Restricted weak upper semicontinuous differentials of convex functions

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Abstract

We characterize the restricted w-upper semicontinuity of the subdifferential of convex functions in terms of the Fenchel biconjugate mapping.

1 Introduction.

Given a convex lower semicontinous function f defined on a real Banach space X, the *subdifferential* of f at $x \in X$ is defined by

$$\partial f(x) := \{x^* \in X^* : \langle y - x, x^* \rangle \leq f(y) - f(x), \forall y \in X\},$$

if $x \in \mathrm{dom}(f)$, while $\partial f(x) = \emptyset$ if $x \in X \setminus \mathrm{dom}(f)$.

A set-valued mapping Φ from a topological space (X, τ') into subsets of another topological space (Y, τ) is said to be $[\tau'$-$\tau]$-*upper semicontinuous* at $x \in X$ if given a τ-open subset W of Y such that $\Phi(x) \subset W$, there exists a τ'-neighbourhood U of x such that $\Phi(U) \subset W$. In this paper we shall always consider X a real Banach space endowed with the norm topology and $Y = X^*$ endowed with a topology τ. We shall write τ-upper semicontinuous instead of $[\|\cdot\|$-$\tau]$-upper semicontinuous.

Given a convex function f on an open subset D of a Banach space X and a point of continuity $x_0 \in D$ of f, it can be proved that $\partial f(x_0)$ is a nonempty,

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w^*-compact and convex subset of X^*, and the mapping $x \mapsto \partial f(x)$ is w^*-upper semicontinuous at x_0.

Gâteaux differentiability and Fréchet differentiability can be characterized in terms of the continuity of the subdifferential mapping: Given a continuous convex function f on an open subset D of a Banach space X and a point $x_0 \in D$, f is Gâteaux differentiable at $x_0 \in A$ if and only if $\partial f(x)$ is a singleton, and f is Fréchet differentiable at x_0 if and only if $\partial f(x)$ is a singleton and the subdifferential mapping $x \mapsto \partial f(x)$ is $\|\cdot\|$-upper semicontinuous at x_0 (for these and related concepts see, for example, [P93]).

If the one sided limit in the definition of the derivative of f at a point x_0 is uniform in every direction, we get a weaker concept than the Fréchet differentiability. More precisely, given a continuous function f defined on an open subset D of a Banach space X, we say (following [FP93] and [GS96]) that f is *strongly subdifferentiable* at $x_0 \in D$ if

$$d^+ f_{x_0}(u) := \lim_{t \to 0+} \frac{f(x_0 + tu) - f(x_0)}{t}$$

is uniform in $\|u\| = 1$. This non-smooth extension of the Fréchet differentiability have found several applications (see for example, [AOPR86], [FP93], [GGS78], [G87], [GMZ95]).

The following definition was introduced in [GGS78]: A set-valued mapping Φ from a Banach space X into the subsets of X^* endowed with the topology τ is said to be *restricted τ-upper semicontinuous* at $x \in X$ if given a τ-neighbourhood W of 0 in X^* there exists an open neighbourhood U of x in X such that $\Phi(U) \subset \Phi(x) + W$.

In [G87] it was proved that given a continuous convex function f defined on an open subset D of a Banach space X, f is strongly subdifferentiable at $x_0 \in D$ if and only if ∂f is restricted $\|\cdot\|$-upper semicontinuous at x_0.

In this note we provide, in the spirit of [GGS78], a characterization of the restricted w-upper semicontinuity of the subdifferential mapping of a convex function by using the Fenchel biconjugate mapping. Notice that a partial characterization was obtained in [GGS78] for the duality mapping (i.e. $x \mapsto \partial\|\cdot\|(x)$). For the use of the concept of restricted w-upper semicontinuity of the subdifferential mapping in questions related to the Asplundness and reflexivity of a Banach space we refer to [BM99], [CP94], [GGS78], [GM96] and references therein.

