Inner characterizations of weakly compactly generated Banach spaces and their relatives

by

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Dedicated to Professor John Horváth on the occasion of his 80th bithday

Abstract. We give characterizations of weakly compactly generated spaces, their subspaces, Vašák spaces, weakly Lindelöf determined spaces as well as several other classes of Banach spaces related to uniform Gâteaux smoothness, in terms of the presence of a total subset of the space with some additional properties. In addition, we describe geometrically, when possible, these classes by means of suitable smoothness or rotundity of the norm. As a consequence, we get new, functional analytic proofs of several theorems on (uniform) Eberlein, Gul'ko and Talagrand compacta.

Introduction

A nonseparable Banach space is usually beyond control unless a kind of coordinate system is available on it. By this we understand sometimes a biorthogonal system or an ordered family of projections —like in Hilbert spaces— sometimes a large weakly compact subset —as in the more general class of reflexive spaces. In fact there is a common ground to both, and it is the existence of a kind of "core" in the space which behaves "almost" as a weakly compact set and provides a "coordinate system" for the dual (evaluations at the points of the core). These ideas were already present in the seminal paper [AL] by D. Amir and J. Lindenstrauss. They proved that *a Banach space X contains a weakly compact total, i.e. linearly dense, subset (if and) only if it contains a total set $\Gamma \subset B_X$ such that its derivative considered in the second dual X^{**} provided with the weak* topology is just $\{0\}$*. Such Banach spaces are called *weakly compactly generated (WCG)*. Observe then that the mapping $x^* \mapsto (\langle \gamma, x^* \rangle;\ \gamma \in \Gamma)$, $x^* \in X^*$, is a weak* to weak continuous linear bounded injection from X^* into $c_0(\Gamma)$. This has several important consequences, in particular for constructing a smooth norm on X.

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Accordingly, J. Lindenstrauss [L] conjectured soon after a close connection between these notions and the geometrical concept of smoothness, asking (problem 9 in [L]) whether the smoothness of a Banach space implies that some superspace of it is WCG. The answer to this question turned out to be negative. However, there are some cases when it is so, proving how far-reaching Lindenstrauss perception was: in particular, it was shown in [FGZ] that *a Banach space X has an equivalent uniformly Gâteaux smooth norm (if and) only if X is a subspace of a Hilbert generated Banach space.*

In this paper we dig in the aforesaid ideas, presenting from those points of view a comprehensive description of WCG Banach spaces and related classes —subspaces of WCG Banach spaces, weakly $\mathcal{K}$-analytic, Vašák, weakly Lindelöf determined and some uniform counterparts— and trying to imitate the aforesaid equivalence for WCG spaces. Along the way, we shall see how a precise description of the core provides a characterization of each class, both in the non-uniform and the uniform settings. In addition, and according to what has been mentioned above, we enrich each statement, when possible, by a geometric information in terms of smoothness and rotundity. Thus these classes of nonseparable Banach spaces, classes which do not assume any unconditional or lattice structure, are described more or less in a uniform way. As a byproduct of the used techniques we also find or rediscover criteria for recognizing (uniform) Eberlein, Gul'ko, and Talagrand compacta among compact subsets of $\Sigma-$products of real lines. The results presented here are in the flavour of the papers [A], [AF], [AL], [AM], [AM1], [Fa], [M], [S], [T], [Tr1], [Tr2].

The paper is organized as follows. Section 1 presents a list of results, together with necessary definitions. Section 2 contains some tools needed; in particular, Šmulyan-like duality statements and a study of Day's norm can be found here. Section 3 contains proofs of the main theorems. Many of the results and proofs are new, others can be found in previous papers by the present authors and colleagues.

Section 1 – List of results

Let M be a nonempty subset of the closed unit ball B_X of a Banach space $(X, \|\cdot\|)$ and let $\varepsilon \geq 0$ be given. We say that the norm $\|\cdot\|$ is $\varepsilon-M-$*smooth* if for every $0 \neq x \in X$

$$\lim_{t\downarrow 0} \frac{1}{t} \sup \big\{ \|x+th\| + \|x-th\| - 2\|x\|;\ h \in M \big\} \leq \varepsilon \|x\|.$$

The norm $\|\cdot\|^*$ on X^*, dual to $\|\cdot\|$, is called $\varepsilon-M-LUR$ if $\limsup_{n\to\infty} \sup \big|\langle M, x_n^* - x^*\rangle\big| \leq \varepsilon \|x^*\|^*$ whenever $x^*, x_n^* \in X^*,\ n \in \mathbb{N}$, and $2\|x^*\|^{*2} + 2\|x_n^*\|^{*2} - \|x^* + x_n^*\|^{*2} \to 0$ as $n \to \infty$. If $\varepsilon = 0$, then we speak about $M-$*smoothness* and $M-LUR$.

