Cantor sets in the dual of a separable Banach space. Applications

M. López-Pellicer* V. Montesinos†‡

Abstract

This survey collects several (classical and more recent) results relating some structural properties of a Banach space X (Asplundness, containing a copy of ℓ_1,...) with the existence of a subset of the unit ball B_{X^*} of the dual that, in the weak star topology, is homeomorphic to the ternary Cantor set. The possibility of finding this set inside the extreme points of B_{X^*} is also considered. Some applications are described.

1 Introduction.

A fundamental result of Ch. Stegall (see [ST75]), says that if X is a Banach space, then X^* has the Radon-Nikodym property (i.e., X is an Asplund space) if and only if every separable subspace of X has a separable dual. A basic tool in his proof was the construction of subsets of X^* which in the weak star topology were homeomorphic to Cantor sets. Originated with this pioneering work, a well-established theory relating some structural properties of a Banach space and the existence in the dual of sets which behave in a dyadic fashion is now of common use in Banach Space Theory (see the references along the following pages).

In this survey we attempt to provide a unified approach to some well-known results and some (maybe not so well-known) others of this type, as well as to some old and recent applications. Proofs are included for the sake of completeness.

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2 Definitions.

By a Banach space we shall understand a real Banach space. B_X will denote its closed unit ball, S_X its unit sphere, X^* its dual space.

Along this note we denote by T the set $\{(n,i) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq 2^n, n \geq 0\}$ and we reserve the symbol Δ (sometime decorated with some extra sub or superscripts) for the topological space $2^{\mathbb{N}}$ endowed with the product topology, the *Cantor ternary set* or, briefly, the *Cantor set.* Δ is a metrizable compact space. Every metrizable and compact space is a continuous image of Δ. Δ can be identified to the subspace of $[0,1]$ (endowed with the usual metric) consisting of all numbers without 1 in its base-3 decimal expansion. Of course, Δ is an uncountable set. By $\Delta_{n,i}$ we denote the dyadic subsets of Δ, $(n,i) \in T$. Let μ be the normalized Haar measure on Δ. Then, $\mu_{n,i}$ will denote the measure μ conditioned to $\Delta_{n,i}$, $(n,i) \in T$. A *Haar system* on Δ is a sequence of functions $\{h_{n,i} : (n,i) \in T\} \subset C(\Delta)$ where $h_{n,i} := \chi_{\Delta_{n,i}}$, the characteristic function of the (n,i) dyadic subset. ω_1 denotes the first ordinal of uncountable cardinal.

3 Cantor sets in the dual of a Banach space.

We describe the behaviour of separable Banach spaces attending to the existence of a subset of the dual unit ball (endowed with the ω^*-topology) homeomorphic to the Cantor set and enjoying some additional properties.

Cantor sets exist in compact metric spaces as soon as they are not small (as sets). This is the content of the next simple lemma:

Proposition 1 *Let $(T, \mathcal{T})$ be a Hausdorff topological space. Let $\emptyset \neq S \subset T$ be a Polish subspace (i.e., a metric d can be defined on S such that (S,d) is a separable complete metric space whose topology coincides with the one induced by $\mathcal{T}$). Let A be an uncountable subset of S. Then, the closure of A in S contains a homeomorphic copy of Δ.*

Proof: The following construction is done in the metric space (S,d): A is separable, hence the subset B of all condensation points of A is again uncountable. Let $x_{0,1}$ be an arbitrary point in B. Select $x_{1,1}$ and $x_{1,2}$ two different points in $B \cap B(x_{0,1};1)$, where $B(x;r) := \{y \in S : d(y,x) < r\}$. Let $0 < r_1 < 1/2$ such that

$$\overline{B}(x_{1,1};r_1) \cap \overline{B}(x_{1,2};r_1) = \phi$$

Choose $x_{2,1}$ and $x_{2,2}$, two different points in $B \cap B(x_{1,1};r_1)$, $x_{2,3}$ and $x_{2,4}$ two different points in $B \cap B(x_{1,2};r_1)$ and $0 < r_2 < 1/2^2$ such that

$$\overline{B}(x_{2,1};r_2) \cap \overline{B}(x_{2,2};r_2) = \phi, \qquad \overline{B}(x_{2,3};r_2) \cap \overline{B}(x_{2,4};r_2) = \phi$$

Proceed in this way to define $\{x_{n,i} : (n,i) \in T\} \subset B$.

This provides a tree-like structure (see Figure 1). Denote $U_{0,1} := B(x_{0,1};1)$, $U_{1,1} := B(x_{1,1};r_1)$, $U_{1,2} := B(x_{1,2};r_1)$, $U_{2,1} := B(x_{2,1};r_2)$, and so on.

Obviously, every "branch" of the tree (i.e., a subset of the tree consisting of a descending path from the very top) is a Cauchy sequence, hence it has a limit in S. The set $\Delta = (\overline{U}_{1,1} \cup \overline{U}_{1,2}) \cap (\overline{U}_{2,1} \cup \overline{U}_{2,2} \cup \overline{U}_{2,3} \cup \overline{U}_{2,4}) \cap \ldots$ of all such limits is readily seen to be homeomorphic to the Cantor ternary set.

$\square$

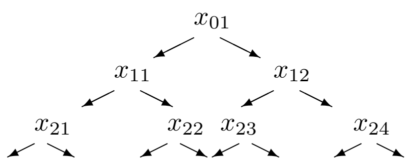

Figure 1: A tree.

The following elementary observation is a slight modification of a lemma in [ST75]:

Lemma 2 *Let X be a separable Banach space. Let S be a nonseparable subset of $(X^*, \|\cdot\|)$. Then there exists $\{x^*_\alpha : 1 \le \alpha < \omega_1\}$ in S, $C > 0$ and $\{x^{**}_\alpha : 1 \le \alpha < \omega_1\}$ in $C.B_{X^{**}}$ such that*

$$\langle x^{**}_\beta, x^*_\alpha \rangle = \begin{cases} 1, & \text{if } \alpha = \beta < \omega_1 \\ 0, & \text{if } \alpha < \beta < \omega_1. \end{cases}$$

Proof: Let x^*_0 be an arbitrary element in S_{X^*}. We can find x^{**}_0 in $S_{X^{**}}$ such that $\langle x^{**}_0, x^*_0 \rangle = 1$. Assume elements x^*_α and x^{**}_α have already been constructed satisfying the requirements for $0 \le \alpha < \beta$, β an ordinal in $[1, \omega_1[$.

The following observation has a simple proof: *Let A be a subset of a Banach space Y such that, for every $\epsilon > 0$ there exists a separable subset D of Y such that $A \subset D + \epsilon.B_Y$. Then, A is separable.*

As $S \subset X^*$ is not separable, it follows that there exists $\epsilon > 0$ such that $S \not\subset \overline{\operatorname{span}}^{\|\cdot\|}\{x^*_\alpha : \alpha < \beta\} + \epsilon.B_{X^*}$. Choose then an element x^*_β in $S \setminus \left(\overline{\operatorname{span}}^{\|\cdot\|}\{x^*_\alpha : \alpha < \beta\} + \epsilon.B_{X^*}\right)$.

By the Separation Theorem, there exists $y^{**}_\beta \in S_{X^{**}}$ such that $\langle y^{**}_\beta, x^*_\alpha \rangle = 0$, $\forall \alpha < \beta$ and $\langle y^{**}_\beta, x^*_\beta \rangle = d \ge \epsilon$, where d is the distance between x^*_β and $\overline{\operatorname{span}}^{\|\cdot\|}\{x^*_\alpha : \alpha < \beta\}$. Define $x^{**}_\beta := d^{-1}.y^{**}_\beta$. It follows that $\langle x^{**}_\beta, x^*_\alpha \rangle = 0$, $\forall \alpha < \beta$, $\langle x^{**}_\beta, x^*_\beta \rangle = 1$, and $\|x^{**}_\beta\| \le C$, where $C := \epsilon^{-1}$. This construction carries on while $\beta < \omega_1$.

$\square$

The following theorem is stated in [ST75]:

Theorem 3 (Stegall [ST75]) *Let X be a separable Banach space. Let $S \subset (X^*, \|\cdot\|)$ be nonseparable and a G_δ subset of (X^*, ω^*). Then, for $\epsilon > 0$ there exist a subset $\Delta \subset S$ which is ω^*-homeomorphic to the Cantor set , a Haar system $\{h_{ni},\ (n,i) \in T\}$ on Δ, $K > 0$, and a sequence $\{x_{ni},\ (n,i) \in T\} \subset C.B_X$, such that $\|x_{ni}\| \leq C$, $(n,i) \in T$, and*

$$\sum_{n=0}^{\infty} \sum_{i=1}^{2^n} \|Ix_{ni} - h_{ni}\| < \epsilon$$

where $I : X \to C(\Delta)$ is the canonical evaluation operator.

Proof: We shall modify appropriately the proof of Proposition 1 to get not only the Cantor set, but also the associated Haar system: As B_{X^*} is obviously a G_δ subset of (X^*, ω^*), by considering $S \cap n.B_{X^*},\ n \in I\!N$ we may and do assume that S is bounded, say $S \subset K.B_{X^*}$ for some $K > 0$. Then (S, ω^*) is a Polish space (S, d). Let $A := \{x^*_\alpha :\ \alpha < \omega_1\}$ and $\{x^{**}_\alpha :\ \alpha < \omega_1\}$ be the sets obtained using Lemma 2. Use the terminology introduced in the proof of Proposition 1. We shall modify the 2^n sets $U_{n,i}$ defined there: Choose $x^*_{\alpha_1} \in U_{n,1}, \ldots,\ x^*_{\alpha_{2^n}} \in U_{n,2^n}$ such that $\alpha_1 > \alpha_i,\ i = 2 \leq i \leq 2^n$. Therefore $\langle x^{**}_{\alpha_1}, x^*_{\alpha_1}\rangle = 1$, $\langle x^{**}_{\alpha_1}, x^*_{\alpha_i}\rangle = 0$, for $2 \leq i \leq 2^n$. Goldstein theorem gives $x_{n,1} \in C.B_X$ such that $\left|\langle x_{n,1}, x^*_{\alpha_1}\rangle - 1\right| < \frac{\epsilon}{2^n}$, $\left|\langle x_{n,1}, x^*_{\alpha_i}\rangle\right| < \frac{\epsilon}{2^n}$, $2 \leq i \leq 2^n$, hence $U_{n,1}, \ldots, U_{n,2^n}$ contain non-void neighborhoods of $x^*_{\alpha_1}, \ldots, x^*_{\alpha_{2^n}}$ in (S, d), say respectively $V_{n,1}, \ldots, V_{n,2^n}$, where

$$\begin{array}{lll} |\langle x_{n,1}, x^*\rangle - 1| < \frac{\epsilon}{2^n}, & \text{for } x^* \in V_{n,1}, & \text{and} \\ |\langle x_{n,1}, x^*\rangle| < \frac{\epsilon}{2^n}, & \text{for } x^* \in V_{n,i}, & 2 \leq i \leq 2^n. \end{array}$$

Now, we choose $x^*_{\beta_1} \in V_{n,1}, \ldots, x^*_{\beta_{2^n}} \in V_{n,2^n}$ in such a way that $\beta_2 > \beta_i$, $1 \leq i \leq 2^n$, $i \neq 2$. Repeat this procedure to get a vector $x_{n,2} \in C.B_X$ and neighborhoods of $x^*_{\beta_1}, \ldots, x^*_{\beta_{2^n}}$ in (S, d), contained in $V_{n,1}, \ldots, V_{n,2^n}$, respectively. After $2^n - 2$ more times we obtain non-void disjoint neighborhoods $W_{n,1}, \ldots, W_{n,2^n}$ in (S, d) with d-diameter less than $1/2^n$ and vectors $x_{n,i} \in C.B_X$, $1 \leq i \leq 2^n$, such that

$$\left|\langle x_{n,i} - \chi_{W_{n,i}}, x^*\rangle\right| < \frac{\epsilon}{2^n}, \qquad \text{for } x^* \in \bigcup_{j=1}^{2^n} W_{n,j}, \qquad 1 \leq i \leq 2^n.$$

Again, define Δ as $\cap_{n=1}^{\infty}\{\cup_{i=1}^{2^n} \overline{W}_{n,i}\}$, where the closures are in (S, d).

