Weakly Uniformly Rotund Banach spaces

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Abstract

The dual space of a WUR Banach space is weakly K-analytic.

A Banach space is said to be weakly uniformly rotund (WUR -for short) if given sequences \((x_n)\) and \((y_n)\) in the unit sphere with \(\|x_n + y_n\| \to 2\) we should have weak-lim \(n(x_n - y_n) = 0\). This notion has become more important since Hájek proved that every WUR Banach space must be Asplund [8]. To obtain this result he uses ideas of Stegall for the equivalence between being an Asplund space and having the Radon-Nikodym property on its dual. Using this result and the Fabian-Godefroy [4] projectional resolution of the identity in the dual of an Asplund space, Fabian, Hájek and Zizler have recently showed that for a WUR Banach space \(E\) the dual space \(E^*\) is a subspace of a WCG Banach space. Indeed they proved that for a Banach space \(E\) to have an equivalent WUR norm is equivalent to the fact that the bidual unit ball \(B_{E^{**}}\), endowed with the weak-* topology, will be a uniform Eberlein compact [5]. Consequently they obtain that \(E\) must be LUR renormable, too [7]. The aim of this note is to provide a direct proof of the fact that every WUR Banach space \(E\) has a dual space \(E^*\) which is weakly K-analytic. This provides a topological approach to Hájek’s result on the Asplundness of the space \(E\) as well as the LUR renorming consequence on \(E\) after [6].

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In this paper, \(E\) will denote a Banach space, \(E^*\) its dual, \(B_E\) its closed unit ball, \(S_E\) its unit sphere.

Definition 1 A Banach space \((E, \| \cdot \|)\) is said to be uniformly Gâteaux differentiable (UGD -for short) if for every \(0 \neq x \in E\),

\[
\lim_{t \to 0} \sup_{\|y\|=1} \frac{\|y + tx\| + \|y - tx\| - 2}{t} = 0.
\]

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The following theorem is the main result of this note:

**Theorem 1** Let $E$ be a Banach space such that $E^*$ has an equivalent (not necessarily dual) UGD norm (in particular, let $E$ be WUR Banach space). Then $E^*$ is weakly K-analytic.

The proof is based on the following assertions

**Fact 1** (Šmulian, see [3, Theorem II.6.7]) The Banach space $E$ is WUR if and only if $E^*$ is UGD.

**Theorem 2** (Talagrand [9]) Let $K$ be a compact space. The following assertions are equivalent:

1. $C(K)$ is weakly K-analytic.
2. There is an increasing mapping $\sigma \to S_\sigma$ from $\mathbb{N}^\mathbb{N}$ (endowed with the product order) in the family of compact subsets of $C(K)$ endowed with the topology of pointwise convergence, such that $\bigcup\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$ separates points of $K$.

**Remark 1** In [1] the validity of the previous theorem for an arbitrary topological space is studied. In particular, for every subset $W$ of a Banach space $E$ it follows that $(W, \text{weak})$ is K-analytic if and only if $W = \bigcup\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$ and every $S_\sigma$ is weakly compact with $S_\sigma \subset S_\gamma$ whenever $\sigma \leq \gamma$ in the product order. This will be the only tool necessary here from the theory of K-analytic spaces.

**Remark 2** From Theorem 1 and [6], see also [3, p. 296], we get that every WUR Banach space admits an equivalent LUR norm.

**Remark 3** From Theorem 1 it follows Hájek’s [8] result asserting that every WUR Banach space is Asplund. Indeed, if we assume that $E$ is also separable the K-analytic structure of $(E^*, \text{weak})$ should imply that $E^*$ is separable too. Let us explain here an easy argument following ideas from [2]: Assume $(E^*, \text{weak})$ is K-analytic. Let $T$ be an usco mapping from $\mathbb{N}^\mathbb{N}$ into the set of subsets of $(E^*, \text{weak})^* = E^*$ (T can be assumed to be increasing by Remark 1). Let $P$ be the natural projection from $(E^*, \text{weak}^*) \times \mathbb{N}^\mathbb{N}$ onto $(E^*, \text{weak})$. Consider the restriction $Q$ of $P$ to $\Sigma := \{(x, \alpha) : (x, \alpha) \in E^* \times \mathbb{N}^\mathbb{N}, x \in T(\alpha)\}$. It is easy to prove that $Q$ is continuous: let $(x^*_i, \alpha_i) \to (x, \alpha) \in \Sigma$. As $\alpha_i \to \alpha$ we can find $\beta \in \mathbb{N}^\mathbb{N}$ such that $x^*_i \to x^* \in \mathbb{N}^\mathbb{N}$ such that $\alpha \leq \beta$ and $\alpha_i \leq \beta$, $\forall i$. Then $x_i \in T(\beta)$, $x \in T(\beta)$, and $x^*_i \text{ weak}^* \to x$, hence $x^*_i \text{ weak} \to x$. Therefore $E^*$ is separable too. See also theorem 2.4 in [9]. With more generality, any submetrizable topological space $X$ is analytic if and only if there is a family of compact sets $\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$ in $X$, $S_\sigma \subset S_\gamma$ whenever $\sigma \leq \gamma$ in the product order and $X = \bigcup\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$, [2, theorem 7].
Proof of Theorem 1. It is well known that $E$ admits an equivalent WUR norm. Then $E^*$ has an equivalent dual UGD norm. Then given $x^* \in S_{E^*}$ and $\epsilon > 0$, there exists $\delta_\epsilon(x^*) > 0$ such that