Given a continuous convex function f on an open convex subset D of a Banach space X we can extend f to a lower semicontinuous convex function on X, denoted again by f, by defining

$$f(x) := \begin{cases} \liminf_{y\to x} f(y) & \text{for } x \in \overline{D}, \\ +\infty & \text{otherwise.} \end{cases}$$

Given a convex, proper, lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ the *Fenchel conjugate* of f is defined by

$$f^*(x^*) := \sup\{\langle x, x^*\rangle - f(x) : x \in X\}.$$

Now f^* is again convex, proper and lower semicontinuous (in fact, lower w^*-semicontinuous). Obviously $\langle x, x^*\rangle \leq f(x) + f^*(x^*)$ for all $x \in X$, $x^* \in X^*$ (and the inequality becomes equality if and only if $x^* \in \partial f(x)$). Moreover, if $\epsilon \geq 0$, then $x^* \in \partial_\epsilon f(x)$ if and only if $f(x) + f^*(x^*) \leq \langle x, x^*\rangle + \epsilon$ (where $\partial_\epsilon f$ denotes de ϵ-subdifferential). Also, $f^{**}{}_{|X} = f$ (see [B83] and [P93]).

2 Preliminary results.

We shall need the following results:

Theorem 2.1 (Brøndsted-Rockafellar) *Suppose that f is a convex proper lower semicontinuous function on the Banach space X. Then given any point $x_0 \in \mathrm{dom}(f)$, $\epsilon > 0$ and any $x_0^* \in \partial_\epsilon f(x_0)$, there exists $x_\epsilon \in \mathrm{dom}(f)$ and $x_\epsilon^* \in X^*$ such that*

$$x_\epsilon^* \in \partial f(x_\epsilon), \quad \|x_\epsilon - x_0\| \leq \sqrt{\epsilon}, \quad \|x_\epsilon^* - x_0^*\| \leq \sqrt{\epsilon}.$$

The Brøndsted-Rockafellar Theorem, together with the local boundedness of the subdifferential mapping at a point of continuity x_0, allows us to interweave the ϵ-subdifferential at x_0 and the subdifferential at a neighbourhood of x_0. The precise relationship is formulated in the next result:

Lemma 2.2 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let x_0 be a point of continuity of f. Then $\forall \epsilon > 0$ there exists $\delta > 0$ such that*

$$\partial f[B(x_0; \delta)] \subset \partial_\epsilon f(x_0) \subset \partial f[B(x_0; \sqrt{\epsilon})] + \sqrt{\epsilon} B_{X^*}.$$

Proof: ∂f is locally bounded at x_0, i.e., there exists $M > 0$ and $N(x_0)$, a neighbourhood of x_0, such that

$$\|x^*\| \leq M, \quad \forall x^* \in \partial f(x), \quad \forall x \in N(x_0).$$

Given $\epsilon > 0$, choose $\delta > 0$ such that

$$B(x_0;\delta) \subset N(x_0), \quad M\delta < \frac{\epsilon}{2}, \quad |f(x) - f(x_0)| < \frac{\epsilon}{2},\ \forall x \in B(x_0;\delta).$$

Let $x^* \in \partial f[B(x_0;\delta)]$, say $x^* \in \partial f(x)$ for some $x \in B(x_0;\delta)$. Then

$$\begin{aligned}&\langle y - x_0, x^*\rangle = \langle y - x, x^*\rangle + \langle x - x_0, x^*\rangle \leq \\ &\quad \leq f(y) - f(x) + \|x^*\|\|x - x_0\| < f(y) - f(x_0) + |f(x_0) - f(x)| + M\delta < \\ &\quad < f(y) - f(x_0) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = f(y) - f(x_0) + \epsilon,\end{aligned}$$

hence $x^* \in \partial_\epsilon f(x_0)$.

The second inclusion is the Brøndsted-Rockafellar Theorem, and does not need the continuity of f at x_0. ∎

The following proposition can be found in [P93]:

PROPOSITION 2.3 *Let $f : D \to \mathbb{R}$ be a convex function on D (a non-empty open and convex subset of X), continuous at $x_0 \in D$. Then, for all $y \in X$,*

$$d^+ f_{x_0}(y) = \sup\{\langle y, x^*\rangle : x^* \in \partial f(x_0)\}$$

and this supremum is attained at some point $x^* \in \partial f(x_0)$.

PROPOSITION 2.4 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex, proper and lower semicontinuous funcion. Then* $\operatorname{epi}(f^{**}) = \overline{\operatorname{epi}(f)}^{w^*}$.

