On restricted weak upper semicontinuous set valued mappings and reflexivity

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Dedicated to Professor Manuel Valdivia.

Abstract

It is known that if a Banach space is quasi smooth (i.e. its duality mapping is restricted weak upper semicontinuous) then its dual has no proper closed norming subspace. Moreover if a Banach space has an equivalent norm whose duality mapping has a graph which contains the graph of a restricted weak upper semicontinuous mapping then the space is Asplund. We prove here that if a Banach space has an equivalent norm whose duality mapping has a graph which contains the graph of a restricted weak upper semicontinuous mapping then its dual has no closed proper norming subspace. We shall apply this theorem in order to give two new characterizations of reflexivity.

È noto che se uno spazio di Banach è quasi-smooth (cioè, la sua applicazione dualità è ristretta debole semicontinua superiormente), allora il suo duale non ha subspazi chiusi normanti propri. Inoltre, se uno spazio di Banach ha una norma equivalente la cui applicazione dualità ha un grafo che contiene superiormente un'applicazione ristretta debole semicontinua superiormente, allora lo spazio è Asplund. Dimostriamo che se uno spazio di Banach ha una norma equivalente la cui applicazione dualità ha un grafo che contiene quello di un'applicazione ristretta debole semicontinua superiormente, allora il suo duale non ha subspazi chiusi normanti propri. Questo teorema viene applicato per dare nuove caratterizzazioni di riflessività.

*Supported in part by DGICYT Grant PB94-0535. AMS Subject classification: 46B10, 46B20. *Keywords and phrases*: Banach space, Asplund space, duality mapping, reflexivity, upper semicontinuous set-valued mapping.

1 Introduction.

One important tool in Banach space theory is the study of several forms of the continuity of the *duality mapping*. For a Banach space X this mapping is defined by

$$\partial \| \cdot \|_X(x) = \{x^* \in S_{X^*} : \langle x, x^* \rangle = 1\} \ , \ x \in S_X,$$

where S_X denotes its unit sphere. This is a particular case of a general definition: If f is a convex continuous function defined on an open subset D of X and $x \in D$, then the *subdifferential* of f at x is defined by

$$\partial f(x) = \{x^* \in S_{X^*} : \langle y - x, x^* \rangle \leq f(y) - f(x), \ \forall y \in D\} \, .$$

Through this paper we shall denote by B_X the closed unit ball of a Banach space and by S_X its unit sphere. When there is no confusion the duality mapping will be denoted by $\partial \| \cdot \|(x)$.

A set valued mapping Φ from a topological space X into nonempty subsets of a topological vector space Y endowed with the topology τ is said to be *τ-upper semicontinuous* at $x \in X$ if given an open subset W of Y containing $\Phi(x)$ there exists an open neighbourhood U of x such that $\Phi(U) \subset W$. Following [G-M] we say that Φ is *restricted τ-upper semicontinuous* at $x \in X$ if given W a neighbourhood of $0 \in Y$ there exists an open neighbourhood U of x such that $\Phi(U) \subset \Phi(x) + W$. In this paper we shall always consider X a Banach space endowed with the norm topology.

Contreras and Payá proved that a Banach space X is an Asplund space if $\partial \| \cdot \|$ is restricted weak upper semicontinuous (they called a Banach space with this property *quasi-smooth* [C-P]). There they proved also several characterizations of reflexivity; see also [F-P] and [G-G-S] for the study of the restricted norm upper semicontinuity of the duality mapping. Giles and Moors [G-M] proved a similar result under a (formally) more general condition: A Banach space X is an Asplund space if it has an equivalent norm whose duality mapping has a graph which contains the graph of a restricted weak upper semicontinuous mapping.

In this paper this weakened condition is used to prove that in such a case the dual of a Banach space has no closed proper norming subspace. A closed subspace N of X^* is *norming* if

$$\|x\| = \sup \{ |\langle x, x^* \rangle| : x^* \in B_N \} \, .$$

By the Hahn-Banach Theorem, a closed subspace N of X^* is norming if and only if B_N is w*-dense in B_{X^*}. Observe that the property that for some equivalent norm the dual contains no proper norming subspace and the property of being Asplund are independent, as is shown in [J-M]. We shall apply this result to give two new characterizations of reflexivity, improving the aforesaid results of Payá and Contreras.

2 A sufficient condition for the dual of a Banach space containing no closed proper norming subspace.

First of all we will prove the separable case. We need the following results:

THEOREM 2.1 ([G-M] Theorem 4.3) *A Banach space is an Asplund space if it has an equivalent norm whose duality mapping has a graph which contains the graph of a restricted weak upper semicontinuous mapping.*

The following lemma is a slight improvement of a lemma of Godefroy.