Theorem 1. *For a Banach space X TFAE:*
(i) X is weakly compactly generated.
(ii) There exists a total set $\Gamma \subset B_X$ such that (1)

$$\forall \varepsilon > 0 \ \ \forall x^* \in X^* \quad \#\{\gamma \in \Gamma :\ \langle \gamma, x^* \rangle > \varepsilon\} < \aleph_0.$$

(1) The reader should not have difficulties in substituting the inequality $\langle \gamma, x^* \rangle > \varepsilon$ in statements about the set Γ in this and forthcoming results with $|\langle \gamma, x^* \rangle| > \varepsilon$.

(iii) *There exist an equivalent norm $|\cdot|$ on X and a total set $M \subset B_X$ such that the double dual norm $|\cdot|^{**}$ on X^{**} is $M-$smooth (the triple dual norm $|\cdot|^{***}$ on X^{***} is $M-$LUR).*

(iv) *X is weakly Lindelöf determined, and there exist an equivalent norm $|\cdot|$ on X and a total set $M \subset B_X$ such that $|\cdot|$ is $M-$smooth.*

Here, (iii) actually characterizes the relative weak compactness of the set M, see the proof of Proposition 3. We recall that a Banach space X is called *weakly Lindelöf determined* (*WLD*) if the dual unit ball B_{X^*} in the weak* topology is a Corson compact. A compact space is called *Corson* if it can be found, up to a homeomorphism, in $\Sigma(\Delta)$ for some uncountable set Δ where

$$\Sigma(\Delta) =: \{u \in \mathbb{R}^{\Delta} :\ \#\{\delta \in \Delta;\ u(\delta) \neq 0\} \leq \aleph_0\},$$

and the topology here is inherited from the product topology of $\mathbb{R}^{\Delta}$. In Theorem 1, (i)$\Leftrightarrow$(ii) is from [AL]. Note that *a Banach space is WCG and Asplund if and only if it is WLD and admits an equivalent Fréchet smooth (i.e. B_X-smooth) norm.*

Given a set Γ in a Banach space X, we say that it *countably supports* X^* if $\#\{\gamma \in \Gamma;\ \langle \gamma, x^*\rangle \neq 0\} \leq \aleph_0$ for every $x^* \in X^*$.

Theorem 2. *For a Banach space X TFAE:*

(i) *X is a subspace of a weakly compactly generated overspace.*

(ii) *There exists a total set $\Gamma \subset B_X$ such that for every $\varepsilon > 0$ there is a decomposition $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n^{\varepsilon}$ such that*

$$\forall n \in \mathbb{N}\ \ \forall x^* \in X^* \quad \#\{\gamma \in \Gamma_n^{\varepsilon};\ \langle \gamma, x^*\rangle > \varepsilon\} < \aleph_0.$$

Moreover, for Γ we can take any total subset of B_X which countably supports X^*.

(iii) *B_{X^*} with the weak* topology is an Eberlein compact.*

(iv) *There exist an equivalent norm $|\cdot|$ on X and a total set $M \subset B_X$ such that for every $\varepsilon > 0$ we can write $M = \bigcup_{n=1}^{\infty} M_n^{\varepsilon}$ and the norm $|\cdot|^{**}$ on X^{**} is $\varepsilon - M_n^{\varepsilon}-$smooth ($|\cdot|^{***}$ on X^{***} is $\varepsilon - M_n^{\varepsilon}-$LUR) for every $n \in \mathbb{N}$.*

(v) *X is WLD, and there exist an equivalent norm $|\cdot|$ on X and a total set $M \subset B_X$ such that for every $\varepsilon > 0$ we can write $M = \bigcup_{n=1}^{\infty} M_n^{\varepsilon}$ and the norm $|\cdot|$ is $\varepsilon - M_n^{\varepsilon}-$smooth for every $n \in \mathbb{N}$.*

A subspace of a WCG Banach space is not necessarily WCG itself. The first counterexample was given in [R]. Theorem 2 enhances results from [FMZ3]. (i)$\Leftrightarrow$(iii) here, together with some results from [AL], yields the following theorem, first obtained by Benyamini, M.E. Rudin and Wage [BRW], and independently by Gul'ko [G]: *A continuous image of an Eberlein compact is Eberlein.* For more references, see [FMZ3].