Proof: First, assume $f \geq 0$. The inclusion $\overline{\operatorname{epi}(f)}^{w^*} \subset \operatorname{epi}(f^{**})$ follows from $\operatorname{epi}(f) \subset \operatorname{epi}(f^{**})$ and the lower w^*-semicontinuity of f^{**}. Let $(x_0^{**}, \lambda_0) \in \operatorname{epi}(f^{**})$. Suppose that $(x_0^{**}, \lambda_0) \notin \overline{\operatorname{epi}(f)}^{w^*}$. By the Hahn-Banach Theorem, there are $x_0^* \in X^*$, $k,\ \alpha,\ \beta \in \mathbb{R}$ such that:

$$\langle x_0^{**}, x_0^*\rangle + k\lambda_0 < \alpha < \beta < \langle x^{**}, x_0^*\rangle + k\lambda, \quad \forall\ (x^{**}, \lambda) \in \overline{\operatorname{epi}(f)}^{w^*}. \qquad (1)$$

From (1) we get $k \geq 0$ (if $k < 0$, it is enough to take $x \in \text{dom}(f)$ and $\lambda \to +\infty$ in order to obtain a contradiction). In particular, from (1), we get $\langle x, x_0^* \rangle + kf(x) > \beta$, for all $x \in \text{dom}(f)$. Take $\epsilon > 0$. Since $f \geq 0$, we get

$$\langle x, -\frac{x_0^*}{k+\epsilon} \rangle - f(x) < -\frac{\beta}{k+\epsilon}, \qquad \forall\, x \in \text{dom}(f),$$

hence $f^*(-\frac{x_0^*}{k+\epsilon}) \leq -\frac{\beta}{k+\epsilon}$. Then

$$\begin{aligned} f^{**}(x_0^{**}) &\geq \langle x_0^{**}, -\frac{x_0^*}{k+\epsilon} \rangle - f^*(-\frac{x_0^*}{k+\epsilon}) \geq \\ &\geq \langle x_0^{**}, -\frac{x_0^*}{k+\epsilon} \rangle + \frac{\beta}{k+\epsilon} = \frac{1}{k+\epsilon}[\beta - \langle x_0^{**}, x_0^* \rangle] > \frac{\beta - \alpha + k\lambda_0}{k+\epsilon}. \end{aligned}$$

If $k = 0$, then $f^{**}(x_0^{**}) > (\beta - \alpha)/\epsilon$. As $\epsilon > 0$ was arbitrary, we get $x_0^{**} \notin \text{dom}(f^{**})$, a contradiction. If $k \neq 0$, since $\epsilon > 0$ was arbitrary, we get $f^{**}(x_0^{**}) \geq (\beta - \alpha + k\lambda_0)/k > \lambda_0$. This contradicts $(x_0^{**}, \lambda_0) \in \text{epi}(f^{**})$.

Now, if $f : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary proper semicontinuous convex function, choose $x_0^* \in \text{dom}(f^*)$. Consider $g : X \to \mathbb{R} \cup \{+\infty\}$ given by $g(x) = f(x) + f^*(x_0^*) - \langle x, x_0^* \rangle$. This function, obviously, is proper, lower semicontinuous and convex. Moreover $\text{dom}(f) = \text{dom}(g)$ and $g \geq 0$. Now, a simple calculation shows $g^{**}(x^{**}) = f^{**}(x^{**}) + f^*(x_0^*) - \langle x^{**}, x_0^* \rangle$ for all $x^{**} \in X^{**}$. By the first part of the proof, the proposition holds for g, and hence for f. ■

Remarks:

1. Note that Goldstine's Theorem is a particular case of the former proposition: It is enough to take as f the indicator function δ_{B_X} of the closed unit ball of X (i.e., $\delta_{B_X}(x) = 0$ if $\|x\| \leq 1$, $\delta_{B_X}(x) = +\infty$ if $\|x\| > 1$), a proper lower semicontinuous convex function. Obviously, f^* is the dual norm. Let $x^{**} \in B_{X^{**}}$. As

$$f^{**}(x^{**}) = \sup\{\langle x^{**}, x^* \rangle - \|x^*\| : x^* \in X^*\} \leq 0 < +\infty,$$

we get $x^{**} \in \text{dom}(f^{**})$. By Proposition 2.4, $\text{dom}(f^{**}) = \overline{\text{dom}(f)}^{w^*} = \overline{B_X}^{w^*}$.