LEMMA 2.1 ([C-P] Lemma 2.2) *Let B be a boundary (i.e. a subset of S_{X^*} with $B\cap\partial\|\cdot\|(x)\neq\emptyset$ for all $x\in X$) such that $B\subset\overline{\text{co}}^{\|\cdot\|}(F+\alpha B_{X^*})$ for some countable subset $F\subset X^*$ and some $0\leq\alpha<1$. Then $\text{lin}(F)$ is norm-dense in X^*.*

The following theorem is the separable case of Theorem 2.3 below.

THEOREM 2.2 (Separable case) *Let X be a separable Banach space such that there exists $\Phi: S_X\to\mathcal{P}(X^*)$ restricted weak upper semicontinuous with the property $\Phi(x)\subset\partial\|\cdot\|(x), \forall\, x\in X$. Then X^* has no closed proper norming subspace.*

PROOF : By Theorem 2.1 X is an Asplund space, so there exists $(x_n)_{n=1}^{\infty}$ dense in S_X such that $\|\cdot\|$ is Fréchet differentiable at each x_n. Let

$$\{x_n^*\}=\Phi(x_n)=\partial\|\cdot\|(x_n).$$

Let N be a norming subspace of X^*. Since B_N is w*-dense in B_{X^*} and B_{X^*} is metrizable in the w*-topology, there exists $\left(x_{n,m}^*\right)_{m=1}^{\infty}\subset B_N$ such that

$$w^*-\lim_{m\to\infty}x_{n,m}^*=x_n^*\ ,\ \forall n\in\mathbb{N}.$$

Let $F=\left\{x_{n,m}^*: n,m\in\mathbb{N}\right\}$, $A=\overline{\text{co}}^{\|\cdot\|}\left(F+\frac{1}{2}B_{X^*}\right)$ and $B=A\cap S_{X^*}$.

It is easy to prove that $B \neq \emptyset$. If B were not a boundary, there would exist $x \in S_X$ such that $B \cap \partial\|\cdot\|(x) = \emptyset$, that is $A \cap \partial\|\cdot\|(x) = \emptyset$. Since A is convex with nonempty $\|\cdot\|$-interior and $\partial\|\cdot\|(x)$ is convex, there exists $z^{**} \in S_{X^{**}}$ such that

$$\sup\left\{\langle z^{**}, a^*\rangle : a^* \in A\right\} \leq \inf\left\{\langle z^{**}, x^*\rangle : x^* \in \partial\|\cdot\|(x)\right\}.$$

We claim that

$$f^* \in F \;\Rightarrow\; \frac{1}{2} + \langle z^{**}, f^*\rangle \leq \inf\left\{\langle z^{**}, x^*\rangle : x^* \in \partial\|\cdot\|(x)\right\}.$$

Let $\eta > 0$, $f^* \in F$. There exists $z^* \in B_{X^*}$ such that $1 - \eta < \langle z^{**}, z^*\rangle$. Since $f^* + \frac{1}{2}z^* \in A$, we have

$$\inf\left\{\langle z^{**}, x^*\rangle : x^* \in \partial\|\cdot\|(x)\right\} \geq \langle z^{**}, f^* + \frac{1}{2}z^*\rangle =$$
$$= \langle z^{**}, f^*\rangle + \frac{1}{2}\langle z^{**}, z^*\rangle > \langle z^{**}, f^*\rangle + \frac{1}{2}(1 - \eta),$$

and it is enough to let $\eta \to 0+$ in order to prove the claim.

Let $W := \left\{y^* \in X^* : |\langle z^{**}, y^*\rangle| < \frac{1}{2}\right\}$, a weak neighbourhood of 0 in X^*. By the claim we have $F \cap (\partial\|\cdot\|(x) + W) = \emptyset$.

Since Φ is restricted weak upper semicontinuous at x, there exists $\epsilon > 0$ such that $\Phi(B(x, \epsilon)) \subset \Phi(x) + \frac{1}{2}W$.

As (x_n) is dense in S_X, there exists $n \in \mathbb{N}$ such that $\|x_n - x\| < \epsilon$. Since $\|\cdot\|$ is Fréchet differentiable at x_n and $\lim_{m\to\infty}\langle x_n, x^*_{n,m}\rangle = \langle x_n, x^*_n\rangle = 1$, then by the Šmulyan characterization of Fréchet differentiability we get $\lim_{m\to\infty}\|x^*_{n,m} - x^*_n\| = 0$. It follows that there exists $f^* \in F$ such that $\|f^* - x^*_n\| < \frac{1}{4}$, hence $\langle z^{**}, f^* - x^*_n\rangle \leq \|f^* - x^*_n\| < \frac{1}{4}$, so $f^* \in x^*_n + \frac{1}{2}W$.