□

The next result states some equivalences in the spirit of the former theorem:

Theorem 4 (Stegall [ST75]) *Let X be a separable Banach space. Then the following assertions are equivalent:*

1. *X is not an Asplund space (i.e., X^* is not separable).*

2. *Given $\epsilon > 0$ and a nonseparable set $S \subset (X^*, \|\cdot\|)$ which is a G_δ subset of (X^*, ω^*), there exists $C > 0$, a subset Δ of (S, ω^*) topologically homeomorphic to the Cantor set, and a subset $\{x_{n,i} :\ (n,i) \in T\}$ of $C.B_X$ such that*
$$\|x_{n,i}|_\Delta - \chi_{\Delta_{n,i}}\|_\infty < \frac{\varepsilon}{2^n},\ (n,i) \in T,$$
(where $\|\cdot\|_\infty$ is the supremum norm of functions on the compact set Δ and χ_D is the characteristic function of a set D).

3. *Given $\epsilon > 0$ and a non-separable set $S \subset (X^*, \|\cdot\|)$ which is a G_δ subset of (X^*, ω^*), there exists $C > 0$, a subset Δ of (S, ω^*) topologically homeomorphic to the Cantor set, and a subset $\{x^{**}_\eta :\ \eta \in \Delta\}$ of $C.B_{X^{**}}$, such that*
$$\langle x^{**}_\eta, \mu\rangle = \delta_{\eta\mu},\ \eta \in \Delta,\ \mu \in \Delta.$$

4. *Given a nonseparable set $S \subset (X^*, \|\cdot\|)$ which is a G_δ subset of (X^*, ω^*), there exists a subset Δ of (S, ω^*) topologically homeomorphic to the Cantor set, which is discrete in the weak topology.*

Proof: (1) $\Rightarrow$ (2) is the previous Theorem 3.

(2) $\Rightarrow$ (3). Each element of $\{x^{**}_\eta :\ \eta \in \Delta\}$ is defined as an ω^*-adherent point (in X^{**}) of a "branch" of the tree-like set $\{x_{n,i} :\ (n,i) \in T\}$.

(3) $\Rightarrow$ (4). This is obvious.

(4) $\Rightarrow$ (1). Δ is also norm discrete. As it is uncountable, X^* is not separable.

$\square$

A more precise requirement on the Cantor set characterizes the separable Banach spaces which contains an isomorphic copy of ℓ_1. This is Theorem 6. The proof, a variation of the proof of Theorem 3, uses the following lemma which substitutes Lemma 2 providing again the building bricks in the process of constructing the Cantor set.

Lemma 5 (Pełczyński [P68]) [1] *Let X be a Banach space containing a copy of ℓ_1. Then X^* contains a copy of $\ell_1(\mathbb{R})$.*

Theorem 6 (Stegall [ST81], Pełczyński [P68], Hagler [HAG73]) *Let X be a separable Banach space. The following conditions are equivalent:*

1. *X contains a copy of ℓ_1.*

2. *X^* contains a copy of $\ell_1(I)$, for I an uncountable set.*

[1] In Corollary 12 a slightly more precise result will be proved, so we omit the proof of this lemma.

3. *Given $\epsilon > 0$, there exists a subset Δ of (B_{X^*}, ω^*), topologically homeomorphic to the Cantor set, and there exists a bounded subset*

$$\{x_{n,i} : \ 1 \leq i \leq 2^{2^n}, \ n \geq 0\}$$

of X such that, if

$$\Sigma_{2^n} := \{-1, 1\}^{2^n} = \{\epsilon_i : \ 1 \leq i \leq 2^{2^n}\},$$

then

$$\left\| x_{n,i} - \sum_{j=1}^{2^n} \epsilon_i(j) . \chi_{\Delta_{n,j}} \right\|_\infty < \epsilon/2^n, \ 1 \leq i \leq 2^{2^n}, \ n \geq 0$$

(where $\| \cdot \|_\infty$ denotes the supremum norm of functions on the compact set Δ).

4. *Given $\epsilon > 0$, there exists a subset Δ of (B_{X^*}, ω^*), topologically homeomorphic to the Cantor set, and there exists a bounded subset*

$$\{x_n : \ n \geq 0\}$$

of X such that

$$\left\| x_n - \sum_{j=1}^{2^n} (-1)^j \chi_{\Delta_{n,j}} \right\|_\infty < \epsilon/2^n, \ n \geq 0.$$

5. *There exists a continuous linear operator from X onto $C(\Delta)$.*

6. *There exists an isomorphism from $C(\Delta)^*$ into X^*.*

Proof: (1) $\Rightarrow$ (2) is Lemma 5.

(2) $\Rightarrow$ (3). Let T be an isomorphism from $\ell_1(I)$ onto a closed subspace Y of X^*, T^* its adjoint. Let $S := \{T(e_i) : \ i \in I\}$, where $\{e_i : \ i \in I\}$ is the canonical basis of $\ell_1(I)$. Repeat the construction of the family $\{U_{n,i} : \ (n,i) \in T\}$ of open sets in (S, ω^*) done in the proof of Proposition 1 and modify those sets, this time using the following argument: Select $T(e_{i_j}) \in U_{n,j}, \ 1 \leq j \leq 2^n$. Take $\epsilon_1 \in \Sigma_{2^n}$. Let $u_1 \in \ell^\infty(I)$ be defined by putting $\epsilon_1(j)$ in position i_j, 0 elsewhere, $1 \leq j \leq 2^n$. Let $y^*_{n,1} := (T^*)^{-1}(u_1) \in Y^*$. Extend $y^*_{n,1}$ to an element $x^{**}_{n,1} \in X^{**}$ preserving the norm. Choose now $x_{n,1} \in X$ with $\|x_{n,1}\| = \|x^{**}_{n,1}\|$ such that

$$\left| \langle x^{**}_{n,1} - x_{n,1}, T(e_{i_j}) \rangle \right| < \frac{\epsilon}{2^{n+1}}, \ 1 \leq j \leq 2^n.$$

Substitute $U_{n,j}$ by

$$\{x^* \in U_{n,j} : \ \left| \langle x_{n,1}, \left[x^* - T(e_{i_j}) \right] \rangle \right| < \frac{\epsilon}{2^{n+1}}\}.$$

Those sets will be renamed $U_{n,j},\ 1 \leq j \leq 2^n$. Repeat the construction, now by using $\epsilon_2 \in S_{2^n}$. After 2^{2^n} steps, we get the appropriate $U_{n,j}$'s and $x_{n,j}$'s. Once the family $\{U_{n,i}:\ (n,i) \in T\}$ is constructed, Δ is defined in the usual way.

(3) $\Rightarrow$ (4). This is obvious.

(4) $\Rightarrow$ (1). It is simple to prove that $\{x_n:\ n \geq 0\}$ is a basic sequence equivalent to the canonical basis of ℓ_1.

(3) $\Rightarrow$ (5). Let $T: X \to C(\Delta)$ be the restriction mapping, $T^*: C(\Delta)^* \to X^*$ its adjoint. Then, for every $\mu \in C(\Delta)^*$ with $\|\mu\| = 1$,

$$\begin{aligned} \|T^*(\mu)\| \geq \frac{1}{K} \sup_{n,i} |\langle x_{n,i}, T^*(\mu)\rangle| = \\ = \tfrac{1}{K} \sup_{n,i} \left|\int_\Delta \langle x_{n,i}, x^*\rangle d\mu(x^*)\right| \geq \\ \geq \tfrac{1}{K} \sup_{n,i} \left[\left|\int_\Delta \chi_{n,i} d\mu(x^*)\right| - \tfrac{\epsilon}{2^n}\right] = \tfrac{1}{K}\|\mu\|, \end{aligned} \tag{1}$$

where $\kappa_{n,i} := \sum_{j=1}^{2^n} \epsilon_i(j)\chi_{\Delta_{n,j}}$ and K is an upper bound for $\{\|x_{n,i}\|:\ 1 \leq i \leq 2^{2^n},\ n \geq 0\}$. Inequalities in (1) prove that T^* is an isomorphism into. It follows that T is a homomorphism with closed range (see, for example, [KÖ79], §33.1 (4)). As it is clear that the range of T is dense in $C(\Delta)$, T is onto.

(5) $\Rightarrow$ (6). This is obvious.

(6) $\Rightarrow$ (2). $\ell_1(\Delta)$ is a subspace of $C(\Delta)^*$, hence it embeds into X^*.

□

The former theorem says, in particular, that in the context of separable Banach spaces, X contains an isomorphic copy of ℓ_1 if and only if the Cantor set Δ can be found in (X^*, ω^*) in such a way that every continuous function on Δ can be extended to a linear continuous function on (X^*, ω^*), that is, an element in X, not increasing substantially the norm. The aim of the next results is to precise the location of the set Δ in this situation, proving that in fact Δ can be found in the set of extreme points of the unit ball of the dual, $[\mathrm{Ext}(B_{X^*}), \omega^*]$ (a G_δ subset of (B_{X^*}, ω^*)). We shall follow [BD78]. Let's start by the following

Lemma 7 (Bourgain-Delbaen [BD78]) *Let E be a topological space with a countable base and $\{f_n : E \to \mathbb{R},\ n \in \mathbb{N}\}$, a family of continuous real mappings such that given an infinite subset $N \subset \mathbb{N}$ there is an $x \in E$ such that*

$$\underline{\lim}_{n\in N} f_n(x) < -1 + \epsilon \quad \text{and} \quad 1 - \epsilon < \overline{\lim}_{n\in N} f_n(x).$$

Then there is a nonvoid closed subset F and an infinite subset M of $\mathbb{N}$ such that for every nonvoid $F-$open subset U there exists m_U in M such that for every m in M, with $m > m_U$ we have

$$\inf_{x\in U} f_m(x) < -1 + \epsilon \quad \text{and} \quad 1 - \epsilon < \sup_{x\in U} f_m(x).$$

Proof: Suppose the sequence $\{f_n\}$ satisfies the following property: given a nonvoid closed subset F of E and an infinite subset N of $I\!N$ there exists a closed set $F' \subset F$, $F' \neq F$ and an infinite subset $N' \subset N$ such that

$$\begin{array}{ll} -1+\epsilon \leq f_n(x), & \forall x \in F \setminus F',\ \forall n \in N', \quad \text{or} \\ f_n(x) \leq 1-\epsilon, & \forall x \in F \setminus F',\ \forall n \in N'. \end{array}$$

(we then say that the couple (F', N') *follows* the couple (F, N)). By transfinite induction we shall determine a decreasing family of closed subsets $\{F_\alpha, \alpha < \omega_1\}$ and a family $\{N_\alpha, \alpha < \omega_1\}$ of infinite subsets of $I\!N$ such that $(F_{\alpha+1}, N_{\alpha+1})$ follows (F_α, N_α) and $\alpha < \beta$ implies that $F_\beta \subset F_\alpha$ and $N_\beta \setminus N_\alpha$ is a finite set. In fact, if the family have been defined until $\alpha < \omega_1$ and $F_\alpha = \emptyset$, then we shall define $F_\beta = F_\alpha$ and $N_\beta = N_\alpha$ for $\alpha < \beta < \omega_1$. If F_α and N_α have been determined for $\alpha < \beta < \omega_1$, and β is not a limit ordinal, apply the aforesaid property to $F := F_{\beta-1}$ and $N := N_{\beta-1}$ to find $F_\beta := F'$ and $N_\beta := N'$. If β is instead a limit ordinal, then define $F_\beta = \cap_{\alpha<\beta} F_\alpha$; to construct N_β we must consider that β is the supremum of the countable set $\{\alpha_1, \alpha_2, \ldots\}$ of ordinals numbers less than β. Therefore β is the limit of the sequence $\{\gamma_n := \sup\{\alpha_i : 1 \leq i \leq n\}\}$. Then we define the infinite set N_β by taking one point in N_{γ_1}, one point in $N_{\gamma_1} \cap N_{\gamma_2}$, and so on.