2. This proposition gives a description of f^{**}, sometimes simpler than the original one.

COROLLARY 2.5 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then, given $x_0 \in X$,*

1. $\partial f(x_0) = \partial f^{**}(x_0) \cap X^*$.

2. *If f is continuous at x_0, f^{**} is also continuous at x_0.*

Proof: (1) is a consequence of two well-known facts (see [P93]): f^{**} induces f on X, and $x_0^* \in \partial f(x_0)$ if and only if $\langle x_0, x_0^* \rangle = f(x_0) + f^*(x_0^*)$.

To prove (2), assume f (but not f^{**}) is continuous at x_0. Let $\mathcal{N}$ be a basis of w^*-open neighbourhoods of 0 in X^{**}. Then there exists $\epsilon > 0$ and $x_N^{**} \in x_0 + N$, $N \in \mathcal{N}$, such that $|f^{**}(x_N^{**}) - f(x_0)| \geq \epsilon$. As $\overline{\mathrm{epi}(f)}^{w^*} = \mathrm{epi}(f^{**})$ and f^{**} is lower semicontinuous, it is possible to choose $x_N \in (x_0 + N) \cap X$ such that $f^{**}(x_N^{**}) \leq f(x_N) < f^{**}(x_N^{**}) + \frac{\epsilon}{2}$, $N \in \mathcal{N}$. It follows that $x_N \xrightarrow{w} x_0$ and $|f(x_N) - f(x_0)| \geq \frac{\epsilon}{2}$, a contradiction. ■

COROLLARY 2.6 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then, given $x_0^{**} \in \mathrm{dom}(f^{**})$,*

$$f^{**}(x_0^{**}) = \inf\{\liminf_i f(x_i) : x_i \subset \mathrm{dom}(f), x_i \xrightarrow{w^*} x_0^{**}\}.$$

Proof: By Proposition 2.4 it is obvious that $\mathrm{dom}(f^{**}) = \overline{\mathrm{dom}(f)}^{w^*}$. Now, given a net $(x_i)_{i\in I} \subset \mathrm{dom}(f)$, $x_i \xrightarrow{w^*} x_0^{**}$, by the w^*-lower semicontinuity of f^{**} we get $f^{**}(x_0^{**}) \leq \liminf_i f(x_i)$. On the other hand, again by Proposition 2.4, given $\epsilon > 0$ we can find a net $(x_i)_{i\in I} \subset \mathrm{dom}(f)$ and $\lambda_i \in \mathbb{R}$ such that $x_i \xrightarrow{w^*} x_0^{**}$, $(x_i, \lambda_i) \in \mathrm{epi}(f)$ and $\lambda_i < f^{**}(x_0^{**}) + \epsilon$. As $f(x_i) \leq \lambda_i$, $i \in I$, we get the conclusion. ■

COROLLARY 2.7 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then, given $x_0 \in \mathrm{dom}(f)$ and $\epsilon > 0$,*

$$\partial_\epsilon f^{**}(x_0) \subset \overline{\partial_{\epsilon+k} f(x_0)}^{X^{***}[w^*]}, \quad \forall k > 0.$$

Proof: Let $x^{***} \in \partial_\epsilon f^{**}(x_0)$. Then $f^{**}(x_0) + f^{***}(x^{***}) \leq \langle x_0, x^{***} \rangle + \epsilon$. It follows that

$$f^{***}(x^{***}) \leq \langle x_0, x^{***} \rangle - f^{**}(x_0) + \epsilon < \langle x_0, x^{***} \rangle - f^{**}(x_0) + \epsilon + \frac{k}{2}.$$

