^2} \end{aligned}\right\} \quad y_0'' - 2x y_0' - 2y_0 = (2+4x^2)e^{x^2} - (2x)\cdot(2x)e^{x^2} - 2e^{x^2} = 0$$

Reducción de orden $y = y_0 \cdot z$ donde

$$z'' + \left(2\frac{y_0'}{y_0} + \frac{q}{p}\right)z' = \frac{f(x)}{p \cdot y_0} \qquad \left|\begin{aligned} q &= -2x \\ p &= 1 \\ f &= 2x\, e^{x^2} \end{aligned}\right.$$

Luego, tomando $w = z'$

$$w' + (4x - 2x)w = \frac{2x}{1} \Rightarrow w' + 2xw = 2x$$

$$w = e^{-\int 2x\,dx}\left(c + \int e^{\int 2x\,dx} \cdot 2x\,dx\right) = e^{-x^2}\left(c + \int 2x\, e^{x^2}\,dx\right)$$

$$= e^{-x^2}(c_1 + e^{x^2}) = c_1 e^{-x^2} + 1$$

$$z = \int w\,dx = \int (c_1 e^{-x^2} + 1)\,dx + c_2 = x + c_1\int e^{-x^2}\,dx + c_2$$

Sol: $y = y_0 \cdot z = e^{x^2}\left(x + c_2 + c_1\int e^{-x^2}\,dx\right) = \underline{c_2 e^{x^2} + c_1 e^{x^2}\int e^{-x^2}\,dx + x e^{x^2}}$

2.- Halla la sol. en serie de potencias de x hasta x^5 de la ED

$$y'' - 2xy' - 3y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2 \sum_{n=1}^{\infty} n a_n x^n - 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(2a_2 - 3a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2n a_n - 3a_n \right] x^n = 0$$

$$a_2 = \frac{3}{2} a_0$$

$$\boxed{a_{n+2} = \frac{2n+3}{(n+2)(n+1)} a_n}$$

$$n=1 \rightarrow a_3 = \frac{5}{3\cdot 2} a_1 = \frac{5}{6} a_1$$

$$n=2 \rightarrow a_4 = \frac{4+3}{4\cdot 3} a_2 = \frac{7}{4\cdot 3} \cdot \frac{3}{2} a_0 = \frac{7}{8} a_0$$

$$n=3 \rightarrow a_5 = \frac{6+3}{5\cdot 4} \cdot a_3 = \frac{9}{5\cdot 4} \cdot \frac{5}{6} a_1 = \frac{3}{8} a_1$$

luego

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + O(x^6)$$

$$= a_0 \left(1 + \frac{3}{2} x^2 + \frac{7}{8} x^4\right) + a_1 \left(x + \frac{5}{6} x^3 + \frac{3}{8} x^5\right) + O(x^6)$$

3- Halla la sol. en serie de potencias alrededor del origen de la ED $y''+4y=\operatorname{sen} x$

$$\operatorname{sen} x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+4a_n\right]x^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Separamos en términos pares e impares

$$\sum_{n=0}^{\infty}\left[(2n+2)(2n+1)a_{2n+2}+4a_{2n}\right]x^{2n}+\sum_{n=0}^{\infty}\left[(2n+3)(2n+2)a_{2n+3}+4a_{2n+1}\right]x^{2n+1}=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)}{(2n+1)!}x^{2n+1}$$

luego

$$(2n+2)(2n+1)a_{2n+2}+4a_{2n}=0 \Rightarrow a_{2(n+1)}=\frac{-2^2}{(2n+2)(2n+1)}a_{2n}=\frac{(-)^{n+1}2^{2(n+1)}}{(2n+2)!}a_0$$

$$(2n+3)(2n+2)a_{2n+3}+4a_{2n+1}=\frac{(-1)^n}{(2n+1)!}$$

$n=0 \to a_2=\frac{-4}{2}a_0=-2a_0$

$n=1 \to a_4=\frac{-4}{4\cdot 3}a_2=\frac{2}{3}a_0$

$$a_{2n+3}=\frac{-2^2}{(2n+3)(2n+2)}a_{2n+1}+\frac{1}{(2n+3)(2n+2)}\cdot\frac{(-1)^n}{(2n+1)!}$$