Given $x \in E \setminus F_\alpha$ let $\gamma + 1$ be the first ordinal α such that $x \notin F_\alpha$. Then $x \in F_\gamma$ and therefore

$$\begin{array}{ll} -1+\epsilon \leq f_n(x), & \forall n \in N_{\gamma+1}, \quad \text{or} \\ f_n(x) \leq 1-\epsilon, & \forall n \in N_{\gamma+1}. \end{array}$$

As the set $N_\alpha \setminus N_{\gamma+1}$ is finite, it follows that for every $x \notin F_\alpha$ there exists a finite set $N'_\alpha(x)$ such that

$$\begin{array}{ll} -1+\epsilon \leq f_n(x), & \forall n \in N_\alpha \setminus N'_\alpha(x), \quad \text{or} \\ f_n(x) \leq 1-\epsilon, & \forall n \in N_\alpha \setminus N'_\alpha(x). \end{array}$$

If in the preceding construction all the F_α were non void we should have a strictly increasing uncountable family of open sets $A_{\alpha+1} = E \setminus F_{\alpha+1}$, $\alpha < \omega_1$, hence E cannot have a countable base. Therefore an empty F_α appears eventually. It follows that there exists an infinite subset $N (= N_\alpha)$ of $I\!N$ such that for every $x \in E$ there exists a finite subset $N'(x) (= N'_\alpha(x))$ of N such that

$$\begin{array}{ll} -1+\epsilon \leq f_n(x), & \forall n \in N \setminus N'(x), \quad \text{or} \\ f_n(x) \leq 1-\epsilon, & \forall n \in N \setminus N'(x). \end{array}$$

and the lemma is proved.

□

Now, the former lemma allows us to construct, given a sequence of continuous functions which “oscillates enough”, a Cantor set and a subsequence which behaves as the canonical basis of ℓ_1 on the dyadic subsets. Precisely we have the following

Proposition 8 (Bourgain-Delbaen [BD78]) *Let K be a compact metric topological space, E a Polish subspace and $0 < \epsilon < 1$. Let (f_n) be a sequence of functions in $B_{C(K)}$ such that $\forall N$ (an infinite subset of $I\!N$), there exists $x \in E$ such that*

$$\left[\underline{\lim}_{n\in N} f_n(x) < -1+\epsilon\right] \text{ and } \left[\overline{\lim}_{n\in N} f_n(x) > 1-\epsilon\right].$$

Then there exists Δ, a subset of E homeomorphic to the Cantor set, and a subsequence (g_n) of (f_n), such that

$$\begin{aligned} g_n(x) &\leq -1+\epsilon, \ \forall x \in \Delta_{n,i}, \ i = 1,3,5,\ldots,2^n-1, \\ g_n(x) &\geq 1-\epsilon, \ \forall x \in \Delta_{n,i}, \ i = 2,4,6,\ldots,2^n, \end{aligned}$$

$n \in I\!N$

Proof: E has a countable basis. Let d be the metric on E for which E is complete. The former proposition gives a non-empty closed subset F of E and an infinite subset N of $I\!N$. Let $F_{0,1} \neq F$ be a closed subset of F. Choose $n_1 \in N$ in such a way that $x_{1,1} \in F \setminus F_{0,1}$ and $x_{1,2} \in F \setminus F_{0,1}$ can be found such that

$$f_{n_1}(x_{1,1}) < -1+\epsilon, \quad f_{n_1}(x_{1,2}) > 1-\epsilon.$$

f_{n_1} is continuous; it is possible then to find two non-empty disjoint closed subsets of F, say $F_{1,1}$ and $F_{1,2}$, both of them with non-empty relatively to F interior, such that

$$f_{n_1}(x) \leq -1+\epsilon, \ \forall x \in F_{1,1}, \quad f_{n_1}(x) \geq 1-\epsilon, \ \forall x \in F_{1,2}.$$

Moreover, $F_{1,1}$ and $F_{1,2}$ can be chosen in such a way that their d-diameter is less than 1. Choose $n_2 \in \{n \in N : \ n > n_1\}$ in such a way that two points, say $x_{2,1}$ and $x_{2,2}$ in $int(F_{1,1})$ and another two points, $x_{2,3}$ and $x_{2,4}$ in $int(F_{1,2})$, can be found verifying

$$f_{n_2}(x_{2,1}) < -1+\epsilon, \ f_{n_2}(x_{2,2}) > 1-\epsilon, \ f_{n_2}(x_{2,3}) < -1+\epsilon, f_{n_2}(x_{2,4}) > 1-\epsilon.$$

As before, it is possible to find four closed and pairwise disjoint subsets $F_{2,1}$, $F_{2,2}$, $F_{2,3}$ and $F_{2,4}$ of F such that $F_{2,1} \cup F_{2,2} \subset F_{1,1}$, $F_{2,3} \cup F_{2,4} \subset F_{1,2}$, having non-empty relatively to F interior, all of them with d-diameter less than 1/2, and

$$f_{n_2}(x) \leq -1+\epsilon, \ \forall x \in F_{2,1} \cup F_{2,3}, \quad f_{n_2}(x) \geq 1-\epsilon, \ \forall x \in F_{2,2} \cup F_{2,4}.$$

Proceeding in the same way, a dyadic structure is constructed defining a Cantor set with all the required properties.

$\square$

From a Cantor set where a sequence of functions oscillate "in a ℓ_1 fashion", as in the former proposition, it is possible to pass to another Cantor set (in fact, a choice of some dyadic subsets from the original one) where the oscillation is, in a sense, "as you like it". This behaviour is in no way anecdotal, as it implies the possibility to map a Banach space onto the space of continuous functions on a Cantor set. The development of these ideas is our next goal:

Lemma 9 (Bourgain-Delbaen [BD78]) *Let Δ be a Cantor set, (f_n) a sequence of continuous functions from Δ into $[-1,1]$ such that, for $n \in I\!N$,*

$$f_n(x) \leq -1+\epsilon,\ \forall x \in \Delta_{n,j},\ j = 1,3,5,\ldots,2^n-1,$$
$$f_n(x) \geq 1-\epsilon,\ \forall x \in \Delta_{n,j},\ j = 2,4,6,\ldots,2^n.$$

Then there exists a Cantor set $\Delta' \subset \Delta$ such that, if

$$\Sigma_{2^n} := \{-1,1\}^{2^n} = \{\epsilon_i,\ 1 \leq i \leq 2^{2^n}\}$$

and (f_n) is renumbered as $(f_{1,1}, f_{1,2}, f_{1,3}, f_{1,4}, \ldots, f_{n,1}, f_{n,2}, \ldots, f_{n,2^{2^n}}, \ldots)$,

$$\epsilon_i(j).f_{n,i}(x) \geq 1-\epsilon,\ \forall x \in \Delta'_{n,j},\ (n,j) \in T,\ n \neq 0 \quad ([2])$$

Proof: The set Δ' will be defined by choosing some of the dyadic subsets of Δ. To clarify the construction, let's index the sets $\Delta_{n,j},\ (n,j) \in T$, $n \neq 0$, as

$$\Delta_0,\ \Delta_1,\ \Delta_{0,0},\ \ \Delta_{0,1}\ \Delta_{1,0},\ \Delta_{1,1},\ \Delta_{0,0,1},\ldots$$

Then, choose the two dyadic subsets of Δ' at level 1 as the rows (omitting the Δ's) in the following matrix:

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}$$

Functions f_1, f_2, f_3, f_4 do the job. Now, choose the four dyadic subsets at level 2 as the rows in the following matrix:

$$\begin{array}{cccc|c|cccc|cccccc|cccc|c}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1
\end{array}$$

Functions $f_5, f_6, \ldots, f_{20}$ oscillate in the required form. Continue in this way and define the Cantor set Δ' as customary by intersection of unions at level n.

$\square$

Putting all the pieces together we get the following

Theorem 10 (Bourgain-Delbaen [BD78]) *Let K be a compact metric space, E a Polish subspace, $0 < \epsilon < 1$, and (f_n) a sequence in $B_{C(K)}$ such that $\forall N \subset I\!N$ (an infinite subset), there exists $x \in E$ such that*

$$\left[\underline{\lim}_{n\in N} f_n(x) < -1+\epsilon\right] \text{ and } \left[\overline{\lim}_{n\in N} f_n(x) > 1-\epsilon\right].$$

Then, there exists a Cantor set $\Delta' \subset E$ such that the following extension property holds:

$$\forall \phi \in C(\Delta') \text{ such that } \|\phi\| < 1-\epsilon, \exists f \in \overline{lin}(f_n) \text{ such that } \|f\| < 1 \text{ and } f\big|_{\Delta'} = \phi.$$

[2] Observe that, if a sequence (f_n) and a Cantor set Δ' as in this lemma are given, just a subsequence of (f_n) provides the alternating ℓ_1-behaviour.

Proof: By Proposition 8 there exists a subsequence (g_n) of (f_n) and a Cantor set $\Delta \subset E$ such that

$$g_n(x) \leq -1+\epsilon,\ \forall x \in \Delta_{n,j},\ j = 1,3,5,\dots,2^n-1,$$
$$g_n(x) \geq 1-\epsilon,\ \forall x \in \Delta_{n,j},\ j = 2,4,6,\dots,2^n,\ n \in I\!N,$$

where $\{\Delta_{n,j} :\ (n,j) \in T,\ n \neq 0\}$ is the family of dyadic subsets of Δ (starting at level 1). By Lemma 9 there is a Cantor set $\Delta' \subset \Delta$ with the properties listed there. Define

$$\kappa_n(x) := \begin{cases} 1, & \text{if } g_n(x) \geq 1-\epsilon, \\ -1, & \text{if } g_n(x) \leq -1+\epsilon. \end{cases},\ \forall x \in \Delta'.$$

Let T be the restriction mapping from $Y := \overline{lin}(g_n)$ into $C(\Delta')$, T^* its conjugate mapping from $(C(\Delta'))^*$ into Y^*. Given $\mu \in (C(\Delta'))^*$,

$$\begin{aligned}\|T^*(\mu)\| = \sup_{y\in B_Y} |\langle y, T^*(\mu)\rangle| = \sup_{y\in B_Y} |\langle T(y), \mu\rangle| \geq \\ \geq \sup\nolimits_{n\in I\!N} \left|\int_{\Delta'} g_n(x^*)d\mu(x^*)\right| = \\ = \sup\nolimits_{n\in I\!N} \left|\int_{\Delta'}(g_n(x^*)-\kappa_n(x^*))d\mu(x^*) + \int_{\Delta'} \kappa_n(x^*)d\mu(x^*)\right| \geq \\ \geq \sup\nolimits_{n\in I\!N} \left|\int_{\Delta'} \kappa_n(x^*)d\mu(x^*)\right| - \epsilon.\|\mu\| = (1-\epsilon).\|\mu\|.\end{aligned} \tag{2}$$

The same argument used in the proof of (3) $\Rightarrow$ (5) in Theorem 6 proves that T is onto. Moreover, from (2) the following property is easily seen to be true: Given $\phi \in C(\Delta')$ such that $\|\phi\| < 1-\epsilon$, there exists $y \in Y$ such that $\|y\| < 1$ and $T(y) = \phi$. This provides the required extension property.