In part *(ii)* of Theorem 2, any total subset Γ of B_X which countably supports X^* has the property listed there. However, this is not so in Theorem 1. Indeed, there exists a WCG Banach space X and a Markushevich basis $(\{x_i\}_{i\in I}, \{x_i^*\}_{i\in I})$ such that $\{x_i\}_{i\in I} \cup \{0\}$ is not weakly compact (see a remark in [FMZ4]). It is easy to prove that $\{x_i\}_{i\in I}$ countably supports X^*. However, condition *(ii)* for Γ in Theorem 1 already implies that $\Gamma \cup \{0\}$ is weakly compact.

A Banach space X is called *Vašák* (also called *weakly countably determined*) if there is a countable family $K_m,\ m \in \mathbb{N}$, of weak* compact sets in X^{**} such that, given any $x \in X$, and any $x^{**} \in X^{**} \setminus X$, there is $m \in \mathbb{N}$ such that $x \in K_m$ and $x^{**} \notin K_m$.

Theorem 3. *For a Banach space X TFAE:*

(i) X is a Vašák space.

(ii) There exist a total set $\Gamma \subset B_X$ and subsets $\Gamma_n \subset \Gamma,\ n \in \mathbb{N}$, with the property

$$\forall \varepsilon > 0 \ \ \forall x^* \in X^* \ \ \forall \gamma \in \Gamma \ \ \exists n \in \mathbb{N} \ \ \text{such that}$$

$$\gamma \in \Gamma_n \quad \text{and} \quad \#\{\gamma' \in \Gamma_n;\ \langle \gamma', x^* \rangle > \varepsilon\} < \aleph_0.$$

Moreover, for Γ we can take any total subset of B_X which countably supports X^*.

If $\operatorname{dens} X = \aleph_1$, then the above is equivalent with:

(iii) There exist an equivalent norm $|\cdot|$ on X, a projectional resolution of the identity $(P_\alpha;\ \omega \le \alpha \le \omega_1)$ on $(X, |\cdot|)$ (see Section 2 for the definition), and a total set $M \subset B_X$, with subsets $\emptyset \ne M_n \subset M,\ n \in \mathbb{N}$, such that for every $\varepsilon > 0$ and every $0 \ne x^* \in X^*$ there is $N \subset \mathbb{N}$ so that $\bigcup_{n \in N} M_n = M$ and $|\cdot|^*$ on X^* is $\varepsilon/|P^*_\alpha x^*|^* - M_n{-}LUR$ at $P^*_\alpha x^*$ for every $\alpha \le \omega_1$ and every $n \in N$.

From (ii) and Proposition 4 we easily get Mercourakis' result [M] that Vašák spaces admit an equivalent norm whose dual norm is strictly convex.

Theorem 3 has a subtler "$\mathcal{K}-$analytic" analogue. For $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $i \in \mathbb{N}$ we put $\sigma|i = (\sigma(1), \dots, \sigma(i))$. Denote $\mathbb{N}^{<\mathbb{N}} = \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \cdots$. We say that a Banach space X is *weakly $\mathcal{K}$-analytic* if there are weak* compact sets $K_s \subset X^{**},\ s \in \mathbb{N}^{<\mathbb{N}}$, such that $X = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{i=1}^{\infty} K_{\sigma|i}$. From (ii) in Theorems 2 and 3 it follows immediately that subspaces of WCG Banach spaces are weakly $\mathcal{K}$-analytic. On the other hand, there are weakly $\mathcal{K}$-analytic Banach spaces which are not subspaces of weakly compactly generated Banach spaces. The first example was given by Talagrand, see, e.g., [F, 4.3].

Theorem 4. *For a Banach space X TFAE:*

(i) X is weakly $\mathcal{K}$-analytic.