By the previous corollary, there exists a net $(x_i^*)_{i\in I}$ in X^* such that $x_i^* \to x^{***}$ in $X^{***}[w^*]$, $f^*(x_i^*) < \langle x_0, x^{***}\rangle - f^{**}(x_0) + \epsilon + \frac{k}{2}$, $\forall i \in I$ and $|\langle x_0, x_i^* - x^{***}\rangle| < \frac{k}{2}$. We get

$$f(x_0) + f^*(x_i^*) < \langle x_0, x^{***}\rangle + \epsilon + \frac{k}{2} < \langle x_0, x_i^*\rangle + \epsilon + k.$$

Then, $x_i^* \in \partial_{\epsilon+k} f(x_0)$, $\forall i \in I$. The conclusion follows. ■

3 A characterization of the restricted w-upper semicontinuity.

If $x \in X$ and $\delta > 0$, we shall denote by $B^{**}(x_0;\delta)$ the open ball in X^{**} of radius δ and centered at x_0.

Now we are ready to prove the main result in this note:

Theorem 3.1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let x_0 be a point of continuity of f. Then the following assertions are equivalent:*

1. *∂f is restricted w-upper semicontinuous at x_0.*
2. *For all N, a w-neighbourhood of 0 in X^*, there is $\epsilon > 0$ such that $\partial_\epsilon f(x_0) \subset \partial f(x_0) + N$.*
3. *$\partial f(x_0)$ is dense in $\partial f^{**}(x_0)$ in $X^{***}[w^*]$.*
4. *$d^+ f^{**}_{x_0}(\cdot) = \sup\{\langle \cdot, x^*\rangle : x^* \in \partial f(x_0)\}$.*

Proof: (1) ⇒ (2): Let N be a convex w-neighbourhood of 0 in X^*. By hypothesis there is $\delta > 0$ such that $\partial f[B(x_0;\delta)] \subset \partial f(x_0) + \frac{1}{2}N$, $\delta B_{X^*} \subset \frac{1}{2}N$. Now, by Lemma 2.2,

$$\partial_{\delta^2} f(x_0) \subset \partial f[B(x_0;\delta)] + \delta B_{X^*} \subset \partial f(x_0) + \frac{1}{2}N + \frac{1}{2}N \subset \partial f(x_0) + N.$$

It is enough to choose $\epsilon = \delta^2$.

(2) $\Rightarrow$ (3): Given a closed neighbourhood N^{**} of 0 in $X^{***}[w^*]$, let $\epsilon > 0$ be as in (2). Then, using Corollary 2.5, Corollary 2.7 and the fact that $\overline{\partial f(x_0)}^{X^{***}[w^*]}$ is compact and $N := N^{**} \cup X^*$ is closed in $X^{***}[w^*]$,

$$\partial f(x_0) \subset \partial f^{**}(x_0) \subset \partial_{\epsilon/2} f^{**}(x_0) \subset \overline{\partial_\epsilon f(x_0)}^{X^{***}[w^*]} \subset$$
$$\subset \overline{\partial f(x_0) + N}^{X^{***}[w^*]} \subset \overline{\partial f(x_0)}^{X^{***}[w^*]} + N^{**}.$$

This proves (3).

(3) $\Rightarrow$ (1): Let N a w-neighbourhood of 0 in X^*. Take a convex w^*-neighbourhood of 0 in X^{***}, N^*, such that $N^* \cap X^* \subset N$. By Corollary 2.5, f^{**} is continuous at x, so ∂f^{**} is upper w^*-semicontinuous at x, hence there exists $\delta > 0$ such that

$$\partial f^{**}(B^{**}(x;\delta)) \subset \partial f^{**}(x) + \frac{1}{2}N^*.$$

By hypothesis, $\partial f^{**}(x) \subset \partial f(x) + \frac{1}{2}N^*$. It follows that

$$\partial f^{**}(B^{**}(x;\delta)) \subset \partial f^{**}(x) + \frac{1}{2}N^* \subset \partial f(x) + \frac{1}{2}N^* + \frac{1}{2}N^* \subset \partial f(x) + N^*.$$

It is now enough to use Corollary 2.5 to get $\partial f(B(x;\delta)) \subset \partial f(x) + N$.