$\square$

There is a natural setting in which the situation described in Theorem 10 appears: The dual unit ball B_{X^*} of a separable Banach space X, endowed with the weak*-topology ω^*, plays the rôle of K, the set of extreme points of B_{X^*} (a G_δ subset of $B_{X^*}[\omega^*]$ (see, for example, [PH66, Proposition 1.3]), hence Polish) the rôle of E, and, as soon as X contains an isomorphic copy of ℓ_1, a sequence (x_n) in X (each x_n viewed as a continuous functions on (B_{X^*}, ω^*)) can be found satisfying the required conditions. Thus, the following theorem holds:

Theorem 11 (Bourgain-Delbaen [BD78]) *Let X be a separable Banach space. Then, the following conditions are equivalent:*

1. *X contains an isomorphic copy of ℓ_1.*
2. *for every $0 < \epsilon < 1$, there exists a subset Δ_ϵ of the set of extreme points Ext(B_{X^*}) of the unit ball B_{X^*} of X^*, which is ω^*-homeomorphic to the Cantor set and such that, given $\phi \in C(\Delta_\epsilon)$ with $\|\phi\| < 1-\epsilon$, there is $x \in X$ so that $\|x\| < 1$ and $\langle x, x^*\rangle = \phi(x^*)$ whenever $x^* \in \Delta_\epsilon$.*

Proof: (2) $\Rightarrow$ (1) as the bounded sequence $\{x_n:\ n\geq 0\}$, where $\langle x_n, x^*\rangle := \sum_{j=1}^{2^n}(-1)^j\chi_{\Delta_{\epsilon n,j}}(x^*)$ whenever $x^*\in\Delta_\epsilon$, is a basic sequence equivalent to the canonical basis of ℓ^1. In order to prove (1) $\Rightarrow$ (2), observe, first of all, that, from a well known result of R. C. James [J51], if a Banach space contains an isomorphic copy of ℓ_1, then it contains another one with a "distortion factor" as small as we want: precisely, given $0<\alpha<1$ there is a sequence (x_n) in X such that $\|x_n\|=1,\ n\in I\!N$, and $\|\sum_{k=1}^n a_kx_k\|\geq\alpha\sum_{k=1}^n|a_k|,\ a_1,\dots a_n\in I\!R,\ n\in I\!N$ (such a sequence is called an $\alpha-\ell_1$-*basis*).

Now, given $0<\epsilon<1$, choose an $(1-\frac{\epsilon}{4})-\ell_1-$basis (x_n). In order to apply Theorem 10, it is enough to prove that for any infinite subset N of $I\!N$ there exists $x^*\in\mathrm{Ext}(B_{X^*})$ such that

$$\underline{\lim}_{n\in N}\langle x_n, x^*\rangle < -1+\epsilon \ \text{ and } \ \overline{\lim}_{n\in N}\langle x_n, x^*\rangle > 1-\epsilon.$$

Let N be an infinite subset of $I\!N$. By the Hahn-Banach Theorem and the fact that (x_n) is an $(1-\frac{\epsilon}{4})-\ell_1-$basis, there exists $y^*\in S_{X^*}$ such that both sets $\{n\in N:\ \langle x_n,y^*\rangle\leq -1+\frac{\epsilon}{4}\}$ and $\{n\in N:\ \langle x_n,y^*\rangle\geq 1-\frac{\epsilon}{4}\}$ are infinite. We need to be able to choose y^* from $\mathrm{Ext}(B_{X^*})$. Choquet's Theorem (see, for example, [PH66, §3]) does the job: there exists a probability measure μ supported by $\mathrm{Ext}(B_{X^*})$ representing y^*. Thus

$$\langle x, y^*\rangle = \int_{\mathrm{Ext}(B_{X^*})}\langle x, x^*\rangle d\mu(x^*),\quad \forall x\in X,$$

and therefore, applying Fatou's Lemma,

$$\int_{\mathrm{Ext}(B_{X^*})}\underline{\lim}_{n\in N}\langle x_n, x^*\rangle d\mu(x^*)\leq \underline{\lim}_{n\in N}\langle x_n, y^*\rangle\leq -1+\frac{\epsilon}{4},$$
$$\int_{\mathrm{Ext}(B_{X^*})}\overline{\lim}_{n\in N}\langle x_n, x^*\rangle d\mu(x^*)\geq \overline{\lim}_{n\in N}\langle x_n, y^*\rangle\geq 1-\frac{\epsilon}{4}.$$

It follows that

$$\int_{\mathrm{Ext}(B_{X^*})}\left[\overline{\lim}_{n\in N}\langle x_n, x^*\rangle - \underline{\lim}_{n\in N}\langle x_n, x^*\rangle\right]d\mu(x^*)\geq 2-\frac{\epsilon}{2}.$$

Hence there exists some $x^*\in\mathrm{Ext}(B_{X^*})$ such that

$$\overline{\lim}_{n\in N}\langle x_n, x^*\rangle - \underline{\lim}_{n\in N}\langle x_n, x^*\rangle\geq 2-\frac{\epsilon}{2}.$$

Since moreover $\underline{\lim}_{n\in N}\langle x_n, x^*\rangle\geq -1$ and $\overline{\lim}_{n\in N}\langle x_n, x^*\rangle\leq 1$ we find that $\underline{\lim}_{n\in N}\langle x_n, x^*\rangle\leq -1+\frac{\epsilon}{2}<-1+\epsilon$ and $\overline{\lim}_{n\in N}\langle x_n, x^*\rangle\geq 1-\frac{\epsilon}{2}>1-\epsilon$, as we wanted to prove.

$\square$

We mentioned before (see Theorem 6) a result of A. Pełczyński and J. Hagler which says that a separable Banach space X contains an isomorphic copy of ℓ_1 if, and only if, there exists a continuous linear surjection $i : X \to C(\Delta)$. Theorem 11 allows to locate Δ in the set of extreme points of the dual unit ball (endowed with the weak*-topology).

We used before (see Lemma 5) another result of A. Pełczyński: if a Banach space contains an isomorphic copy of ℓ_1 then X^* contains an isomorphic copy of $\ell_1(\mathbb{R})$ (it is not different to say that X^* contains an isomorphic copy of $\ell_1(\Delta)$, Δ, as always, the Cantor set). Theorem 11 allows us to choose the canonical basis of $\ell_1(\Delta)$ from the extreme points of the dual unit ball:

Corollary 12 (Bourgain-Delbaen [BD78]) *Let X be a Banach space containing an isomorphic copy of ℓ_1. Then, for every $0 < \epsilon < 1$, there is an embedding $i_\epsilon : \ell_1(\Delta) \to X^*$ such that $i_\epsilon(e_\delta) \in Ext(B_{X^*})$, $\forall \delta \in \Delta$, where $\{e_\delta : \ \delta \in \Delta\}$ denotes the canonical basis of $\ell_1(\Delta)$, and $\|i_\epsilon(l)\| \geq (1-\epsilon)\|l\|$, $\forall l \in \ell_1(\Delta)$.*

Proof: If X is separable, given $\epsilon > 0$ choose $\Delta := \Delta_\epsilon$ as in the preceding theorem. Define $i := i_\epsilon : \ell_1(\Delta) \to X^*$ by $i(e_\delta) := \delta$, $\forall \delta \in \Delta$. This embedding satisfies the requirement: Let $\sum_{i=1}^n a_i e_{\delta_i}$ be an element in $\ell_1(\Delta)$. Find $k \in \mathbb{N}$ and Δ_{k,j_i}, $1 \leq i \leq n$ different dyadic subsets of Δ such that $\delta_i \in \Delta_{k,j_i}$, $1 \leq i \leq n$. Let $0 < \eta < (1-\epsilon)$ be an arbitrary positive number. Define $\Phi \in C(\Delta)$ as $\Phi := (1-\epsilon-\eta)\sum_{i=1}^n \operatorname{sign}(a_i)\chi_{\Delta_{k,j_i}}$. By Theorem 11 there exists $x \in X$ such that $\|x\| < 1$ and $\langle x, \delta\rangle = \Phi(\delta)$, $\forall \delta \in \Delta$. Then

$$\langle x, i\left(\sum_{i=1}^n a_i e_{\delta_i}\right)\rangle = \langle x, \sum_{i=1}^n a_i \delta_i\rangle = \sum_{i=1}^n a_i \langle x, \delta_i\rangle = \sum_{i=1}^n a_i \Phi(\delta_i) = (1-\epsilon-\eta)\sum_{i=1}^n |a_i|.$$

So, $\|i(\sum_{i=1}^n a_i\delta_i)\| \geq (1-\epsilon-\eta)\sum_{i=1}^n |a_i|$. As η was arbitrary, $\|i(\sum_{i=1}^n a_i\delta_i)\| \geq (1-\epsilon)\sum_{i=1}^n |a_i|$. It follows that $\|i(l)\| \geq (1-\epsilon)\|l\|$, $\forall l \in \ell_1(\Delta)$.

In general, let Y be a subspace of X isomorphic to ℓ^1. $Y^* = X^*/Y^\perp$ and let $q : X^* \to X^*/Y^\perp$ be the quotient mapping. The first part of the proof provides $\Delta \subset \mathrm{Ext}(B_{Y^*})$. By the Krein-Milman Theorem, there is an extreme point u_δ in B_{X^*} such that $q(u_\delta) = \delta$, $\forall \delta \in \Delta$. Now the embedding $i : \ell_1(\Delta) \to X^*$ given by $i(e_\delta) = u_\delta$, $\forall \delta \in \Delta$, satisfies, obviously, the requirements.

□

4 Extreme points and James Boundaries.

In [ST75], Ch. Stegall asked whether the set of extreme points of the dual unit ball B_{X^*} of a separable Banach space X with nonseparable dual was itself nonseparable. This question was solved (in the affirmative):

Theorem 13 *A dual Banach space X^* with a separable set of extreme points of B_{X^*} is itself separable.*

with different degrees of generality by several authors ([HAY76], [KF76], [MU76], [BOU78], [BD78]). In particular, it follows from Corollary 12: If the Banach space contains an isomorphic copy of ℓ_1, apply the corollary. If not, it is well known that B_{X^*} is the $\|\cdot\|$-closed convex hull of its extreme points and again the result follows.

The result just mentioned (if a Banach space X does not contain an isomorphic copy of ℓ_1 then B_{X^*} is the $\|\cdot\|$-closed convex hull of its extreme points) has a long history: it seems that it appears for the first time in the particular case of $X^* := \ell_1$ [LI66]. In the same year it was proved for separable duals in [BP66]. Once it was known that RNP implied KMP, then the classical Dunford-Pettis Theorem [DP40] gives that a separable dual has RNP, hence KMP. Rosenthal fundamental work on Banach spaces containing ℓ_1 (see, for example, [RO74]) includes the fact quoted here. R. Haydon, in [HAY76], proves that a weak*-compact convex set K in X^* for which $\mathrm{Ext}(K)$ is a norm separable set, has RNP, K itself is norm separable and $K = \overline{conv}(\mathrm{Ext}(K))$, incidentally solving also Stegall's problem mentioned at the previous paragraph.

A fruitful way to extend considerations on extreme points (in particular, the aforesaid question) to more general situations is to consider that the set $\mathrm{Ext}(B_{X^*})$ is a particular case of a *James boundary*, that is, a subset B of B_{X^*} with the following property: For all $x \in X$, there exists $b^* \in B$ such that $\|x\| := \sup\{\langle x, x^*\rangle : \ x^* \in B_{X^*}\} = \langle x, b^*\rangle$.