$$n=0 \to a_3=\frac{-4}{6}a_1+\frac{1}{6}\frac{1}{1!}=\boxed{\frac{-2}{3}a_1+\frac{1}{6}}$$

$$n=1 \to a_5=\frac{-4}{5\cdot 4}\cdot a_3+\frac{1}{5\cdot 4}\cdot\frac{-1}{3!}=\frac{-1}{5}\left(\frac{-2}{3}a_1+\frac{1}{6}\right)-\frac{1}{6\cdot 5\cdot 4}=\frac{2}{15}a_1-\frac{(1+1/4)}{6\cdot 5}$$

$$=\boxed{\frac{2}{15}a_1-\frac{1}{24}}$$

$$y=a_0\left(1-2x^2+\frac{2}{3}x^4\right)+a_1\left(x-\frac{2}{3}x^3+\frac{2}{15}x^5\right)+\frac{x^3}{6}-\frac{x^5}{24}+O(x^6)$$

Halla la sol. en serie de potencias hasta x^5 de la EDO

$$y'' + 2x\,y' + 3y = \text{sen}(3x)$$

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} a_n n\cdot x^{n-1}, \quad y''=\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

$$\text{sen}(3x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}3^{2n+1}x^{2n+1}$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + 2\sum_{n=1}^{\infty} a_n n x^n + 3\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}3^{2n+1}\cdot x^{2n+1}$$

$$(a_2\cdot 2\cdot 1 + 3a_0) + \sum_{n=1}^{\infty}\underbrace{\left(a_{n+2}(n+2)(n+1) + 2n\,a_n + 3a_n\right)}_{(n+2)(n+1)a_{n+2}+(2n+3)a_n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}3^{2n+1}\cdot x^{2n+1}$$

$$(2a_2+3a_0) + \sum_{n=0}^{\infty}\left(((2n+1)+2)((2n+1)+1)\,a_{2n+1+2} + (2(2n+1)+3)\,a_{2n+1}\right)x^{2n+1}$$

$$+\sum_{n=1}^{\infty}\left(\left((2n+2)(2n+1)\right)a_{2n+2} + \left(2\cdot(2n)+3\right)a_{2n}\right)x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}3^{2n+1}x^{2n+1}$$

$$2a_2+3a_0=0 \rightarrow \boxed{a_2=-\frac{3}{2}a_0}$$

$$(2n+2)(2n+1)\cdot a_{2(n+1)} + (4n+3)\,a_{2n} = 0 \rightarrow a_{2(n+1)} = -\frac{4n+3}{(2n+2)(2n+1)}a_{2n}$$

$$\boxed{a_4 = -\frac{4+3}{4\cdot 3}a_2 = } = \frac{7}{12}\cdot\frac{3}{2} = \boxed{\frac{7}{12}a_0}$$

$$(2n+3)(2n+2)a_{2n+3} + (4n+5)a_{2n+1} = \frac{(-1)^n}{(2n+1)!}3^{2n+1}$$

$n=0 \rightarrow 3\cdot 2\cdot a_3 + 5a_1 = 3 \rightarrow \boxed{a_3 = \frac{1}{2} - \frac{5}{6}a_1}$

$n=1 \rightarrow 5\cdot 4\,a_5 + 9\,a_3 = -\frac{1}{3!}\cdot 3^3 = -\frac{9}{2} \rightarrow a_5 = \frac{1}{20}\left(-\frac{9}{2} - 9a_3\right) = \frac{-9}{20}\left(\frac{1}{2}+\frac{1}{2}-\frac{5}{6}a_1\right) = \frac{-9}{20}+\frac{3}{8}a_1$

(4-) Resuelve : a) $y'' + 2xy' - y = 0$, $\quad y(0)=1,\ y'(0)=0.$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2\sum_{n=1}^{\infty} n\, a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(2a_2 - a_0) + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + 2n\,a_n - a_n\right]x^n = 0$$

$$a_2 = \frac{1}{2}a_0$$

$$\boxed{a_{n+2} = \frac{1-2n}{(n+2)(n+1)}a_n}$$

$$n=1 \rightarrow a_3 = \frac{-1}{3\cdot 2}a_1 = \frac{-1}{6}a_1$$

$$n=2 \rightarrow a_4 = \frac{1-4}{4\cdot 3}a_2 = \frac{-1}{4}\cdot\frac{1}{2}a_0 = \frac{-1}{8}a_0$$

$$n=3 \rightarrow a_5 = \frac{1-6}{5\cdot 4}a_3 = \frac{-5}{5\cdot 4}\cdot\frac{-1}{6}a_1 = \frac{1}{24}a_1$$

$$y = a_0\left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{24}x^5\right) + O(x^6)$$

$$\left.\begin{array}{l} y(0)=1 \Rightarrow a_0 = 1 \\ y'(0)=0 \Rightarrow a_1 = 0 \end{array}\right\} \Rightarrow \boxed{y = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + O(x^6)}$$