Since $\|x_n - x\| < \epsilon$ and $\|\cdot\|$ is Fréchet differentiable at x_n

$$\{x^*_n\} = \partial\|\cdot\|(x_n) = \Phi(x_n) \subset \Phi(B(x, \epsilon)) \subset \Phi(x) + \frac{1}{2}W.$$

It follows that $f^* \in \Phi(x) + \frac{1}{2}W + \frac{1}{2}W \subset \Phi(x) + W$. This contradicts $F \cap (\partial\|\cdot\|(x) + W) = \emptyset$. Hence B is a boundary. By Lemma 2.1 $\mathrm{lin}(F)$ is norm-dense in X^*. Since $F \subset B_N$ we have $\mathrm{lin}(F) \subset N$ and because N is $\|\cdot\|$-closed, we get $N = X^*$. ■

We need the following simple lemma in order to prove the general case:

LEMMA 2.2 *Let $\Phi : S_X \to \mathcal{P}(X^*)$ be a restricted weak upper semicontinuous mapping such that $\Phi(x) \subset \partial\|\cdot\|_X(x)$ for all $x \in X$. Let Y be a closed subspace of X. Define $\Psi : S_Y \to \mathcal{P}(Y^*)$ as $\Psi(y) = \{x^*|_Y : x^* \in \Phi(y)\}$. Then*

i) $\Psi(y) \subset \partial\|\cdot\|_Y(y)$ for all $y \in Y$.

ii) Ψ is restricted weak upper semicontinuous.

PROOF : i) Let $x^* \in \Phi(y) \subset \partial\|\cdot\|_X(y)$. Then $\langle y, x^*|_Y\rangle = \langle y, x^*\rangle = 1$. Moreover,

$$1 = \|x^*\| \geq \|x^*|_Y\| \geq \langle y, x^*|_Y\rangle = 1.$$

This implies $x^*|_Y \in S_{Y^*}$.

ii) Let $y \in Y$ and W a weak neighbourhood of 0 in Y^*. Since $x^* \mapsto x^*|_Y$ is continuous for the weak topologies there exists V, a weak neighbourhood of 0 in X^*, such that $\{x^*|_Y : x^* \in V\} \subset W$. Since Φ is restricted weak upper semicontinuous, there exists $\delta > 0$ such that $\Phi(B(y,\delta)) \subset \Phi(y) + V$. Now it is easy to prove that $\Psi(B(y,\delta)) \subset \Psi(y) + W$. ■

We will use several results of Godefroy and Kalton relative to the ball topology. Given a Banach space we define the *ball topology* b_X as the coarsest topology in X so that every closed ball is b_X-closed (see, for instance, [G-K] and [D-B]).

LEMMA 2.3 ([G-K] Theorem 2.4 and Proposition 2.5) *Let X be a Banach space and let $x^* \in X^*$. Then*

i) If $x^*|_{B_X}$ is b_X-continuous then x^* belongs to all norming subspaces.

ii) If x^* belongs to all norming subspaces and X is separable, then $x^*|_{B_X}$ is b_X-continuous.

iii) If for all separable closed subspaces Y of X, $x^*|_{B_Y}$ is b_Y-continuous, then $x^*|_{B_X}$ is b_X-continuous.

This lemma says, in particular, that the property "the dual of a Banach space contains no proper norming subspace" is separably determined: Let X a Banach space such that for every closed separable subspace, its dual has no proper norming subspaces. Let N be a proper norming subspace of X^*. Given $x^* \in X^* \setminus N$, $x^*|_{B_X}$ is not b_X-continuous, hence there exists a closed separable subspace Y of X such that $x^*|_{B_Y}$ is not b_Y-continuous. It follows that Y^* contains a proper norming subspace, a contradiction.