The following theorem is due to Godefroy (and it contains, again, as a particular case, a solution to the aforesaid question):

Theorem 14 (Godefroy [GO87]) *Let X be a separable Banach space. Let BD be a James boundary for X. Assume there is F, a countable subset of X^*, and $0 < \alpha < 1$ such that $BD \subset F + \alpha . B_{X^*}$. Then $X^* = \overline{lin}^{\|\cdot\|}(F)$ (in particular, X^* is separable).*

It was remarked in [PP80] that Theorem 13 is a simple consequence of what the authors called the *duality criterion of separable Banach spaces* [PP74], and that a similar and more general result follows from the *duality criterion of WCG Banach spaces* [PL78]. Precisely, Petunin and Plichko obtained the following results:

Theorem 15 (Petunin and Plichko [PP80]) *Let X^* has a norm separable James boundary B. Then $\overline{span}^{\|\cdot\|}(B) = X^*$, hence X^* is norm separable.*

Theorem 16 (Petunin and Plichko [PP80]) *Let X^* has a* weakly compact generated *James boundary B (i.e., there exists a relatively weakly compact subset K of X^* such that $B \subset \overline{span}^{\|\cdot\|}(K)$. Then $\overline{span}^{\|\cdot\|}(B) = X^*$, hence X^* is a WCG space.*

Theorem 17 (Petunin and Plichko [PP80]) *Let X^* has a James boundary B such that $\overline{span}^{\|\cdot\|}(B)$ has a weak star angelic dual. Then $\overline{span}^{\|\cdot\|}(B) = X^*$, hence X^* has a weak star angelic dual.*

Godefroy, in [GO87] (see also [GO87M]), proves Theorem 14 and similar results using Simons inequality. Let us show how the former theorems can be obtained in this way:

Proof of Theorem 16: We shall prove something a little bit more precise, namely that in the aforesaid circumstances, $\overline{conv}^{\|\cdot\|}(B) = B_{X^*}$. To this end, assume this will fail. It is possible then to find $x_0^{**} \in S_{X^{**}}$, $\alpha < \beta$ and $x_0^* \in B_{X^*} \setminus \overline{conv}^{\|\cdot\|}(B)$ such that

$$|\langle x_0^{**}, b^* \rangle| \leq \alpha < \beta < |\langle x_0^{**}, x_0^* \rangle|, \quad \text{for all } b^* \in B.$$

The set K, as it is bounded, can be chosen in B_{X^*}. Take $K \cup \{x_0^*\}$, a weakly relatively compact subset of X^*. It follows that

$$K_0 := \overline{\Gamma(K \cup \{x_0^*\})}^{\omega},$$

(where Γ denotes the absolutely convex hull) is weakly compact, contained in B_{X^*} and, obviously, $B \subset \overline{\text{span}}^{\|\cdot\|}(K_0)$. Denote $[K_0] := \overline{\text{span}}^{\|\cdot\|}(K_0)$, a closed subspace of X^*. Let $j : [K_0] \to X^*$ be the canonical injection. Its adjoint mapping, $q : X^{**} \to X^{**}/[K_0])^{\perp}$ is the canonical quotient mapping from X^{**} onto $[K_0]^*$. q is a $\|\cdot\| - \|\cdot\|$-continuous, $\omega - \omega$-continuous and $\omega^* - \omega^*$-continuous mapping. It is simple to prove that $q(B_{X^{**}}) = B_{[K_0]^*}$. B_X is ω^*-dense in $B_{X^{**}}$, hence $q(B_X)$ is ω^*-dense in $q(B_{X^{**}}) = B_{[K_0]^*}$. Consider the Banach space $[K_0]$ and its dual $[K_0]^* = X^{**}/[K_0]^{\perp}$. By a well-known theorem (see, for example, [FL80], Thm. 3.7), $([K_0]^*, \omega^*)$ is angelic. $q(B_X)$ is ω^*-relatively compact, so every element in its closure $\overline{q(B_X)}^{\omega^*} = B_{[K_0]^*}$ is the limit of a sequence in $q(B_X)$. In particular, $q(x_0^{**}) \in q(B_{X^{**}}) = B_{[K_0]^*}$, so there exists a sequence (x_n) in B_X such that $q(x_n) \xrightarrow{\omega^*} q(x_0^{**})$. This means

$$\langle x_0^{**} - x_n, x^* \rangle \to 0, \text{ when } n \to \infty, \quad \text{for all } x^* \in [K_0].$$

As $\langle x_n, x_0^* \rangle \to \langle x_0^{**}, x_0^* \rangle > \beta$ when $n \to \infty$, we may and do assume $\langle x_n, x_0^* \rangle > \beta$ for all $n \in I\!N$.

From now on we consider all elements in X^{**} as functions on B, i.e., elements in $\ell^{\infty}(B)$. Let $C := \{x \in B_X : \langle x, x_0^* \rangle \geq \beta\}$. C is a superconvex subset of $\ell^{\infty}(B)$ (see, for example, [JAM74]), and, as B is a boundary, every element in C attains its supremum on B.

By Simons inequality,

$$\alpha \geq \sup_B x_0^{**} \geq \inf_{x \in C} \{\sup_B x\} = \inf\{\|x\| : x \in C\} \geq \beta,$$

a contradiction. It follows that $\overline{conv}^{\|\cdot\|}(B) = B_{X^*}$.

□

Remark An obvious modification of the proof allows us to state the following result:

Theorem 18 *Let X be a Banach space, $B \subset B_{X^*}$ a James boundary for X. Assume $B \subset F$, F a closed subspace of X^* such that F^* is ω^*- angelic. Then $\overline{conv}^{\|\cdot\|}(B) = B_{X^*}$.*

A topological space E is called *K-analytic* if it is a continuous image of a $K_{\sigma\delta}$ subset of a compact space. A subspace A of a compact K is called *countably determined* if there exists a sequence (A_n) of compacta in K such that for every $x \in A$ there exists a subset $M \subset \mathbb{N}$ for which $x \in \cap_{n\in M} A_n \subset A$. Let X be a Banach space. A subset $E \subset X$ is called *weakly K-analytic* if it is K-analytic in the weak topology.

The following theorem is due to Talagrand:

Theorem 19 (Talagrand [TA79]) *Let X be a Banach space. The following conditions are equivalent:*

1. *X contains a weakly K-analytic (resp., weakly countably determined) subset E such that $\overline{conv}^{\|\cdot\|}(E) = X$.*

2. *X is weakly K-analytic (resp., weakly countably determined).*

Every weakly K-analytic space is weakly countably determined [TA79]. Every weakly countably determined space has a weak star angelic dual. The preceding result gives, for example, the result of Petunin and Plichko (see [PP74]) that if a James boundary B is contained in the closed span of a weakly K-analytic (resp. weakly countably determined) subset M of X^*, then $\overline{conv}^{\|\cdot\|}(B) = X^*$, and, in particular, X^* is weakly K-analytic (resp., weakly countably determined) (see, also [GO87] and [SAA77]).

5 Some Applications.

Stegall's result on the existence of a Cantor set in the dual of a separable Banach space with a non-separable dual (Theorem 4) has, certainly, many applications. We mention here some classical and another more recent ones.

5.1 RNP and KMP.

Among the former, the equivalence between the Radon-Nikodym (RNP) and the Krein-Milman Properties (KMP) in the dual of a Banach space [HM75] is, may be, the first to come to mind: it was a pioneer result of J. Lindenstrauss (see, for example, [PH74]) that the RNP (every bounded set in X has a denting point) implies the KMP (every convex closed subset of X is the closed convex hull of the set of its extreme points). The converse is still an open problem. W. Schachermayer [SC85] has proved that both properties are equivalent in case of Banach spaces isomorphic to their squares. In a dual Banach space, both properties coincide. The proof relies on Stegall's construction:

Theorem 20 (Huff/Morris [HM75]) *Let X be a Banach space. Then, X^* has RNP if, and only if, it has KMP.*

Proof: That RNP $\Rightarrow$ KMP is a general fact, as it was mentioned earlier. Assume now that X^* has not RNP. As it is well known, X is not an Asplund space ([ST75]), hence it has a separable subspace Y such that Y^* is not separable. By Theorem 4, given $\epsilon > 0$, there exists a subset Δ of (B_{Y^*}, ω^*), topologically homeomorphic to the Cantor set, and a subset $\{x_{n,i} : (n,i) \in T\}$ of $(1+\epsilon)B_Y$ such that

$$\|x_{n,i} - \chi_{\Delta_{n,i}}\|_\infty < \epsilon/2^n, \ (n,i) \in T$$

To simplify the notation, let's renumber the ordered double-index set

$$\{(0,1),(1,1),(1,2),(2,1),(2,2),(2,3),(2,4),(3,1),\ldots\}$$

as $\{1,2,3,\ldots\}$. The restriction mapping $T : Y \to C(\Delta)$ can be viewed as a mapping from Y into $L^\infty(\mu)$ (as usual μ denotes the normalized Haar measure on Δ). By the injective property of this last space, T can be extended to a mapping (let's keep the name) $T : X \to L^\infty(\mu)$. Its adjoint, T^*, maps $(L^\infty(\mu))^*$ into X^*. Measures $\mu_{n,i}, \ (n,i) \in T$ (now listed as $\mu_n, \ n \in I\!N$) are elements in $(L^\infty(\mu))^*$ (in fact, a tree there), so we get a bounded tree $\{x_n^* := T^*(\mu_n), \ n \in I\!N\}$ in X^*.

Define $C := \overline{conv}^{\omega^*}(\{\mu_n, \ n \in I\!N\})$ and $D := \overline{conv}^{\omega^*}(\{x_n^*, \ n \in I\!N\})$. Let K be the subset of D of all x^* such that $\lim_{n\to\infty}\langle x_n, x^*\rangle = 0$. It is simple to prove that K is a bounded, convex and norm-closed subset of X^*. We claim that K has no extreme points. To prove the claim, observe that K is an extremal subset of D (i.e., $x^* \in K, \ y^* \in K$ as soon as $(x^*+y^*)/2 \in K, \ x^* \in D, \ y^* \in D$), hence any possible extreme point x^* of K is an extreme point of D, so there is an extreme point λ of C such that $T^*(\lambda) = x^*$. By Milman's Theorem (see, for example, [KÖ69, §24]), $\lambda \in \overline{\{\mu_n : \ n \in I\!N\}}^{\omega^*} \setminus \{\mu_n : \ n \in I\!N\}$, as every μ_n is certainly not an extreme point.

It is clear that $\mu_n(\chi_{\Delta_m}) \in \{0,1\}, \ n \geq m$, hence $\lambda(\chi_{\Delta_m}) \in \{0,1\}, \ \forall m \in I\!N$. As λ is a probability measure, $\lambda(\chi_{\Delta_m}) = 1$ infinitely often. Then we have

$$\langle x_m, x^*\rangle = \langle x_m, T^*(\lambda)\rangle =$$

$$= \int_\Delta \chi_{\Delta_m} d\lambda + \int_\Delta (T(x_m) - \chi_{\Delta_m}) d\lambda = 1 + \int_\Delta (T(x_m) - \chi_{\Delta_m}) d\lambda$$

(infinitely often). It is plain now that x^* does not belong to K, a contradiction. This proves the theorem.