(ii) There exist a total set $\Gamma = \Gamma_\emptyset \subset B_X$, with subsets $\Gamma_s \subset \Gamma,\ s \in \mathbb{N}^{<\mathbb{N}}$, such that $\Gamma = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{i=1}^{\infty} \Gamma_{\sigma|i}$ and having the property

$$\forall \varepsilon > 0 \ \ \forall x^* \in X^* \ \ \forall \sigma \in \mathbb{N}^{\mathbb{N}} \quad \exists i \in \mathbb{N} \quad \#\{\gamma \in \Gamma_{\sigma|i};\ \langle \gamma, x^* \rangle > \varepsilon\} < \aleph_0.$$

Moreover, for Γ we can take any total subset of B_X which countably supports X^*.

(iii) There exist an upper semicontinuous multivalued mapping $\varphi : \mathbb{N}^{\mathbb{N}} \to (B_X, w)$, with total range, and an equivalent norm $|\cdot|$ on X such that for every $\sigma \in \mathbb{N}^{\mathbb{N}}$ the set $\varphi(\sigma)$ is nonempty and weakly compact, and $|\cdot|$ is $\varphi(\sigma){-}$smooth ($|\cdot|^*$ on X^* is $\varphi(\sigma){-}LUR$).

A weakly $\mathcal{K}$-analytic space X is Vašák. This follows from the very definition: let $K_s \subset X^{**},\ s \in \mathbb{N}^{<\mathbb{N}}$, be the family of w^*-compact subsets of X^{**} witnessing that X is weakly $\mathcal{K}$-analytic. It is countable and given $x \in X,\ x^{**} \in X^{**} \setminus X$, there exists $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $i \in \mathbb{N}$ such that $x \in K_{\sigma|i}$ and $x^{**} \notin K_{\sigma|i}$. Of course, this can also be seen using (ii) in Theorems 3 and 4. On the other hand, there are Vašák spaces which are not weakly $\mathcal{K}$-analytic. The first example was given again by Talagrand (see, e.g., [F, 7.4] and references therein).

Theorem 5. *For a Banach space X TFAE:*
(i) X is a subspace of a weakly Lindelöf determined space.
(ii) There exists a total set $\Gamma \subset B_X$ which countably supports X^*, that is,

$$\forall x^* \in X^* \quad \#\big\{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle \neq 0\big\} \leq \aleph_0.$$

(iii) X is weakly Lindelöf determined.

From conditions (ii) in Theorems 4 and 5 we easily get that Vašák spaces are WLD. Of course, (i)⇒(iii) has been known since Gul'ko (and later, independently, Valdivia [V]) proved the deep fact that continuous images of Corson compacta are Corson. Here we do not rely on this fact, and that is the reason why we reprove that subspaces of WLD spaces are themselves WLD. We do not have any geometrical assertion in Theorem 5. Actually, there exists a WLD space of the form $C(K)$ such that the compact K contains no dense G_δ metrizable set [AM, Theorem 3.6]. Hence, by [F, Theorem 2.2.3 and Corollary 4.2.5], such $C(K)$ admits no equivalent Gâteaux smooth norm (and, in particular, by the remark following Theorem 3, this space is not Vašák).

Next, we shall focus on subtler relatives of the WCG spaces. In what follows, we shall consider a kind of uniformity when dealing with compactness or smoothness. Let us recall that a WCG space X is, according to the interpolation theorem of Davies, Figiel, Johnson and Pełczyński, *reflexive generated*, that is, there are a reflexive space R and a linear bounded mapping $T : R \to X$ with dense range, see, e.g., [F, Theorem 1.2.3]. Analogously, we say that X is *Hilbert generated* if there are a Hilbert space H and a linear bounded mapping $T : H \to X$ with dense range. A compact space is called *uniform Eberlein* if it can be found, up to a homeomorphism, in a Hilbert space provided with the weak topology.

Let $\emptyset \neq M \subset B_X$ and let $\varepsilon \geq 0$ be given. We say that the norm $\|\cdot\|$ on X is *uniformly $\varepsilon - M-$smooth* if

$$\lim_{t \downarrow 0} \frac{1}{t} \sup\big\{\|x+th\| + \|x-th\| - 2;\ x \in X,\ \|x\| = 1,\ h \in M\big\} \leq \varepsilon.$$

The norm $\|\cdot\|^*$ on X^*, dual to $\|\cdot\|$, is called *uniformly $\varepsilon - M-$rotund* if $\limsup_{n\to\infty} \sup\big|\langle M, x_n^* - y_n^*\rangle\big| \leq \varepsilon$ whenever $x_n^*, y_n^* \in B_{X^*}$, $n \in \mathbb{N}$, and $2\|x_n^*\|^{*2} + 2\|y_n^*\|^{*2} - \|x_n^* + y_n^*\|^{*2} \to 0$ as $n \to \infty$. If $\varepsilon = 0$, then we speak about *uniform $M-$smoothness* and *uniform $M-$rotundity* respectively. The norm $\|\cdot\|$ is called *uniformly Gâteaux smooth* if it is uniformly $\{h\}-$smooth for every $h \in X$.

