Some remarks on ℓ^1-sequences

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Abstract

We use a result of Bourgain and Delbaen on extreme points in duals of separable Banach spaces to characterize separable Banach spaces containing isomorphic copies of ℓ^1 in terms of extreme points. We also study the weak star closure of a bounded subset A of a separable X Banach space in X^{**} in terms of the existence of a sequence in A equivalent to the canonical basis of ℓ^1.

1 Introduction

Banach spaces containing a copy of ℓ^1 have been characterized in several ways, both in the separable and the non-separable case. In the second situation, a theorem due to Haydon [HA76] says that a Banach space does not contain a copy of ℓ^1 if, and only if, every $x^{**} \in X^{**}$ has the Point of Continuity Property ($\equiv$PCP), i.e., given any weak star closed subset F of B_{X^*}, the closed dual unit ball, the restriction of x^{**} to F has at least a point of weak star continuity.

Accordingly, if the Banach space X contains a copy of ℓ^1, an element x^{**} in X^{**} and a weak star closed subset of B_{X^*} can be found violating the PCP. It is possible, at least in the separable case, to precise the location of the pathological subset of the dual. This is done in Theorem 2. Theorem 7 relates the weak star closure of a bounded subset A of X in the bidual to the existence of sequences in A equivalent to the canonical basis of ℓ^1.

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A basic tool is a result due to Bourgain and Delbaen [BD78] (Theorem 1), which locates a Cantor Set in the set $Ext(B_{X^*})$ of extreme points of the dual unit ball (endowed with the weak star topology, w^*) of a separable Banach space containing a copy of ℓ^1, satisfying an appropiate extension property:

Theorem 1 (Bourgain-Delbaen [BD78]) *Let X be a separable Banach space. Then, if X contains an isomorphic copy of ℓ_1, there exists, for every $\epsilon > 0$, a subset Δ_ϵ of $(Ext(B_{X^*}), w^*)$ homeomorphic to the Cantor Ternary Set such that the following extension property holds: Given f, a real continuous function on (Δ_ϵ, w^*) such that $\|f\| < 1-\epsilon$, there exists $x \in X$, $\|x\| < 1$, such that the restriction of x to Δ_ϵ, $x|_{\Delta_\epsilon}$, coincides with f.*

In the sequel, Δ (sometimes decorated with upper or subscripts) will always denote a Cantor Ternary Set, and $\Delta_{n,i}$, $i = 1, 2, \ldots, 2^n$, $n = 0, 1, 2, \ldots$ its dyadic subsets (i.e., up to a homeomorphism,

$$\begin{aligned}\{\Delta_{0,1}, \Delta_{1,1}, \Delta_{1,2}, \Delta_{2,1}, \Delta_{2,2}, \Delta_{2,3}, \Delta_{2,4}, \ldots\} :=&\\ \{[0,1] \cap \Delta, [0,1/3] \cap \Delta, [2/3,1] \cap \Delta,&\\ [0,1/9] \cap \Delta, [2/9,3/9] \cap \Delta, [6/9,7/9] \cap \Delta, [8/9,1] \cap \Delta, \ldots\},&\end{aligned}$$

where $\Delta := \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{2^n} \Delta_{n,i}$).

2 Checking ℓ^1 on extreme points.

Theorem 2 *Let X be a separable Banach space. The following statements are equivalent:*

1. *X contains an isomorphic copy of ℓ_1.*
2. *There exists Δ, a subset of $(Ext(B_{X^*}), w^*)$ homeomorphic to the Cantor Ternary Set, and there exists $x^{**} \in X^{**}$ such that $x^{**}|_\Delta$ has no point of continuity.*

Proof. (1) $\Rightarrow$ (2). By Theorem 1, given $\epsilon > 0$ it is possible to choose an appropiate Δ_ϵ and a sequence (x_n) in B_X such that

$$\langle x_n, x^* \rangle = (-1)^i (1-\epsilon),\ x^* \in \Delta_{n,i},\ i = 1, 2, \ldots, 2^n,\ n = 0, 1, 2, \ldots$$

Let $\mathcal{U}$ be a non-trivial ultrafilter on $I\!N$. Let

$$x^{**} := \omega^* - \lim_{\mathcal{U}} x_n \in X^{**}.$$

We shall prove that x^{**} restricted to (Δ_ϵ, w^*) has no point of continuity: Assume, on the contrary, that $x^{**}|_{\Delta_\epsilon}$ is continuous at $x_0^* \in \Delta_\epsilon$. As x^{**} takes only two values ($1-\epsilon$ and $-1+\epsilon$), it is possible then to find a dyadic subset $\Delta_{n_0,i}$ where x^{**} is a constant (say $1-\epsilon$). It is easy now to find $x^* \in \Delta_{n_0,i}$ such that $\langle x_n, x^* \rangle = -1+\epsilon,\ \forall n > n_0$. Now, there exists $U \in \mathcal{U}$ such that $\langle x_n, x^* \rangle = \langle x^{**}, x^* \rangle,\ \forall n \in U$. This is a contradiction, as $\{n \in I\!N : \ n > n_0\} \cap U \neq \emptyset$.