(4.) Resuelve: $y'' + e^x y' + (1-x^2)y = 0 \qquad y(0)=1,\ y'(0)=0$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad \sum_{n=0}^{\infty} A_n \sum_{n=0}^{\infty} B_n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} A_k B_{n-k} \right) = \sum_{n=0}^{\infty} C_n$$

$$\sum_{n=0} (n+2)(n+1) a_{n+2} x^n + \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} x^n \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n}_{\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (k+1) a_{k+1} \cdot \frac{1}{(n-k)!} \right) x^n} + \sum_{n=0}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Big(2a_2 + \underbrace{\sum_{k=0}^{0} \frac{(k+1)}{(0-k)} a_{k+1}}_{a_1} + a_0\Big) x^0 + \Big(3 \cdot 2 a_3 + \underbrace{\sum_{k=0}^{1} \frac{(k+1)}{(1-k)} a_{k+1}}_{a_1 + 2a_2} + a_1\Big) x^1$$

$$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} \frac{k+1}{(n-k)!} a_{k+1} + a_n - a_{n-2} \right] x^n = 0$$

$n=0 \rightarrow 2a_2 + a_1 + a_0 = 0$

$n=1 \rightarrow 6a_3 + 2a_2 + 2a_1 = 0$

$n=2 \rightarrow 4 \cdot 3 a_4 + \left(\frac{1}{2!} a_1 + \frac{2}{1!} a_2 + \frac{3}{0!} a_3 \right) + a_2 - a_0 \big) = 0$

$12 a_4 + 3a_3 + 3a_2 + \frac{1}{2} a_1 - a_0 = 0$

$\left. \begin{array}{l} y(0) = 1 \rightarrow a_0 = 1 \\ y'(0) = 0 \rightarrow a_1 = 0 \end{array} \right\}$

$n=0 \rightarrow a_2 = \frac{-1}{2} a_0$

$n=1 \rightarrow a_3 = \frac{-1}{3} a_2 = \frac{1}{6} a_0$

$n=2 \rightarrow 12 a_4 + \frac{3}{6} a_0 + \frac{3}{2} a_0 - a_0 = 0 \rightarrow a_4 = \frac{1}{6} a_0$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$$\boxed{y = 1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + O(x^5)}$$

5.- Hallar la sol. en serie de pot. alrededor de $x_0=3$ hasta $(x-x_3)^4$ de

$$y'' + 2xy = 0$$

$$t = x-3 \rightarrow x = t+3$$

$$D^2y + 2(t+3)y = 0$$

$$y = \sum_{n=0}^{\infty} a_n t^n \qquad y'' = \sum_{n=2}^{\infty} a_n n (n-1) t^{n-2}$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) t^n + 2\sum_{n=0}^{\infty} a_n t^{n+1} + 6\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) + 6a_n\right) t_n + \sum_{n=1}^{\infty} 2a_{n-1} t^n = 0$$

$$2a_2 + 6a_0 + \sum_{n=1}^{\infty} \left(a_{n+2}(n+2)(n+1) + 6a_n + 2a_{n-1}\right) t^n = 0$$

$$a_2 = -3a_0$$

$$a_{n+2} = -\frac{6a_n + 2a_{n-1}}{(n+2)(n+1)} \quad \Bigg| \quad \begin{array}{l} n=1 \rightarrow a_3 = -\dfrac{6a_1 + 2a_0}{6} = -a_1 - \dfrac{1}{3}a_0 \\ n=2 \rightarrow a_4 = -\dfrac{6a_2 + 2a_1}{6} = -\dfrac{-18a_0 + 2a_1}{6} = -\dfrac{1}{3}a_1 + 3a_0 \end{array}$$