Theorem 6. *For a Banach space X TFAE:*
(i) X is a subspace of a Hilbert generated space.
(ii) There exists a total set $\Gamma \subset B_X$ such that for every $\varepsilon > 0$ we have a decomposition $\Gamma = \bigcup_{n=1}^\infty \Gamma_n^\varepsilon$ satisfying

$$\forall n \in \mathbb{N} \quad \forall x^* \in B_{X^*} \quad \#\big\{\gamma \in \Gamma_n^\varepsilon;\ \langle \gamma, x^* \rangle > \varepsilon\big\} < n.$$

Moreover, for Γ we can take any total subset of B_X which countably supports X^*.

(iii) (B_{X^*}, w^*) *is a uniform Eberlein compact.*
(iv) X *admits an equivalent uniformly Gâteaux smooth norm.*

This theorem enhances results from [FGZ]. (i)⇔(iii) here, together with some results from [BRW], immediately yields the following theorem, first obtained by Benyamini, M.E. Rudin and Wage [BRW]: *A continuous image of a uniform Eberlein compact is uniform Eberlein.* For more details, see [FGZ].

Theorem 7. *For a Banach space* X *TFAE:*
(i) X *is both weakly compactly generated and a subspace of a Hilbert generated space.*
(ii) There exists a total set $\Gamma \subset B_X$ *having the properties from (ii) in Theorems 1 and 6.*
(iii) X *admits an equivalent norm which is uniformly Gâteaux smooth and (another equivalent norm which is)* $M-$*smooth for some total set* $M \subset B_X$.

Note that Theorem 6 describes a class larger than Theorem 7. This is demonstrated by the famous Rosenthal's counterexample [R]. For new counterexamples see [AM1].

Theorem 8. *For a Banach space* X*, with density less than* $\aleph_{\omega_1}$*, TFAE:*
(i) There exists an equivalent norm on X *which is uniformly* $M-$*smooth for some total subset* $M \subset B_X$.
(ii) There exists a total subset $\Gamma \subset B_X$ *such that for every* $\varepsilon > 0$ *there is* $\kappa(\varepsilon) \in \mathbb{N}$ *satisfying*

$$\forall x^* \in B_{X^*} \quad \#\big\{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle > \varepsilon\big\} < \kappa(\varepsilon).$$

That Theorem 7 describes a class larger than Theorem 8 does is shown, under the continuum hypothesis, in [FGHZ]; actually there exists a reflexive uniformly Gâteaux smooth space not satisfying (i) in Theorem 8.

We do not know of any characterization of Hilbert generated spaces via a cardinality condition for a total set $\Gamma \subset B_X$ in the spirit of the above theorems. However, we have instead the following equivalence.

Theorem 9. *Let* $1 < p < +\infty$ *and* $q = \frac{p}{p-1}$*. For a Banach space* X *TFAE:*
(i) X *is* $\ell_p(\Delta)-$ *generated for a suitable set* Δ *with* $\#\Delta = \operatorname{dens} X$.
(ii) There exists a total set $\Gamma \subset B_X$ *such that*

$$\forall x^* \in B_{X^*} \quad \sum_{\gamma \in \Gamma} |\langle \gamma, x^* \rangle|^q \le 1.$$

If $\operatorname{dens} X = \aleph_1$*, then the above is equivalent with:*
(iii) X *is* $Y-$*generated for a suitable Banach space* Y *whose norm has modulus of smoothness of power type* p.

We recall here the well known result, due to Pisier, that *a Banach space admits an equivalent norm whose modulus of smoothness is of power type* p *for some* $1 < p < +\infty$ *if (and only if) it is superreflexive* [P]. A space satisfying Theorem 8 and not Theorem 9 is $X = \left(\sum_{n=1}^{\infty} \ell_{r_n}(\Gamma)\right)_{\ell_2}$, where $\{r_n :\ n \in \mathbb{N}\}$ is a dense subset of $(1, +\infty)$ and Γ is uncountable. This follows from Pitt's Theorem, see [F$^{\sim}$, Prop. 6.25].