(2) $\Rightarrow$ (1). This is a consequence of Corollary III.3.3 in [DGZ93].

□

Remark. In fact, the former proof allows to say that if X is a separable Banach space with an isomorphic copy of ℓ^1, something a little bit more precise is true: $x^{**} \in X^{**}$ and $\Delta_\epsilon \in Ext(B_{X^*})$ can be found such that for every point $x_0^* \in \Delta_\epsilon$ there exists a ω^*-neighbourhood of x_0^* in (Δ_ϵ, w^*) where the oscillation of x^{**} is greater or equal than $2-2\epsilon$. This will be used in the sequel.

Rainwater's Theorem [RA63] says that a bounded sequence in a Banach space X weakly converges to a point u in X if it converges on the extreme points of the dual unit ball. The result is not longer true if u belongs to $X^{**} \setminus X$. This can be easily seen in $C(K)$, K a compact space such that a regular Borel measure with non-separable support exits on K. Then, $\overline{lin}^{\|\cdot\|}(Ext(B_{M(K)})) \neq M(K)$, where $M(K)$ denotes the dual space of $C(K)$. The Separation Theorem gives $u \in S_{X^{**}}$ which vanishs on $Ext(B_{M(K)})$, and u is, on the extreme points of $B_{M(K)}$, the limit of the zero sequence. However, the validity of Rainwater's Theorem "going to the bidual" relates, at least in the separable case, to Banach spaces without isomorphic copies of ℓ^1: A classical result characterizes those spaces: (a) [OR75]*A separable Banach space X does not contain an isomorphic copy of ℓ^1 if and only if every point $x^{**} \in X^{**}$ is the weak star limit of a sequence (x_n) in X.* Using the former theorem we can formulated the next

Corollary 3 *Let X be a separable Banach space. The following are equivalent:*

1. *The space X does not contain an isomorphic copy of ℓ^1.*

2. *For every $x^{**} \in X^{**}$ there exists a sequence (x_n) in X which converges to x^{**} on the extreme points of B_{X^*}.*

Proof. (1) $\Rightarrow$ (2) is trivial after the result (a). To prove (2) $\Rightarrow$ (1) assume X contains an isomorphic copy of ℓ^1. By Theorem 2 there exists $x^{**} \in X^{**}$ and $\Delta \subset Ext(B_{X^*})$ such that $x^{**}|_{(\Delta,w^*)}$ has no point of continuity. By (2), there exists a sequence (x_n) in X such that $\langle x^{**} - x_n, x^* \rangle \to 0$ as $n \to \infty$ for every $x^* \in Ext(B_{X^*})$. As (Δ, w^*) is metrizable, Baire Great Theorem (see, for example, [DGZ93], Theorem I.4.1) says that x^{**} has points of continuity when restricted to any non-empty closed subset of (Δ, w^*), which is plainly false.

$\square$

Remark. In the result (a) it is possible to substitute the statement on points of the bidual by the following one: *every point $x^{**} \in B_{X^{**}}$ is the weak star limit of a sequence in B_X*. Analogously, it is possible to substitute (2) in Corollary 3 by *For every $x^{**} \in B_{X^{**}}$ there exists a sequence (x_n) in B_X which converges to x^{**} on the extreme points of B_{X^*}.*

A subset BD of B_{X^*} is called a *boundary* (more precisely, a *James boundary*) if every $x \in X$ attains its supremum on B_{X^*} at some point of BD. Using Simons inequality, G. Godefroy (see, for example, [DGZ93], Lemma I.5.10) proved the following statement: (b) *Let X be a Banach space. Assume that every $x^{**} \in B_{X^{**}}$ is the weak star limit of a sequence (x_n) in B_X. Then* $\overline{conv}^{\|\cdot\|}(BD) = B_{X^*}$. In case the Banach space X is separable, it is enough to have convergence of sequences on extreme points. This is the content of the following

Corollary 4 *Let X be a separable Banach space and let $BD \subset B_{X^*}$ be a James boundary. Assume that every element $x^{**} \in B_{X^{**}}$ is the limit of a sequence (x_n) in X for the topology of the pointwise convergence on extreme points of B_{X^*}. Then $\overline{conv}^{\|\cdot\|}(BD) = B_{X^*}$.*

Proof. According to Corollary 3 the space X does not contains an isomorphic copy of ℓ^1. It follows that every point $x^{**} \in B_{X^{**}}$ is the weak star limit of a sequence (x_n) in B_X. The result now is a consequence of (b).