$$\|y^* + tx^*\| + \|y^* - tx^*\| \leq 2 + \epsilon|t|, \text{ if } |t| < \delta_\epsilon(x^*) \text{ and } y^* \in S_{E^*}.$$ 

Given a positive integer $p$ define

$$S_p(\epsilon) := \{x^* \in S_{E^*} : \delta_\epsilon(x^*) > \frac{1}{p}\}.$$ 

Obviously,

$$S_1(\epsilon) \subset S_2(\epsilon) \subset \ldots \subset S_p(\epsilon) \subset \ldots$$

and $\cup_{p=1}^\infty S_p(\epsilon) = S_{E^*}$. Let $\alpha = (a_n) \in \mathbb{N}^\mathbb{N}$. Define

$$S_\alpha := \cap_{n=1}^\infty S_{a_n}(\frac{1}{n}).$$

We have

$$S_{E^*} = \cup\{S_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\},$$

and

$$S_\alpha \subset S_\beta, \text{ whenever } \alpha = (a_n) \leq \beta = (b_n) (\text{i.e., } a_n \leq b_n, \forall n).$$

This sets will give us the K-analytic structure of $E^*$ in the weak topology.

Indeed, we have the following

Claim 1 Given $x^{**} \in B_{E^{**}}$, $\epsilon > 0$ and $\alpha = (a_n) \in \mathbb{N}^\mathbb{N}$, there is $x \in B_E$ such that

$$\langle x^{**} - x, x^* \rangle < \epsilon, \forall x^* \in S_\alpha.$$

Proof of the claim: Find $n \in \mathbb{N}$ such that $\frac{3}{n} < \epsilon$. Pick $y^* \in S_{E^*}$ such that

$$\langle x^{**}, y^* \rangle > 1 - \frac{1}{na_n}.$$ 

Find $x \in B_E$ such that

$$\langle x, y^* \rangle > 1 - \frac{1}{na_n}.$$ 

Let $x^* \in S_\alpha$. Since $x^* \in S_{a_n}(\frac{3}{n})$

$$\|y^* + \frac{1}{a_n} x^*\| + \|y^* - \frac{1}{a_n} x^*\| \leq 2 + \frac{1}{na_n}.$$ 

In particular we have

$$\langle x^{**}, y^* + \frac{1}{a_n} x^* \rangle + \langle x, y^* - \frac{1}{a_n} x^* \rangle \leq 2 + \frac{1}{na_n} \quad (1)$$

hence

$$\frac{1}{a_n} \langle x^{**} - x, x^* \rangle \leq 2 + \frac{1}{na_n} - \langle x^{**}, y^* \rangle - \langle x, y^* \rangle < \frac{3}{na_n} < \frac{\epsilon}{a_n}$$
and so
\[ \langle x^{**} - x^*, x^* \rangle < \epsilon, \forall x^* \in S_\alpha. \]
By interchanging \( x^{**} \) and \( x \) in (1), we get
\[ |\langle x^{**} - x, x^* \rangle| < \epsilon, \forall x^* \in S_\alpha. \]
and this proves the claim.

To finish the proof of the Theorem, observe that, by the Claim, each \( S_\alpha \) is weakly relatively compact since it is weak*–relatively compact. Thus, we have
\[ S_{E^*} \subset \bigcup \{ S_\alpha^{\text{weak}} : \alpha \in \mathbb{N}^\mathbb{N} \} := W \]
and \( W \) is weakly K-analytic in \( E^* \) [Theorem 2 and Remark 1].

Consider the map
\[ (W, \text{weak}) \times [0, +\infty[ \xrightarrow{\Psi} (E^*, \text{weak}) \]
given by \( \Psi(x^*, t) := t x^* \). \( \Psi \) is continuous, \([0, +\infty[ \) is a Polish space, \( (W, \text{weak}) \times [0, +\infty[ \) is K-analytic and \( \Psi(W \times [0, +\infty[) = E^* \), so \((E^*, \text{weak})\) is itself K-analytic. q.e.d.

**References**


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