THEOREM 2.3 *Let X be a Banach space such there exists $\Phi : S_X \to \mathcal{P}(X^*)$ restricted weak upper semicontinuous such that $\Phi(x) \subset \partial\|\cdot\|(x)$ $\forall x \in S_X$. Then X^* has no closed proper norming subspace.*

Proof : Let Y be a closed separable subspace of X. By Lemma 2.2 we get that there exists a restricted weak upper semicontinuous mapping $\Psi : S_Y \to \mathcal{P}(Y^*)$, such that $\Psi(y) \subset \partial\| \cdot \|_Y(y)$. By Theorem 2.2, Y^* has no proper closed norming subspace. By Lemma 2.3, X^* has no proper closed norming subspace either. ■

3 Two characterizations of reflexivity.

We shall use the following well known fact: If f is a convex continuous function defined on an open convex subset of a Banach space X then ∂f is always upper semicontinuous for the w*-topology.

A Banach space X has the *finite-infinite intersection property* ($IP_{f,\infty}$) if for every collection of closed balls in X with empty intersection there is a finite subcollection with empty intersection. It is easy to prove that if X is reflexive then has the property $IP_{f,\infty}$: Let $\{B_\alpha\}_{\alpha\in I}$ be a collection of balls such that for every finite $F \subset I$, $\cap_{\alpha\in F} B_\alpha \neq \emptyset$. By w*-compactness there exists $x^{**} \in \cap_{\alpha\in I} \overline{B_\alpha}^{w^*}$. Since X is reflexive we get $\cap_{\alpha\in I} B_\alpha \neq \emptyset$.

The following lemma is due to Godefroy [G]:

Lemma 3.1 *Let X be a Banach space which has the property $IP_{f,\infty}$ and X^* has no closed proper norming subspace. Then X is reflexive.*

Now we are ready to prove the first characterization of reflexivity:

Theorem 3.1 *Let X be a Banach space. Then X is reflexive if and only if X has the property $IP_{f,\infty}$ and there exists a restricted weak upper semicontinuous mapping $\Phi : S_X \to \mathcal{P}(X^*)$ such that $\Phi(x) \subset \partial\| \cdot \|(x)$ $\forall x \in X$.*

Proof : If X is reflexive, then X has the property $IP_{f,\infty}$ and $\partial\| \cdot \|(x)$ is weak upper semicontinuous $\forall x \in S_X$.

The converse follows from Lemma 3.1 and Theorem 2.3. ■

Now the following fact is obvious:

Corollary 3.1 *Let X be a Banach space with the property $IP_{f,\infty}$. Then the following statements are equivalent:*

i) X is reflexive.

ii) X is quasi smooth.

iii) There exists $\Phi : S_X \to \mathcal{P}(X^*)$ restricted weak upper semicontinuous such that $\Phi(x) \subset \partial\| \cdot \|(x)$ $\forall x \in X$.

In order to prove the second characterization of reflexivity we will use the following lemma due also to Godefroy and Kalton.

LEMMA 3.2 ([G-K] Theorem 8.2) *Let X be a Banach space and W a bounded subset of X such that for every equivalent norm in X, W is closed in the respective ball topology. Then W is w-compact.*

THEOREM 3.2 *A Banach space X is reflexive if and only if for all equivalent norm $\|\cdot\|$ in X there exists $\Phi : S_X \to \mathcal{P}(X^*)$ restricted weak upper semicontinuous such that $\Phi(x) \subset \partial\|\cdot\|(x)$ $\forall x \in X$.*

PROOF : The only if part is obvious. To prove the converse, assume first that X is separable. By Lemma 3.2 it is enough to prove that B_X is b_X-closed for all equivalent norm in X. Let $\|\cdot\|$ be an equivalent norm in X. By hypothesis there exists $\Phi : S_X \to \mathcal{P}(X^*)$ restricted weak upper semicontinuous such that $\Phi(x) \subset \partial\|\cdot\|(x)$ $\forall x \in X$. Hence X^* has no proper closed norming subspace, so, by Lemma 2.3, $x^*|_{B_X}$ is b_X-continuous for all $x^* \in X^*$.

Let $(x_n)_{n=1}^{\infty} \subset B_X$, such that $x_n \to x$ in the b_X-topology. It follows that $x_n \to x$ in the w-topology, hence $x \in B_X$ so B_X is b_X-closed for all equivalent norm, and the theorem is proved in the separable case.

In the general case, by Eberlein's Theorem, it is enough to prove that if Y is a closed separable subspace then Y is reflexive. Let $\|\cdot\|_Y$ an equivalent norm in Y. Let $\|\cdot\|$ be an equivalent norm in X which extends $\|\cdot\|_Y$. By Lemma 2.2 there exists a restricted weak upper semicontinuous mapping $\Psi : S_Y \to \mathcal{P}(Y^*)$ such that $\Psi(y) \subset \partial\|\cdot\|_Y(y)$, and by the separable case Y is reflexive. ∎

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