$\square$

An interesting result due to E. and P. Saab characterizes Banach spaces with isomorphic copies of ℓ^1 in terms of the oscillation of an element of X^{**} on weak star slices of a certain subset of X^*. Precisely, we have

Theorem 5 (Saab-Saab [SS83]) *Let X be a Banach space. Then, the following statements are equivalent:*

1. *X contains an isomorphic copy of ℓ^1.*
2. *There exists $x^{**} \in X^{**}$, K, a non-empty subset of B_{X^*} and $\epsilon > 0$ such that*

$$osc(x^{**}, S) \geq \epsilon, \quad \textit{for all non-empty weak star section } S \textit{ of } K,$$

where $osc(x^{**}, S)$ denotes the oscillation of a function x^{**} on a set S.

Now, Theorem 2 allows us to precise the location of the subset K in the former theorem in case of a separable Banach space X. Just use the previous remark.

Theorem 6 *Let X be a separable Banach space. Then, the following statements are equivalent:*

1. *X contains an isomorphic copy of ℓ^1.*
2. *There exists $x^{**} \in X^{**}$, $K \subset (Ext(B_{X^*}), w^*)$, homeomorphic to the Cantor Ternary Set, and $\epsilon > 0$ such that*

$$osc(x^{**}, S) \geq \epsilon, \quad \textit{for all non-empty weak star section } S \textit{ of } K.$$

3 Closures of bounded subsets of X in X^{**} and ℓ^1-sequences.

Let A be a bounded subset of a Banach space X. Let's denote by $\overset{**}{A}$ the closure of A in (X^{**}, w^*), by $\ddot{A}$ the sequential closure (i.e., the set of all limits of sequences) of A in (X^{**}, w^*), and by $\tilde{A}$ the set $\bigcup\{\overset{**}{N} \colon N \subset A, \ N \text{ countable}\}$.

The following theorem relates those closures with the existence of a sequence in A equivalent to the canonical basis of ℓ^1:

Theorem 7 *Let X be a separable Banach space. Let A be a bounded subset of X. Then, the following statement are equivalent:*

1. *A does not contain any sequence equivalent to the canonical basis of ℓ^1.*
2. *Every $a^{**} \in \overset{**}{A}$ is a Borel function on (B_{X^*}, w^*).*
3. *Every $a^{**} \in \overset{**}{A}$ is a First Baire Class function on (B_{X^*}, w^*).*
4. *$\overset{**}{A} = \dddot{A}$.*
5. *$\tilde{A} = \dddot{A}$.*

Proof. (1) $\Rightarrow$ (3): Let $a^{**} \in \overset{**}{A}$. Assume a^{**} is not a First Baire Class function on (B_{X^*}, w^*). This is a Polish topological space, hence a^{**} satisfies the Discontinuity Criterium on a countable subset $N \subset (B_{X^*}, w^*)$ ([RO77]). Precisely, there exists $r \in I\!R$ and $\delta > 0$ such that for every non-empty open set O in (N, w^*), there exists x_1^* and x_2^* in O such that

$$\langle a^{**}, x_1^* \rangle \leq r, \quad \langle a^{**}, x_2^* \rangle \geq r + \delta. \tag{1}$$

We shall construct a tree of subsets of N by repeatingly using inequalities (1): let's start by taking $N_1 := N$. We can then find $x_{1,1}^*$ and $x_{1,2}^*$ in N_1 such that $\langle a^{**}, x_{1,1}^* \rangle \leq r$, $\langle a^{**}, x_{1,2}^* \rangle \geq r + \delta$. Choose $a_1 \in A$ such that $|\langle a^{**} - a_1, x_{1,i}^* \rangle| < \delta/4,\ i = 1, 2$. a_1 defines disjoint open neighbourhoods (say $N_{1,1}$ and $N_{1,2}$) of $x_{1,1}^*$ and $x_{1,2}^*$ in N, respectively. Again by (1) it is possible to find $x_{2,1}^*,\ x_{2,2}^*$ in $N_{1,1}$ and $x_{2,3}^*,\ x_{2,4}^*$ in $N_{1,2}$ such that

$$\begin{gathered}\langle a^{**}, x_{2,1}^* \rangle \leq r, \quad \langle a^{**}, x_{2,2}^* \rangle \geq r + \delta \\ \langle a^{**}, x_{2,3}^* \rangle \leq r, \quad \langle a^{**}, x_{2,4}^* \rangle \geq r + \delta.\end{gathered}$$