$\square$

5.2 Spaces with a GD norm nowhere FD.

Let's recall the classical definitions of differentiability of the norm and related notions:

Definition 21 *The norm $\|\cdot\|$ in a Banach space X is* Gâteaux differentiable *(in short, GD) if, for every $x \neq 0$ in X and every $h \in X$, the following limit exists:*

$$\lim_{t\to 0} \frac{\|x+th\| - \|x\|}{t}.$$

The norm is called Fréchet differentiable *(briefly, FD) if the former limit is, for every $x \neq 0$ in X, uniform in $h \in S_X$. If, instead, given a direction h, the limit is uniform in $x \in S_X$, the norm is called* uniformly Gâteaux differentiable *(in short, UGD).*

Definition 22 *A Banach space is* uniformly rotund *(in short, UR) [*weakly uniformly rotund *(briefly, WUR)] whenever given two sequences (x_n) and (y_n) in S_X, such that $\|\frac{1}{2}(x_n + y_n)\|$ converges to 1, then $x_n - y_n$ converges to 0 in norm [weakly].*

A classical result due to Šmulyan says that a norm is UGD if and only if its dual norm is W^*UR, and is WUR if and only if its dual norm is UGD (see, for example, [DGZ93, Theorem II.6.7]).

Answering in the affirmative an old problem of S. Mazur, R. Phelps (see [PH89]) gave an example (the space ℓ_1 endowed with an equivalent norm) of a Banach space with a norm GD nowhere FD. Later, the following characterization of such class of separable Banach spaces was given in [DGHZ87] (see, also, [DGZ93, Theorem III.1.9]):

Theorem 23 (Deville/Godefroy/Hare/Zizler [DGHZ87]) *Let X be a separable Banach space. The following conditions are equivalent:*

1. *X has an equivalent GD norm nowhere FD.*
2. *There exists $\epsilon > 0$ and a convex ω^*-compact subset K of X^* such that every non-empty ω^*-open subset of K has $\|\cdot\|$-diameter $> \epsilon$.*
3. *There exists in X an equivalent rough GD norm.*

The class of spaces described by the theorem above are in between the separable Banach spaces containing a copy of ℓ_1 and the separable Banach spaces with a non-separable dual. This is the statement in the next result. The implications cannot be reversed, as it can be deduced from examples in [GMS]. We shall return to this later on.

Theorem 24 *Let X be a separable Banach space. Let's consider the following properties:*

1. *X contains a copy of ℓ_1.*
2. *X has an equivalent GD norm nowhere FD.*
3. *X has a non-separable dual ($\equiv$ X is not Asplund).*

Then, $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$ and none of these implications can be reversed [3].

[3] The class of separable Banach spaces X having an equivalent GD norm nowhere FD and its location between the separable Banach spaces containing ℓ_1 and the separable Banach spaces with a non-separable dual raises the following question, in the light of characterizations for the later classes given in Theorems 6 and 4, respectively: is it possible to characterize this class by the existence of a suitable Cantor set in (B_{X^*}, ω^*), together with a bounded sequence of functions from X?

Proof: Assume (1). By Theorem 6, given $\epsilon > 0$ (say $\epsilon = 1/3$), there exists a Cantor set Δ of (B_{X^*}, ω^*) with the properties listed there. Let $K := \overline{conv}^{\omega^*}(\Delta)$, a ω^*-closed convex subset of B_{X^*}. Let A be a non-empty relatively open subset of $K[\omega^*]$. A certainly contains a convex combination $x^* := \sum_{i=1}^n \alpha_i x_i^*$ of elements $x_i^* \in \Delta,\ 1 \leq i \leq n$. As A is open, it is possible to choose $y_i^*,\ 1 \leq i \leq n$, in Δ, such that $y^* := \sum_{i=1}^n \alpha_i y_i^* \in A$. Select $m \in I\!N$, $m > 3$, big enough such that each x_i^* and y_j^* lie in different $\Delta_{m,k}$'s. It is plain now that $\|x^* - y^*\| > 1/3$. This proves, using Theorem 23, that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is clear from the definition of Asplund space.

$\square$

The following theorem has been called Great Baire Theorem (see, for example, [DGZ93]):

Theorem 25 (Baire) *Let Z be complete metric space, X a normed space, $f : Z \to X$ a mapping from Z into X. The following assertions are equivalent:*

1. *The restriction of f to any non-empty closed subset F of Z has a point of continuity.*
2. *The restriction of f to any non-empty closed subset F of Z has a dense G_δ subset of points of continuity.*
3. *f is the pointwise limit of a sequence (f_n) of continuous functions*
$$f_n : Z \to X,\ n \in I\!N.$$
4. *Given any non-empty closed subset F of Z, and any $\epsilon > 0$, there exists a non-empty relatively open subset G of F where the oscillation of f is less than ϵ.*

Observe that, in case of a separable Banach space X, (B_{X^*}, ω^*) is a compact metric space. Let $f := Id(\equiv Identity) : (B_{X^*}, \omega^*) \to (X^*, \|\cdot\|)$. Condition 4 is the definition of norm fragmentability of (B_{X^*}, ω^*). Therefore, it is equivalent to the fact that the restriction of f to any non-empty closed subset of (B_{X^*}, ω^*) has a point of continuity.

A Banach space X has PCP (*point of continuity property*) whenever every non-empty closed and bounded subset S of X has a weak-norm point of continuity (for the identity mapping).

A space has CPCP (*convex point of continuity property*) whenever S can be taken to be convex in the former definition.

A dual Banach space X^* has C*PCP whenever every ω^*-compact and convex subset C has a point of continuity for $Id : (C, \omega^*) \to (C, \|\cdot\|)$.

Observe that P*CP is just RNP, from Stegall's results [ST75],[ST81].

A Banach space is called a *Phelps space* whenever every real continuous and convex function on X which is GD, is FD at the points of a dense subset. It is the same thing to say that every equivalent GD norm is FD at the points of a dense subset. Therefore, Theorem 23 characterizes the separable Banach spaces which are not Phelps:

Theorem 26 (Deville/Godefroy/Hare/Zizler [DGHZ87]) *A separable Banach space X is a Phelps space if, and only if, X^* has C^*PCP.*

Examples. In [GMS] two examples are provided which allow to separate the class of Phelps separable Banach spaces from Asplund spaces and spaces which do not contain a copy of ℓ_1. To be more precise, there exists a separable Banach space which is not Asplund but its dual has C*PCP, so it has no GD rough norm. On the other hand, there exists a separable Banach space X such that its dual has not C*PCP but X does not contains a copy of ℓ_1.

The first example is JT ($\equiv$ James tree), a space of real functions defined on a dyadic tree. Now, in T consider the partial ordering in which $(0,1)$ is the minimal element and every element (n,i) has exactly two immediate successors $(n+1,2i-1)$ and $(n+1,2i)$. Elements in JT are those functions $f : T \to I\!R$ such that

$$\|f\| := \sup\{(\sum_{j=1}^{k}(\sum_{\mathcal{S}_j}|f(n,i)|)^2)^{1/2}\},$$

where the supremum is taken on all finite families $(\mathcal{S}_j)_{j=1}^{k},\ k = 1,2,\dots$ of segments pairwise disjoint in T. A *segment* is a finite totally ordered subset of T formed by consecutive elements.

The second example has a more difficult description. It provides a separable Banach space X such that it has a separable dual X^*, X^{**} has not C*PCP and X^{**}/X is reflexive. It follows that X^* has an equivalent GD nowhere FD norm. However, X^* does not contain a copy of ℓ_1.

5.3 Banach spaces with WUR norm and Asplund spaces.

It was an open question for some time whether any Banach space with WUR norm (see Definition 22) should be Asplund. The answer is affirmative: the following theorem is due to P. Hájek [HAJ96], and its proof uses Theorem 4:

Theorem 27 (Hájek [HAJ96]) *Every Banach space with WUR norm is Asplund.*

Proof: Let X be a separable Banach space such that X^* is not separable. By Theorem 4, and given $\epsilon > 0$, there exists a subset Δ of (B_{X^*}, ω^*), topologically homeomorphic to the Cantor set, and a subset $\{x_{n,i} :\ (n,i) \in T\}$ of $(1+\epsilon)B_X$ such that

$$\|x_{n,i} - \chi_{\Delta_{n,i}}\|_\infty < \epsilon/2^n,\ (n,i) \in T.$$

Let's define a partial ordering in $I\!N$: say $n_1 \ll n_2$ if it is possible to find a descending path from n_1 to n_2 along arrows in Figure 2.

We shall order (and renumber) vectors $x_{n,i}$ and dyadic subsets $\Delta_{n,i}$ accordingly. This time we shall allow branches to start at any level $n \geq 0$, and we shall understand that any branch goes down forever. Let $\mathcal{B}$ be the set of all branches. Given $b \in \mathcal{B}$, let x_b^* be the element in Δ defined by such a branch: precisely

$$x_b^* := \cap_{n \in b} \Delta_n.$$

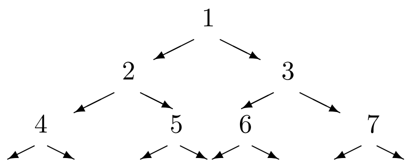

Figure 2: A partial ordering in $I\!\!N$.

Claim: *Let $b = \{b_1 \ll b_2 \ll \ldots\}$ be a branch in $\mathcal{B}$. Let $\delta > 0$. Then, there exists natural numbers n_b and m_b and convex combinations*

$$x_b := \sum_{b_1 \ll n \ll n_b} \alpha_n x_n, \quad y_b := \sum_{n_b \ll n \ll m_b} \alpha_n x_n,$$

such that

$$2\|x_b\|^2 + 2\|y_b\|^2 - \|x_b + y_b\|^2 < \delta.$$

To prove the Claim, given $b \in \mathcal{B}$ let's define $M_b : b \to I\!\!R$ by

$$M_b(n) := \inf\{\|x\| : \ x = \sum_{n \ll i \ll m} \gamma_i x_i \text{ (a convex combination) }, \ m \in b\}.$$

Obviously, $M_b(n) \leq 1 + \epsilon$, $\forall n \in b$, and $M_b(\cdot)$ increases along b.

Let $\rho > 0$. Choose a natural number n_ρ and a convex combination $x_b := \sum_{n_\rho \ll n \ll n_b} \alpha_n x_n$ (this defines n_b) such that

$$\sup_{k \in b} M_b(k) - \rho < M_b(n_\rho) \leq \|x_b\| < M_b(n_\rho) + \rho \leq \sup_{k \in b} M_b(k) + \rho.$$

Choose now a convex combination $y_b := \sum_{n_b \ll n \ll m_b} \beta_n x_n$ such that

$$\sup_{k \in b} M_b(k) - \rho < M_b(n_\rho) \leq M_b(n_b) \leq \|y_b\| < M_b(n_b) + \rho \leq \sup_{k \in b} M_b(k) + \rho.$$

It follows that

$$\sup_{k \in b} M_b(k) - \rho < M_b(n_\rho) \leq \|\frac{x_b + y_b}{2}\| < \frac{1}{2}\left[\|x_b\| + \|y_b\|\right] \leq \sup_{k \in b} M_b(k) + \rho.$$

Choosing ρ small enough,

$$2\|x_b\|^2 + 2\|y_b\|^2 - \|x_b + y_b\|^2 < \delta,$$

so the Claim is true.

To finish the proof of the theorem, let (δ_n) be a null decreasing sequence of real numbers. Choose first a branch b_1 starting at 1. Take $\delta := \delta_1$ and apply the Claim to the branch b_1: we get x_1 (a convex combination with indices between 1 and n_1) and y_1 (again a convex combination with indices between n_1 and m_1).

Let $b_2 \in \mathcal{B}$ the branch starting at n_1 and disjoint with indices in y_1. Let $\delta := \delta_2$ and apply again the Claim. Follow this procedure... Finally, let b_0 be the branch starting at 1 and containing $\{n_1, n_2, n_3, \ldots\}$. The element $x^*_{b_0}$ satisfies

$$\begin{cases} \langle x_1, x^*_{b_0} \rangle \approx 1, \\ \langle y_1, x^*_{b_0} \rangle \approx 0, \end{cases}$$

$$\begin{cases} \langle x_2, x^*_{b_0} \rangle \approx 1, \\ \langle y_2, x^*_{b_0} \rangle \approx 0, \end{cases} \quad \text{better approximation than before,}$$

$$\ldots$$

hence

$$2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0,$$

while $x_n - y_n$ does not converge to zero in the topology ω^*, so the norm is not WUR. Obviously, the non-separable case is a consequence of this.