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$$

$$= a_0 + a_1 t - 3a_0 t^2 - \left(a_1 + \tfrac{1}{3}a_0\right) t^3 + \left(-\tfrac{1}{3}a_1 + 3a_0\right) t^4$$

$$y = a_0\left(1 - 3(x-3)^2 - \tfrac{1}{3}(x-3)^3 + 3(x-3)^4\right) + a_1\left((x-3) - (x-3)^3 - \tfrac{1}{3}(x-3)^4\right)$$

4) Halla la relación de recurrencia de los coefs. de la sol en serie de potencias de x de

$$(1-x^2)y'' - 2xy' + 6y = 0, \qquad -1<x<1$$

y halla los 5 primeros términos de la solución

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} a_n n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n \cdot n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^n - 2\sum_{n=1}^{\infty} a_n n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^n - 2\sum_{n=1}^{\infty} a_n n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$(a_2 \cdot 2 + 6a_0) + (a_3 \cdot 6 - 2a_1 + 6a_1)x +$$

$$+\sum_{n=2}^{\infty}\left[a_{n+2}(n+2)(n+1) \underbrace{- a_n \cdot n(n-1) - 2na_n + 6a_n}_{(-n^2-n+6)a_n}\right]x^n = 0$$

$$2a_2 + 6a_0 = 0 \rightarrow a_2 = -3a_0$$

$$6a_3 + 4a_1 = 0 \rightarrow a_3 = \frac{-2}{3}a_1$$

$$a_{n+2}(n+2)(n+1) - (n^2+n-6)a_n = 0 \rightarrow \boxed{a_{n+2} = \frac{n^2+n-6}{(n+2)(n+1)}a_n}$$

$$a_4 = \frac{4+2-6}{4\cdot 3}a_2 = 0$$

$$\boxed{y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = a_0 + a_1x - \frac{1}{3}a_0x^2 - \frac{2}{3}a_1x^3 = a_0(1-3x^2) + a_1\left(x - \frac{2}{3}x^3\right)}$$

Sabemos que $P_k(x)$ (pol. de Legendre de orden k) es sol. particular de la ED

$$(1-x^2)y'' - 2xy' + k(k+1)y = 0$$

Escribe la sol. general en términos de $P_k(x)$

$y_0 = P_k(x)$

$p = 1-x^2$

$q = -2x$

$$y_0(x) = P_k(x) \Rightarrow y = y_0 \cdot Z$$

$$Z'' + \left(2\frac{P_k'}{P_k} + \frac{-2x}{1-x^2}\right)Z' = 0$$

$$Z' = C_1 \exp\left(-\int\left(2\frac{P_k'}{P_k} + \frac{-2x}{1-x^2}\right)dx\right) = C_1 e^{-(\ln P_k^2 + \ln(1-x^2))} = \frac{C_1}{(1-x^2)P_k^2(x)}$$

$$Z = C_1 \int \frac{dx}{(1-x^2)P_k^2(x)} + C_2$$

$$\boxed{y = C_2 P_k(x) + C_1 P_k(x) \int \frac{dx}{(1-x^2)P_k^2(x)}}$$

Halla la rel. de recurr. de los coefs. de la sol. en serie de potencias de x de

$$\cancel{(1-x^2)}\; y'' - 2xy' + 6y = 0$$

y halla los 5 primeros términos de la sol.

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 2\sum_{n=1}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$(2a_2 + 6a_0) + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (-2n+6)a_n\right]x^n = 0$$

$$a_2 = -3a_0$$

$$\boxed{a_{2n+2} = \frac{6-2n}{(n+2)(n+1)}\, a_n}$$ rel. recurr.

$$n=1 \to a_3 = \frac{6-2}{3\cdot 2}a_1 = \frac{2}{3}a_1$$

$$n=2 \to a_4 = \frac{6-4}{4\cdot 3}a_2 = \frac{2}{4\cdot 3}(-3a_0) = -\frac{1}{2}a_0$$

$$n=3 \to a_5 = \frac{6-6}{5\cdot 4}a_3 = 0 = a_7 = a_9 = \cdots = a_{2n+1}$$

$$n=4 \to a_6 = \frac{6-8}{6\cdot 5}\cdot a_4 = \frac{-2}{6\cdot 5}\cdot\frac{-1}{2}a_0 = \frac{1}{30}a_0$$

$$y = a_0\left(1 - 3x^2 - \frac{1}{2}x^4 + \frac{1}{30}x^6 + \cdots\right) + a_1\left(x + \frac{2}{3}x^3\right)$$

$$H_3(x) = -12x + 8x^3 \Rightarrow \boxed{\begin{matrix} a_0 = 0 \\ a_1 = -12 \end{matrix}} \Rightarrow \boxed{y(x) = H_3(x)}$$