Choose now $a_2 \in A$ such that $|\langle a^{**} - a_2, x_{2,i}^* \rangle| < \delta/4,\ i = 1, 2, 3, 4$. Proceed in this way to find a sequence (a_n) in A and a tree $N_{n,i},\ i = 1, 2, \ldots, 2^n,\ n = 1, 2, \ldots$ of subsets of N. The sequence (a_n) is equivalent to the canonical basis of ℓ^1 (see, for example, [DI84], Proposition XI,2), a contradiction.

(3) $\Rightarrow$ (4): Let $\mathcal{B}^1(P)$ be the space of all First Baire Class functions on the Polish space $P := (B_{X^*}, w^*)$, endowed with the pointwise topology $\mathcal{T}_p$. This is an angelic space ([RO77]). By hypothesis, $\overset{**}{A}$ is a compact subset of $(\mathcal{B}^1(P), \mathcal{T}_p)$, hence $\overset{**}{A} = \dddot{A}$.

(4) $\Rightarrow$ (5): This is obvious, as $A \subset \ddot{A} \subset \tilde{A} \subset \overset{**}{A}$.

(5) $\Rightarrow$ (1): Let (a_n) be a sequence in A equivalent to the canonical basis of ℓ^1. Let $\mathcal{U}$ be a non-trivial ultrafilter in $I\!N$. Let $u \in X^{**}$ be the weak star limit of (a_n) along the ultrafilter.

Claim: *There exists a non-empty closed subset M of (B_{X^*}, w^*) such that the restriction of u to M has no point of continuity.*

Proof of the claim: Let $Y := \overline{lin}\{a_n : n = 1, 2, \ldots\} \subset X$. Denote by q the canonical mapping from X^* onto Y^*. Let $T : \ell^1 \to Y$ be an isomorphism such that $Te_n = a_n,\ n = 1, 2, \ldots$. The set $\Delta := \{(x_n) :\ x_n = \pm 1, n \in I\!N\} \subset (B_{\ell^\infty}, w^*)$ is homeomorphic to the Cantor Ternary Set and the oscillation of $e^{**} := \lim_{\mathcal{U}}(e_n)$ on a non-empty open subset of (Δ, ω^*) is 2. Let $D := (T^*)^{-1}(\Delta)$. D is a weak star compact subset of $r.B_{Y^*}$ for some $r > 0$. Let N be a compact subset of $(r.B_{X^*}, w^*)$ such that $q(N) = D$, and N is minimal for this two properties. As $T^{**}(e^{**}) = u$, it follows that $osc(u, O) = 2$ for every non-empty open subset O of (N, ω^*). Then $M := \frac{1}{r}N$ is a non-empty closed subset of (B_{X^*}, ω^*) such that $osc(u, V) = \frac{2}{r}$ on every non-empty open subset V of (M, ω^*), hence $u|_{(M,\omega^*)}$ has no point of continuity. This proves the Claim.

Now, (B_{X^*}, w^*) is a Polish space. We can use Baire Great Theorem (see, for example, [DGZ93], Theorem I.4.1) to conclude that u is not the weak star limit of a sequence in X. It follows that $u \notin \ddot{A}$. However, $u \in \tilde{A}$, a contradiction.

(3) $\Rightarrow$ (2): This is obvious, as every First Baire Class function is also Borel.

(2) $\Rightarrow$ (5): Let $a^{**} \in \tilde{A}$. There exists a sequence (a_n) in A such that $a^{**} \in \overset{**}{\{a_n\}}$. As every element of $\overset{**}{\{a_n\}}$ is Borel, we can use Lemma III.3.4 in [DGZ93] in order to find a subsequence (a_{n_k}) of (a_n) such that $\omega^* - \lim_k a_{n_k} = a^{**}$. It follows that $a^{**} \in \overset{\cdots}{\{a_n\}}$.

□

Remark. Each of the conditions in the former theorem implies, of course, that $\overset{**}{A} = \tilde{A}$. However, this condition is not equivalent to the others: let X be a separable Banach space with an isomorphic copy of ℓ^1. Let (a_n) be a sequence in X equivalent to the canonical basis of ℓ^1. Obviously $\overset{**}{A} = \tilde{A}$, where $A := \{a_n :\ n = 1, 2, \ldots\}$, but none of the (equivalent) conditions of the former theorem are satisfied.

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