(3) $\Leftrightarrow$ (4): By Proposition 2.3 and Corollary 2.5,

$$d^+ f^{**}_{x_0}(\cdot) = \sup\{\langle \cdot, x^{***}\rangle : x^{***} \in \partial f^{***}(x_0)\}.$$

Now, using the Hahn-Banach Theorem, the equivalence should be obvious. ■

Note that the equivalence (1) $\Leftrightarrow$ (2) is valid not only for the restricted w-upper semicontinuity, but for the τ-upper upper semicontinuity, τ a Hausdorff topology weaker than the norm-topology. More precisely:

THEOREM 3.2 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let $x_0 \in X$ be a point of continuity of f. If τ is a topology on X^* weaker than the norm topology, then the following assertions are equivalent:*

1. *∂f is restricted τ-upper semicontinuous at x_0.*

2. *For every τ-neighbourhood N of 0 in X^*, there is $\epsilon > 0$ such that $\partial_\epsilon f(x_0) \subset \partial f(x_0) + N$.*

Proof: For (1) $\Rightarrow$ (2) the same proof used in the previous theorem, (1) $\Rightarrow$ (2), works. To prove (2) $\Rightarrow$ (1), use Lemma 2.2. ■

This characterization can be considered as the analogue of the Šmulyan Test.

It is well known that if the dual norm of X^* is locally uniformly rotund, then the norm of X is Fréchet differentiable. The next proposition, that uses the previous theorem, extends this result. Note that the Fenchel conjugate of the norm of a Banach space X is the indicator function of B_{X^*}.

PROPOSITION 3.3 *Let $f : D \to \mathbb{R}$ be a convex, continuous function defined on D, a non-empty open subset of X. Let $x_0 \in D$. If τ is a Hausdorff topology on X^* weaker than the norm topology, then the following assertions are equivalent:*

1. *f is Gâteaux differentiable at x_0 and ∂f is restricted τ-upper semicontinuous at x_0.*

2. *For all τ-neighbourhood N of 0 in X^* and $x^* \in \partial f(x_0)$, there exists $\delta = \delta(x^*, N)$ such that*

$$f(x_0) + \frac{1}{2}(f^*(x^*) + f^*(y^*)) - \delta < \frac{1}{2}\langle x_0, x^* + y^*\rangle \Rightarrow y^* \in x^* + N.$$

Proof: (1) $\Rightarrow$ (2). Let N be a τ-neighbourhood of 0 in X^* and $\{x^*\} = \partial f(x_0)$. By the previous theorem there exists $\epsilon > 0$ such that $\partial_\epsilon f(x_0) \subset x^* + N$. Take $y^* \in X^*$ such that

$$f(x_0) + \frac{1}{2}(f^*(x^*) + f^*(y^*)) - \frac{\epsilon}{2} < \frac{1}{2}\langle x_0, x^* + y^*\rangle.$$

A simple calculation shows that $f(x_0) + f^*(y^*) < \langle x_0, y^*\rangle + \epsilon$. It follows that $y^* \in \partial_\epsilon f(x_0) \subset x^* + N$.

(2) $\Rightarrow$ (1). First, we shall prove that f is Gâteaux differentiable at x_0. If not, there would exist $x_1 \neq x_2$ in $\partial f(x_0)$. Choose a τ-neighbourhood N of 0 in X^* such that $(x_1^* + N) \cap (x_2^* + N) = \emptyset$. Let $\delta_i = \delta_i(x_i^*, N)$ be as in (2)

($i = 1, 2$) and let $\delta := \min\{\delta_1, \delta_2\}$. Take $y^* \in \partial_\delta f(x_0)$. A simple calculation shows that

$$f(x_0) + \frac{1}{2}(f^*(x_i^*) + f^*(y^*)) - \delta_i < \frac{1}{2}\langle x_0, x_i^* + y^* \rangle,$$

for $i = 1, 2$. By hypothesis, $y^* \in (x_1^* + N) \cap (x_2^* + N)$, a contradiction.

Now, let N be a τ-neighbourhood of 0 in X^*. Since f is Gâteaux differentiable at x_0, $\partial f(x_0) = \{x_0^*\}$. Given N and x_0^*, we get $\delta = \delta(x_0^*, N)$ as in (2). It is easy to prove that $\partial_\delta f(x_0) \subset x_0^* + N = \partial f(x_0) + N$. By Theorem 3.2, ∂f is restricted τ-upper semicontinuous at x_0. ■

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