Sabemos que $H_k(x)$ (pol. de Hermite de grado k) es sol. particular de la ED

$$y'' - 2xy' + 2ky = 0$$

Escribe la sol. general en términos de $H_k(x)$

$$\begin{cases} y_0 = H_k(x) \\ p = 1 \\ q = -2x \end{cases} \Rightarrow \quad Z'' + \left(2\frac{H_k'}{H_k} + \frac{-2x}{1}\right) Z' = 0$$

luego

$$Z' = C_1 e^{-\int \left(2\frac{H_k'}{H_k} - 2x\right)dx} = C_1 e^{-2\ln H_k + x^2} = C_1 \frac{e^{x^2}}{H_k^2}$$

e integrando

$$Z = C_1 \int \frac{e^{x^2}}{H_k^2}\, dx + C_2$$

y la sol. $y = y_0 \cdot Z$ es

$$\boxed{y = C_2 H_k(x) + C_1 H_k(x) \int \frac{e^{x^2}}{H_k^2(x)}\, dx}$$

Sabemos que $L_k(x)$ (pol. de Laguerre de orden k) es sol. particular de la ED

$$x y'' + (1-x) y' + k y = 0$$

Escribe la sol. general en términos de $L_k(x)$

$$\left\lfloor \begin{array}{l} y_0 = L_k(x) \\ p = x \\ q = 1-x \end{array} \right. \Rightarrow Z'' + \left(2\frac{L_k'}{L_k} + \frac{1-x}{x}\right) Z' = 0$$

$$Z' = C_1 e^{-\int\left(2\frac{L_k'}{L_k} + \frac{1-x}{x}\right)dx} = C_1 e^{-2\ln(L_k) - \ln x + x} = C_1 \frac{e^x}{x \cdot L_k^2(x)}$$

$$Z = C_1 \int \frac{e^x}{x L_k^2(x)} dx + C_2$$

$$\boxed{y = C_2 L_k(x) + C_1 L_k \int \frac{e^x}{x L_k^2(x)} dx}$$

▶*Find the power series solutions about $z = 0$ of*

$$4zy'' + 2y' + y = 0.$$

Dividing through by $4z$ to put the equation into standard form, we obtain

$$y'' + \frac{1}{2z}y' + \frac{1}{4z}y = 0. \tag{16.17}$$

and on comparing with (16.7) we identify $p(z) = 1/(2z)$ and $q(z) = 1/(4z)$. Clearly $z = 0$ is a singular point of (16.17), but since $zp(z) = 1/2$ and $z^2q(z) = z/4$ are finite there, it is a regular singular point. We therefore substitute the Frobenius series $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$ into (16.17). Using (16.13) and (16.14), we obtain

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} + \frac{1}{2z}\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1} + \frac{1}{4z}\sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0,$$

which, on dividing through by $z^{\sigma-2}$, gives

$$\sum_{n=0}^{\infty}\left[(n+\sigma)(n+\sigma-1) + \tfrac{1}{2}(n+\sigma) + \tfrac{1}{4}z\right] a_n z^n = 0. \tag{16.18}$$

If we set $z = 0$ then all terms in the sum with $n > 0$ vanish, and we obtain the indicial equation

$$\sigma(\sigma - 1) + \tfrac{1}{2}\sigma = 0,$$

which has roots $\sigma = 1/2$ and $\sigma = 0$. Since these roots do not differ by an integer, we expect to find two independent solutions to (16.17), in the form of Frobenius series.