□

Using the fact that a separable Banach space with separable dual can be renormed to be WUR [Z71] (see, also [DGZ93, p. 65]), the following holds:

Theorem 28 *Let X be a separable Banach space. The following conditions are equivalent:*

1. *X is Asplund.*
2. *X has an equivalent WUR norm.*

Quite recently, using Hájek's Theorem 27, M. Fabian, P. Hájek and V. Zizler proved [FHZ] that a WUR Banach space has a dual which is a subspace of a WCG ($\equiv$ weakly compactly generated), and moreover that it has an equivalent WUR norm, if and only if the closed unit ball in the bidual, endowed with the topology ω^*, is a uniform Eberlein compact (i.e., homeomorphic to a weakly compact subset of a Hilbert space).

5.4 WUR spaces and weakly K-analytic spaces.

The introduction of the concept of a K-analytic space is due to G. Choquet. He defines a K-analytic topological space as the continuous image of a $K_{\sigma\delta}$ subset of a compact space (see, also, page 16). As it has been mentioned already, M. Talagrand, in [TA79], considers Banach spaces which are K-analytic in its weak topology (together with a generalization, due to L. Vašák [VA81], the spaces called WCD ($\equiv$ weakly countable determined)). Ch. Stegall gives an alternative definition of a K-analytic space as the image by an usco (i.e., upper semicontinuous and compact valued) mapping of $\mathbb{N}^{\mathbb{N}}$. If the domain is a subset of $\mathbb{N}^{\mathbb{N}}$ then the class corresponds to the WCD spaces. B. Cascales [CA87] proves, among some other things, that in a locally convex space such that its Mackey topology is angelic (alternatively, that it is quasicomplete in its Mackey topology) it is the same to say that the space is K-analytic and that there exits an increasing (for the pointwise order) family of compact subsets with indices in $\mathbb{N}^{\mathbb{N}}$. A Banach space in its weak topology satisfies any of the required conditions.

A weakly K-analytic Banach space is, of course, WCD. Theorem 29 says that if X is a Banach space such that its dual has a norm (not necessarily dual norm) UGD, then X^* is weakly K-analytic, so X^* is WCD and, henceforth, X has an equivalent LUR norm such that its dual norm is also LUR (see [FT90]). From there, it follows that X has a norm which is simultaneously LUR and FD. Hájek's result (Theorem 27) follows, too, because if X^* is weakly K-analytic, X is Asplund.

A way to prove the last assertion (using ideas in [CO91]) is the following: Without loss of generality, we can assume that X is a separable Banach space. Consider the existence of a increasing mapping with compact values

$$T : \mathbb{N}^{\mathbb{N}} \to (B_{X^*}, \omega)$$

which covers the space B_{X^*}. Then, the natural projection

$$P : \mathbb{N}^{\mathbb{N}} \times (B_{X^*}, \omega^*) \to (B_{X^*}, \omega)$$

has a continuous restriction to $S := \{(\alpha, x^*) : \ x^* \in T(\alpha)\}$ hence (B_{X^*}, ω) is separable, so X^* is also separable.

We have the following result:

Theorem 29 (Moltó/Montesinos/Orihuela/Troyanski [MMOT98]) *Let X be a Banach space such that in X^* there exists an equivalent (not necessarily dual) UGD norm. Then X^* is weakly K-analytic.*

Using some ideas in the proof of the previous theorem, M. Fabian and V. Zizler have proved recently the following density theorem, from which, again, Hájek's result (Theorem 27) follows:

Theorem 30 (Fabian/Zizler [FZ]) *Assume that a Banach space X has uniformly Gâteaux smooth norm. Let $\|\cdot\|$ be any equivalent norm on X. Then the density of X is equal to the ω^* density of the dual unit ball $B_{X^*} := \{x^* : \|x^*\| \leq 1\}$.*

Quite recently, G. Godefroy provided a wonderful elementary proof of Theorem 27:

Proof of Theorem 27: (Godefroy) As it was mentioned in the Introduction, it is enough to show that if the norm of a separable Banach space X is WUR, then X^* is separable.

For $n \in \mathbb{N}$ put

$$V_n = \{x^* \in B_{X^*} : \ |\langle x - y, x^* \rangle| \leq \frac{1}{3} \text{ whenever } x, y \in B_X \text{ are such that } \|x + y\| \geq 2 - \tfrac{1}{n}\}.$$

Note that $B_{X^*} = \bigcup V_n$.

Since (B_{X^*}, ω^*) is a metric compact space, for every $n \in I\!N$ there is a countable ω^*-dense set S_n in V_n. We claim that $\overline{\text{span}}\{\bigcup S_n\} = X^*$.

Assume that this is not the case and find $x^{**} \in S_{X^{**}}$ such that $\langle x^{**}, x^* \rangle = 0$ for all $x^* \in \bigcup S_n$. Choose $x_0^* \in S_{X^*}$ such that $\langle x^{**}, x_0^* \rangle > \frac{8}{9}$. Assume that $x_0^* \in V_{n_0}$. Choose $\{x_\alpha\}$ a net in B_X such that $x^{**} = \omega^* - \lim x_\alpha$. From the ω^*-lower semicontinuity of the second dual norm, assume that α_0 is such that $\|x_\alpha + x_\beta\| \geq 2 - \frac{1}{n}$ for every $\alpha \geq \alpha_0$ and $\beta \geq \alpha_0$. Hence $|\langle (x_\alpha - x_\beta), x^* \rangle)| \leq \frac{1}{3}$ for all $\alpha \geq \alpha_0$ and $\beta \geq \alpha_0$ and $x^* \in V_{n_0}$.

It follows that $|\langle x^{**} - x_{\alpha_0}, x^* \rangle| \leq \frac{1}{3}$ for all $x^* \in V_{n_0}$. Thus for $x^* \in S_{n_0}$, we have

$$
\begin{aligned}
&|\langle x_{\alpha_0}, x^* - x_0^* \rangle| = \\
&\quad |\langle x^{**}, x^* \rangle - \langle x^{**}, x_0^* \rangle + \langle x_{\alpha_0}, x^* \rangle - \langle x^{**}, x^* \rangle + \langle x^{**}, x_0^* \rangle - \langle x_{\alpha_0}, x_0^* \rangle| \geq \\
&\quad |\langle x^{**}, x^* \rangle - \langle x^{**}, x_0^* \rangle| - |\langle x_{\alpha_0}, x^* \rangle - \langle x^{**}, x^* \rangle| - |\langle x^{**}, x_0^* \rangle - \langle x_{\alpha_0}, x_0^* \rangle| \geq \\
&\qquad \geq \frac{8}{9} - \frac{6}{9} = \frac{2}{9},
\end{aligned}
$$

which contradicts the fact that $x_0^* \in \overline{S_{n_0}}^{\omega^*}$. This completes the proof.

□

5.5 Checking ℓ_1 on extreme points.

Banach spaces containing a copy of ℓ_1 have been characterized in several ways, both in the separable and the non-separable case. In the second situation, a theorem due to Haydon [HA76] says that a Banach space does not contain a copy of ℓ_1 if, and only if, every $x^{**} \in X^{**}$ has the Point of Continuity Property ($\equiv$PCP), i.e., given any weak star closed subset F of B_{X^*}, the closed dual unit ball, the restriction of u to F has at least a point of weak star continuity.

It is possible, at least in the separable case, to precise the location of the pathological subset of the dual (if the Banach space contains a copy of ℓ_1). This is done in Theorem 31 by using the aforesaid result of Bourgain and Delbaen (Theorem 11).

Theorem 31 *Let X be a separable Banach space. The following statements are equivalent:*

1. *X contains an isomorphic copy of ℓ_1.*
2. *There exists Δ, a subset of $(Ext(B_{X^*}), \omega^*)$ homeomorphic to the Cantor ternary set, and there exists $x^{**} \in X^{**}$ such that $x^{**}|_\Delta$ has no point of continuity.*

Proof. (1) $\Rightarrow$ (2). By Theorem 11, given $\epsilon > 0$ it is possible to choose an appropriate $\Delta_\epsilon \subset \text{Ext}(B_{X^*})$ and a sequence (x_n) in B_X such that

$$\langle x_n, x^* \rangle = (-1)^i (1 - \epsilon), \;\; x^* \in \Delta_{n,i}, \;\; (n, i) \in T.$$

Let $\mathcal{U}$ be a non-trivial ultrafilter on $I\!N$. Let

$$x^{**} := \omega^* - \lim_{\mathcal{U}} x_n \in X^{**}.$$

We shall prove that x^{**} restricted to $(\Delta_\epsilon, \omega^*)$ has no point of continuity: Assume, on the contrary, that $x^{**}|_{\Delta_\epsilon}$ is continuous at $x_0^* \in \Delta_\epsilon$. As x^{**} takes only two values ($1-\epsilon$ and $-1+\epsilon$), it is possible then to find a dyadic subset $\Delta_{n_0,i}$ where x^{**} is a constant (say $1-\epsilon$). It is easy now to find $x^* \in \Delta_{n_0,i}$ such that $\langle x_n, x^* \rangle = -1+\epsilon$, $\forall n > n_0$. Now, there exists $U \in \mathcal{U}$ such that $\langle x_n, x^* \rangle = \langle x^{**}, x^* \rangle$, $\forall n \in U$. This is a contradiction, as $\{n \in I\!N : \ n > n_0\} \cap U \neq \emptyset$.

(2) $\Rightarrow$ (1). This is a consequence of Corollary III.3.3 in [DGZ93].

□

Remark. In fact, the former proof allows to say that if X is a separable Banach space with an isomorphic copy of ℓ_1, something a little bit more precise is true: $x^{**} \in X^{**}$ can be found such that for every point $x_0^* \in \Delta_\epsilon$ there exists a ω^*-neighborhood of x_0^* in $(\Delta_\epsilon, \omega^*)$ where the oscillation of x^{**} is greater or equal than $2-2\epsilon$. This will be used in the sequel.

Rainwater's Theorem says that a bounded sequence in a Banach space X weakly converges to a point u in X if it converges on the extreme points of the dual unit ball. The result is not longer true if u belongs to $X^{**} \setminus X$. This can be easily seen in $C(K)$, K a compact space such that a regular Borel measure with non-separable support exits on K. Then, $\overline{lin}^{\|\cdot\|}(\mathrm{Ext}(B_{M(K)})) \neq M(K)$, where $M(K)$ denotes the dual space of $C(K)$. The Separation Theorem gives $u \in S_{X^{**}}$ which vanishes on $\mathrm{Ext}(B_{M(K)})$, and u is, on the extreme points of $B_{M(K)}$, the limit of the zero sequence. However, the validity of Rainwater's Theorem "going to the bidual" relates, at least in the separable case, to Banach spaces without isomorphic copies of ℓ_1: A classical result characterizes those spaces: (a) [OR75]*a separable Banach space X does not contain an isomorphic copy of ℓ_1 if and only if every point $x^{**} \in X^{**}$ is the weak star limit of a sequence (x_n) in X.* Using the former theorem we can formulated the next

Corollary 32 *Let X be a separable Banach space. The following are equivalent:*

1. *The space X does not contain an isomorphic copy of ℓ_1.*
2. *For every $x^{**} \in X^{**}$ there exists a sequence (x_n) in X which converges to x^{**} on the extreme points of B_{X^*}.*

Proof. (1) $\Rightarrow$ (2) is trivial after the result (a). To prove (2) $\Rightarrow$ (1) assume X contains an isomorphic copy of ℓ_1. By Theorem 31 there exists $x^{**} \in X^{**}$ and $\Delta \subset \mathrm{Ext}(B_{X^*})$ such that $x^{**}|_{(\Delta,\omega^*)}$ has no point of continuity. Assume that there exists a sequence (x_n) in X such that $\langle x^{**} - x_n, x^* \rangle \to 0$ as $n \to \infty$ for every $x^* \in \mathrm{Ext}(B_{X^*})$. As (Δ, ω^*) is metrizable, Baire Great Theorem (Theorem 25) says that x^{**} has points of continuity when restricted to any non-empty closed subset of (Δ, ω^*), which is plainly false.