Demanding that the coefficients of z^n vanish separately in (16.18), we obtain the recurrence relation

$$(n+\sigma)(n+\sigma-1)a_n + \tfrac{1}{2}(n+\sigma)a_n + \tfrac{1}{4}a_{n-1} = 0. \tag{16.19}$$

If we choose the larger root, $\sigma = 1/2$, of the indicial equation then (16.19) becomes

$$(4n^2 + 2n)a_n + a_{n-1} = 0 \quad \Rightarrow \quad a_n = \frac{-a_{n-1}}{2n(2n+1)}.$$

Setting $a_0 = 1$, we find $a_n = (-1)^n/(2n+1)!$, and so the solution to (16.17) is given by

$$\begin{aligned} y_1(z) &= \sqrt{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^n \\ &= \sqrt{z} - \frac{(\sqrt{z})^3}{3!} + \frac{(\sqrt{z})^5}{5!} - \cdots = \sin\sqrt{z}. \end{aligned}$$

To obtain the second solution we set $\sigma = 0$ (the smaller root of the indicial equation) in (16.19), which gives

$$(4n^2 - 2n)a_n + a_{n-1} = 0 \quad \Rightarrow \quad a_n = -\frac{a_{n-1}}{2n(2n-1)}.$$

Setting $a_0 = 1$ now gives $a_n = (-1)^n/(2n)!$, and so the second (independent) solution to (16.17) is

$$y_2(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^n = 1 - \frac{(\sqrt{z})^2}{2!} + \frac{(\sqrt{4})^4}{4!} - \cdots = \cos\sqrt{z}.$$

PROBLEMAS RESUELTOS

1. Resolver por series $2x^2y'' - xy' + (x^2+1)y = 0$.

Sustituyendo

$$y = A_0x^m + A_1x^{m+1} + A_2x^{m+2} + \cdots\cdots + A_nx^{m+n} + \cdots$$

$$y' = mA_0x^{m-1} + (m+1)A_1x^m + (m+2)A_2x^{m+1} + \cdots\cdots + (m+n)A_nx^{m+n-1} + \cdots$$

$$y'' = (m-1)mA_0x^{m-2} + (m+1)mA_1x^{m-1} + (m+1)(m+2)A_2x^m + \cdots + (m+n-1)(m+n)A_nx^{m+n-2} + \cdots$$

en la ecuación diferencial dada se obtiene

$$(m-1)(2m-1)A_0x^m + m(2m+1)A_1x^{m+1} + \{[(m+2)(2m+1)+1]A_2 + A_0\}x^{m+2} + \cdots$$
$$+ \{[(m+n)(2m+2n-3)+1]A_n + A_{n-2}\}x^{m+n} + \cdots\cdots = 0.$$

Ahora bien, todos los términos, excepto los dos primeros, se anulan si $A_2, A_3, \cdots$ satisfacen la fórmula de recurrencia

1) $$A_n = -\frac{1}{(m+n)(2m+2n-3)+1}A_{n-2}, \quad n \geq 2.$$

Las raíces de la ecuación determinante, $(m-1)(2m-1) = 0$, son $m = \frac{1}{2}, 1$, y para cada uno de los dos valores se anula el primer término. Sin embargo, como para ninguno de estos valores se anula el segundo término, se tomará $A_1 = 0$. Mediante 1) se deduce que $A_1 = A_3 = A_5 = \cdots = 0$. Se tiene, pues,

$$\bar{y} = A_0x^m\left(1 - \frac{1}{(m+2)(2m+1)+1}x^2 + \frac{1}{[(m+2)(2m+1)+1][(m+4)(2m+5)+1]}x^4 - \cdots\cdots\right)$$

satisface $$2x^2\bar{y}'' - x\bar{y}' + (x^2+1)\bar{y} = (m-1)(2m-1)A_0x^m$$

y el miembro de la derecha se anulará cuando $m = \frac{1}{2}$ o bien $m = 1$.

Cuando $m = \frac{1}{2}$ y $A_0 = 1$ se tiene $y_1 = \sqrt{x}(1 - x^2/6 + x^4/168 - x^6/11088 + \cdots\cdots)$

y cuando $m = 1$, con $A_0 = 1$, se tiene $y_2 = x(1 - x^2/10 + x^4/360 - x^6/28080 + \cdots\cdots)$.

Luego la solución completa es

$$y = Ay_1 + By_2$$
$$= A\sqrt{x}(1 - x^2/6 + x^4/168 - x^6/11088 + \cdots) + Bx(1 - x^2/10 + x^4/360 - x^6/28080 + \cdots).$$

Como $x = 0$ es el único punto singular finito, la serie converge para todos los valores finitos de x.

2. Resolver por series $3xy'' + 2y' + x^2y = 0$.

Sustituyendo y, y' e y'' como en el problema anterior se tiene

$$m(3m-1)A_0x^{m-1} + (m+1)(3m+2)A_1x^m + (m+2)(3m+5)A_2x^{m+1} + [(m+3)(3m+8)A_3 + A_0]x^{m+2}$$
$$+ \cdots\cdots + [(m+n)(3m+3n-1)A_n + A_{n-3}]x^{m+n-1} + \cdots\cdots = 0.$$

Todos los términos a partir del tercero se anularán si $A_3, A_4, \cdots$ satisfacen la fórmula de recurrencia

$$A_n = -\frac{1}{(m+n)(3m+3n-1)}A_{n-3}, \quad n \geq 3.$$

Las raíces de la ecuación determinante $m(3m-1)=0$ son $m=0,\ 1/3$. Como con ninguna se anulan los términos segundo y tercero se tomará $A_1 = A_2 = 0$. Entonces, utilizando la fórmula de recurrencia, $A_1 = A_4 = A_7 = \cdots = 0$ y $A_2 = A_5 = A_8 = \cdots = 0$. Luego la serie

1) $$\bar{y} = A_0 x^m \left(1 - \frac{1}{(m+3)(3m+8)} x^3 + \frac{1}{(m+3)(m+6)(3m+8)(3m+17)} x^6 - \cdots\cdots\right)$$

satisface $$3x\bar{y}'' + 2\bar{y}' + x^2\bar{y} = m(3m-1)A_0 x^{m-1}.$$

Para $m=0$, con $A_0 = 1$, se obtiene de 1) $y_1 = 1 - x^3/24 + x^6/2448 - \cdots\cdots$

y para $m = 1/3$, con $A_0 = 1$, se obtiene $y_2 = x^{1/3}(1 - x^3/30 + x^6/3420 - \cdots\cdots)$.

La solución completa es

$$y = Ay_1 + By_2 = A(1 - x^3/24 + x^6/2448 - \cdots\cdots) + Bx^{1/3}(1 - x^3/30 + x^6/3420 - \cdots\cdots).$$

La serie converge para todos los valores finitos de x.

RAICES IGUALES DE LA ECUACION DETERMINANTE.

3. Resolver por series $xy'' + y' - y = 0$.

Sustituyendo y, y' e y'' como en los anteriores Problemas 1 y 2 se obtiene

$$m^2A_0x^{m-1} + [(m+1)^2A_1 - A_0]x^m + [(m+2)^2A_2 - A_1]x^{m+1} + \cdots\cdots + [(m+n)^2A_n - A_{n-1}]x^{m+n-1} + \cdots\cdots = 0.$$

Todos los términos, excepto el primero, se anulan si $A_1, A_2, \cdots$ satisfacen la fórmula de recurrencia

1) $$A_n = \frac{1}{(m+n)^2} A_{n-1}, \qquad n \geqq 1.$$

Así,

$$\bar{y} = A_0x^m\left(1 + \frac{1}{(m+1)^2}x + \frac{1}{(m+1)^2(m+2)^2}x^2 + \frac{1}{(m+1)^2(m+2)^2(m+3)^2}x^3 + \cdots\cdots\right)$$

satisface

2) $$x\bar{y}'' + \bar{y}' - \bar{y} = m^2A_0x^{m-1}.$$

Las raíces de la ecuación determinante son $m = 0, 0$. Por tanto, corresponde solamente una solución en serie que satisfaga 2) con $m=0$. Sin embargo, considerando a $\bar{y}$ como una función de las variables independientes x y m,

$$\frac{\partial \bar{y}'}{\partial m} = \frac{\partial}{\partial m}\left(\frac{\partial \bar{y}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \bar{y}}{\partial m}\right) = \left(\frac{\partial \bar{y}}{\partial m}\right)'$$

y $$\frac{\partial \bar{y}''}{\partial m} = \frac{\partial}{\partial m}\frac{\partial}{\partial x}\left(\frac{\partial \bar{y}}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial}{\partial m}\left(\frac{\partial \bar{y}}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \bar{y}}{\partial m}\right) = \left(\frac{\partial \bar{y}}{\partial m}\right)''.$$

y derivando 2) parcialmente respecto de m, se tiene

3) $$x\left(\frac{\partial \bar{y}}{\partial m}\right)'' + \left(\frac{\partial \bar{y}}{\partial m}\right)' - \left(\frac{\partial \bar{y}}{\partial m}\right) = 2mA_0x^{m-1} + m^2A_0x^{m-1}\ln x.$$

De 2) y 3) se deduce que $y_1 = \bar{y}\big|_{m=0}$ y $y_2 = \dfrac{\partial \bar{y}}{\partial m}\Big|_{m=0}$ son soluciones de la ecuación diferencial dada.