$\square$

Remark. In the result (a) it is possible to substitute the statement on points of the bidual by the following one: *every point* $x^{**} \in B_{X^{**}}$ *is the weak star limit of a sequence in* B_X. Analogously, it is possible to substitute 2. in Corollary 32 by *For every* $x^{**} \in B_{X^{**}}$ *there exists a sequence* (x_n) *in* B_X *which converges to* x^{**} *on the extreme points of* B_{X^*}.

In Section 4 the concept of a James boundary was introduced. Let's recall that a subset BD of B_{X^*} is called a *boundary* (more precisely, a *James boundary*) if every $x \in X$ attains its supremum on B_{X^*} at some point of BD. Using Simons inequality, G. Godefroy proved the following statement (see, also, Theorem 14): b) *Let* X *be a Banach space. Assume that every* $x^{**} \in B_{X^{**}}$ *is the weak star limit of a sequence* (x_n) *in* B_X. *Then* $\overline{conv}^{\|\cdot\|}(BD) = B_{X^*}$. In case the Banach space X is separable, it is enough to have convergence of sequences on extreme points. This is the content of the following

Corollary 33 *Let* X *be a separable Banach space and let* $BD \subset B_{X^*}$ *be a James boundary. Assume that every element* $x^{**} \in B_{X^{**}}$ *is the limit of a sequence* (x_n) *in* X *for the topology of the pointwise convergence on extreme points of* B_{X^*}. *Then* $\overline{conv}^{\|\cdot\|}(BD) = B_{X^*}$.

Proof. According to Corollary 32 the space X does not contains an isomorphic copy of ℓ_1. It follows that every point $x^{**} \in B_{X^{**}}$ is the weak star limit of a sequence (x_n) in B_X. The result now is a consequence of (b).

$\square$

An interesting result due to E. and P. Saab characterizes Banach spaces with isomorphic copies of ℓ_1 in terms of the oscillation of an element of X^{**} on weak star slices of a certain subset of X^*. Precisely, we have

Theorem 34 (Saab-Saab [SS83]) *Let* X *be a Banach space. Then, the following statements are equivalent:*

1. X *contains an isomorphic copy of* ℓ_1.
2. *There exists* $x^{**} \in X^{**}$, K, *a non-empty subset of* B_{X^*} *and* $\epsilon > 0$ *such that*

   $$osc(x^{**}, S) \geq \epsilon, \quad \textit{for all non-empty weak star section } S \textit{ of } K,$$

   where $osc(x^{**}, S)$ *denotes the oscillation of a function* x^{**} *on a set* S.

Now, Theorem 31 allows us to precise the location of the subset K in the former theorem in case X is a separable Banach space. Just use the previous remark.

Theorem 35 *Let X be a separable Banach space. Then, the following statements are equivalent:*

1. *X contains an isomorphic copy of ℓ_1.*
2. *There exists $x^{**} \in X^{**}$, $K \subset (Ext(B_{X^*}), \omega^*)$, homeomorphic to the Cantor ternary set, and $\epsilon > 0$ such that*

$$osc(x^{**}, S) \geq \epsilon, \quad \textit{for all non-empty weak star section } S \textit{ of } K.$$

5.6 Closures in X^{**} and ℓ_1-sequences.

Let A be a bounded subset of a Banach space X. Let's denote by $\overset{**}{A}$ the closure of A in (X^{**}, ω^*), by $\ddot{A}$ the sequential closure (i.e., the set of all limits of sequences) of A in (X^{**}, ω^*), and by $\tilde{A}$ the set $\bigcup\{\overset{**}{N}\colon\ N \subset A,\ N \text{ countable}\}$.

The following theorem relates those closures with the existence of a sequence in A equivalent to the canonical basis of ℓ_1:

Theorem 36 *Let X be a separable Banach space. Let A be a bounded subset of X. Then, the following statement are equivalent:*

1. *A does not contain any sequence equivalent to the canonical basis of ℓ_1.*
2. *Every $a^{**} \in \overset{**}{A}$ is a Borel function on (B_{X^*}, ω^*).*
3. *Every $a^{**} \in \overset{**}{A}$ is a First Baire Class function on (B_{X^*}, ω^*).*
4. *$\overset{**}{A} = \dddot{A}$.*
5. *$\tilde{A} = \dddot{A}$.*

Proof. (1) $\Rightarrow$ (3): Let $a^{**} \in \overset{**}{A}$. Assume a^{**} is not a First Baire Class function on (B_{X^*}, ω^*). This is a Polish topological space, hence a^{**} satisfies the Discontinuity Criterion on a countable subset $N \subset (B_{X^*}, \omega^*)$ ([RO77]). Precisely, there exists $r \in I\!R$ and $\delta > 0$ such that for every non-empty open set O in (N, ω^*), there exists x_1^* and x_2^* in O such that

$$\langle a^{**}, x_1^* \rangle \leq r, \quad \langle a^{**}, x_2^* \rangle \geq r + \delta. \tag{3}$$

We shall construct a tree of subsets of N by repeatedly using inequalities (3): let's start by taking $N_1 := N$. We can then find $x_{1,1}^*$ and $x_{1,2}^*$ in N_1 such that $\langle a^{**}, x_{1,1}^* \rangle \leq r$, $\langle a^{**}, x_{1,2}^* \rangle \geq r + \delta$. Choose $a_1 \in A$ such that $|\langle a^{**} - a_1, x_{1,i}^* \rangle| < \delta/4$, $i = 1, 2$. a_1 defines disjoint open neighborhoods (say $N_{1,1}$ and $N_{1,2}$) of $x_{1,1}^*$ and $x_{1,2}^*$, respectively. Again by (3) it is possible to find $x_{2,1}^*$, $x_{2,2}^*$ in $N_{1,1}$ and $x_{2,3}^*$, $x_{2,4}^*$ in $N_{1,2}$ such that

$$\langle a^{**}, x_{2,1}^* \rangle \leq r, \quad \langle a^{**}, x_{2,2}^* \rangle \geq r + \delta$$
$$\langle a^{**}, x_{2,3}^* \rangle \leq r, \quad \langle a^{**}, x_{2,4}^* \rangle \geq r + \delta.$$

Choose now $a_2 \in A$ such that $|\langle a^{**} - a_2, x^*_{2,i}\rangle| < \delta/4,\ i = 1,2,3,4$. Proceed in this way to find a sequence (a_n) in A and a tree $N_{n,i},\ (n,i) \in T,\ n \neq 0$ of subsets of N. The sequence (a_n) is equivalent to the canonical basis of ℓ_1 (see, for example, [DI84], Proposition XI,2), a contradiction.

(3) $\Rightarrow$ (4): Let $\mathcal{B}^1(P)$ be the space of all First Baire Class functions on the Polish space $P := (B_{X^*}, \omega^*)$, endowed with the pointwise topology $\mathcal{T}_p$. This is an angelic space ([RO77]). By hypothesis, $\overset{**}{A}$ is a compact subset of $(\mathcal{B}^1(P), \mathcal{T}_p)$, hence $\overset{**}{A} = \overset{\cdots}{A}$.

(4) $\Rightarrow$ (5): This is obvious, as $A \subset \overset{\cdots}{A} \subset \tilde{A} \subset \overset{**}{A}$.

(5) $\Rightarrow$ (1): Let (a_n) be a sequence in A equivalent to the canonical basis of ℓ_1. Let $\mathcal{U}$ be a non-trivial ultrafilter in $I\!N$. Let $u \in X^{**}$ be the weak star limit of (a_n) along the ultrafilter.

Claim: *There exists a non-empty closed subset M of (B_{X^*}, ω^*) such that the restriction of u to M has no point of continuity.*

Proof of the claim: Let $Y := \overline{lin}\{a_n :\ n \in I\!N\} \subset X$. Denote by q the canonical mapping from X^* onto Y^*. Let $T : Y \to \ell_1$ be an isomorphism. The set $\Delta := \{(x_n) :\ x_n = \pm 1, n \in I\!N\} \subset (B_{\ell^\infty}, \omega^*)$ is homeomorphic to the Cantor ternary set. Let $D := T^*(\Delta)$. D is a weak star compact subset of Y^*, and we may and do assume $D \subset B_{Y^*}$. Let M be a) a compact subset of (B_{X^*}, ω^*) and b) $q(M) = D$, and such that M is minimal for this two properties. Assume $u|_M$ has a point of continuity, say $x^*_0 \in M$. Then, for every $\epsilon > 0$ there exists $N(x^*_0) \subset M$, an open neighborhood of x^*_0 in (M, ω^*), such that $osc(u, N(x^*_0)) < \epsilon$. By minimality, it is simple to prove that there exists a non-empty open subset V of (D, ω^*) such that $V \subset q(N(x^*_0))$ and then

$$osc(u, V) \leq osc(u, N(x^*_0)) < \epsilon.$$

There exists an open subset W of (Δ, ω^*) such that $T^*[W] = V$. Starting from an arbitrary $\delta > 0$, it is possible to choose $\epsilon > 0$ in such a way that the former construction with such an ϵ produces V and W with the property that $osc(w, W) < \delta$, where $w := \omega^* - \lim_{\mathcal{U}}(e_n) \in (\ell^\infty)^*$ and $(e_n)_{n=1}^\infty$ is the canonical basis of ℓ_1. Now, by the argument in the proof of Theorem 31, $osc(w, W) = 2$, and we reach a contradiction. This proves the Claim.

Now, (B_{X^*}, ω^*) is a Polish space. We can use Baire Great Theorem (Theorem 25) to conclude that u is not the weak star limit of a sequence in X. It follows that $u \notin \overset{\cdots}{A}$. However, $u \in \tilde{A}$, a contradiction.

(3) $\Rightarrow$ (2): This is obvious, as every First Baire Class function is also Borel.

(2) $\Rightarrow$ (5): Let $a^{**} \in \tilde{A}$. There exists a sequence (a_n) in A such that $a^{**} \in \overset{**}{\{a_n\}}$. As every element of $\overset{**}{\{a_n\}}$ is Borel, we can use Lemma III.3.4 in [DGZ93] in order to find a subsequence (a_{n_k}) of (a_n) such that $\omega^* - \lim_k a_{n_k} = a^{**}$. It follows that $a^{**} \in \overset{\cdots}{\{a_n\}}$.

$\square$

Remark. Each of the conditions in the former theorem implies, of course, that $\overset{**}{A} = \tilde{A}$. However, this condition is not equivalent to the others: let X be a separable Banach space with an isomorphic copy of ℓ_1. Let (a_n) be a sequence in X equivalent to the canonical basis of ℓ_1. Obviously $\overset{**}{A} = \tilde{A}$, where $A := \{a_n : n \in I\!N\}$, but none of the (equivalent) conditions of the former theorem are satisfied.

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Manuel López-Pellicer.
Depto. Matemática Aplicada (ETSIA), Apartado 22012,
E-46071 Valencia (SPAIN).
mlopezpe@mat.upv.es

Vicente Montesinos.
Depto. Matemática Aplicada (ETSIT), Apartado 22012,
E-46071 Valencia (SPAIN).
vmontesinos@mat.upv.es