Tomando $A_0 = 1$ se halla

20.2. Halle la solución general cerca de $x = 0$ de $2x^2y'' + 7x(x+1)y' - 3y = 0$.

En este caso $P(x) = 7(x+1)/2x$ y $Q(x) = -3/2x^2$; entonces, $x = 0$ es un punto especial regular y puede aplicarse el método de Frobenius. Sustituyendo *(1)*, *(2)* y *(3)* del Problema 20.1 en la ecuación diferencial y combinando términos, obtenemos

$$\begin{aligned} & x^{\lambda}[2\lambda(\lambda-1)a_0 + 7\lambda a_0 - 3a_0] \\ & \quad + x^{\lambda+1}[2(\lambda+1)\lambda a_1 + 7\lambda a_0 + 7(\lambda+1)a_1 - 3a_1] + \cdots \\ & \qquad + x^{\lambda+n}[2(\lambda+n)(\lambda+n-1)a_n + 7(\lambda+n-1)a_{n-1} + 7(\lambda+n)a_n - 3a_n] \\ & \qquad\quad + \cdots \\ & = 0 \end{aligned}$$

Dividiendo por x^{λ} y simplificando obtenemos

$$\begin{aligned} & (2\lambda^2 + 5\lambda - 3)a_0 + x[(2\lambda^2 + 9\lambda + 4)a_1 + 7\lambda a_0] + \cdots \\ & \quad + x^n\{[2(\lambda+n)^2 + 5(\lambda+n) - 3]a_n + 7(\lambda+n-1)a_{n-1}\} + \cdots \\ & = 0 \end{aligned}$$

Descomponiendo en factores el coeficiente de a_n e igualando cada coeficiente a cero, encontramos

$$(2\lambda^2 + 5\lambda - 3)a_0 = 0 \tag{1}$$

y, para $n \geqq 1$,

$$[2(\lambda+n) - 1][(\lambda+n) + 3]a_n + 7(\lambda+n-1)a_{n-1} = 0$$

o

$$a_n = \frac{-7(\lambda+n-1)}{[2(\lambda+n)-1][(\lambda+n)+3]}a_{n-1} \tag{2}$$

De *(1)* bien sea $a_0 = 0$ ó

$$2\lambda^2 + 5\lambda - 3 = 0 \tag{3}$$

Es conveniente mantener a_0 arbitrario; por lo tanto necesitamos λ para satisfacer la ecuación índice *(3)*. Las raíces de *(3)* son $\lambda_1 = \frac{1}{2}$ y $\lambda_2 = -3$. Como $\lambda_1 - \lambda_2 = \frac{7}{2}$, la solución se da por *(20.3)* y *(20.4)*. Sustituyendo $\lambda = \frac{1}{2}$ en *(2)* y simplificando, obtenemos

$$a_n = \frac{-7(2n-1)}{2n(2n+7)}a_{n-1} \qquad (n \geqq 1)$$

Entonces,

$$a_1 = -\frac{7}{18}a_0, \qquad a_2 = -\frac{21}{44}a_1 = \frac{147}{792}a_0, \qquad \ldots$$

y

$$y_1(x) = a_0x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots\right)$$

Sustituyendo $\lambda = -3$ en *(2)* y simplificando, obtenemos

$$a_n = \frac{-7(n-4)}{n(2n-7)}a_{n-1} \qquad (n \geqq 1)$$

Entonces $a_1 = -\frac{21}{5}a_0, \quad a_2 = -\frac{7}{3}a_1 = \frac{49}{5}a_0, \quad a_3 = -\frac{7}{3}a_2 = -\frac{343}{15}a_0, \quad a_4 = 0$

y, como $a_4 = 0$, $a_n = 0$ para $n \geqq 4$. Entonces

$$y_2(x) = a_0x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right)$$

La solución general es

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= k_1x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots\right) + k_2x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right) \end{aligned}$$

donde $k_1 = c_1a_0$ y $k_2 = c_2a_0$.