We do not know if the cardinality restrictions in Theorems 8 and 9 can be removed. Note however that there do exist statements dependent on the cardinality: a nonseparable Sobczyk's theorem fails to hold if the density of the space is $\aleph_\omega$ or more [ACGJM].

Theorems 2, 3, 4, and 6 have the following topological consequences. A compact space K is called *Gul'ko* (*Talagrand*) if the space $C(K)$ is Vašák (weakly $\mathcal{K}-$analytic).

Theorem 10. *Let Γ be an uncountable set and let $K \subset \Sigma(\Gamma) \cap [-1,1]^\Gamma$ be a compact subset.*

(a) K is a (uniform) Eberlein compact if and only if for every $\varepsilon > 0$ there is a decomposition $\Gamma = \bigcup_{n=1}^\infty \Gamma_n^\varepsilon$ such that

$$\forall n \in \mathbb{N} \;\; \forall k \in K \quad \#\{\gamma \in \Gamma_n^\varepsilon;\; |k(\gamma)| > \varepsilon\} < \aleph_0 \quad (< n).$$

(b) K is a Gul'lko compact if and only if there are sets $\Gamma_n \subset \Gamma,\; n \in \mathbb{N}$, such that

$$\forall \varepsilon > 0 \;\; \forall k \in K \;\; \forall \gamma \in \Gamma \;\; \exists n \in \mathbb{N} \quad \text{such that}$$

$$\gamma \in \Gamma_n \quad \text{and} \quad \#\{\gamma' \in \Gamma_n;\; |k(\gamma')| > \varepsilon\} < \aleph_0.$$

(c) K is a Talagrand compact if and only if there are sets $\Gamma_s,\; s \in \mathbb{N}^{<\mathbb{N}}$, such that $\Gamma = \bigcup_{\sigma \in \mathbb{N}^\mathbb{N}} \bigcap_{j=1}^\infty \Gamma_{\sigma|j}$ and

$$\forall \varepsilon > 0 \;\; \forall k \in K \;\; \forall \sigma \in \mathbb{N}^\mathbb{N} \;\; \exists j \in \mathbb{N} \quad \#\{\gamma \in \Gamma_{\sigma|j};\; |k(\gamma)| > \varepsilon\} < \aleph_0.$$

Of course, every Eberlein compact is a Talagrand compact, which, in turns, is Gul'ko. Finally, every Gul'ko compact is Corson. Here, (a) was originally proved by Farmaki [Fa] using combinatiorial methods, see also [AF]; if the compact K above comes from an adequate family $\mathcal{A}$, that is $K = \{\chi_A;\; A \in \mathcal{A}\}$, the result was proved already by Talagrand [T]. (b) is from [FMZ4].

Behind many of our considerations, there originally was the concept of Markushevich, see [F$^\sim$, Definition 6.2.3] and [FMZ4]. However, in the meantime we realized that what we actually need is only the bottom part of it, that is, any total set $\Gamma \subset B_X$ which countably supports X^*.

Finally, we summarize the relationship between the aforesaid classes. The number in brackets refer to the theorem where the characterization of the class is given, so the meaning of the acrostic can be easily understood. The non obvious implications easily follow from the corresponding conditions (*ii*). None of the arrows can be reversed.

$$\begin{array}{ccccccccc}
WCG(1) & \Rightarrow & SWCG(2) & \Rightarrow & W\mathcal{K}A(4) & \Rightarrow & WCD(3) & \Rightarrow & WLD(5) \\
\Uparrow & & \Uparrow & & & & & & \\
HG(9) & \Rightarrow & SHG(6) & & & & & & \\
 & & \Uparrow & & & & & & \\
 & & WCG \;\&\; SHG(7) & & & & & & \\
 & & \Uparrow & & & & & & \\
 & & UMS(8) & & & & & &
\end{array}$$

Section 2 – Tools

For the geometry of Banach spaces we refer to the books [F~], [DGZ] and [Di]. We shall also follow the notation used in these texts. A main tool for the arguments in Section 3 will be a machinery of projectional resolutions of the identity due to J. Lindenstrauss. Let $(X, \|\cdot\|)$ be a nonseparable Banach space and denote by μ the first ordinal whose cardinality is equal to the density of X. A *projectional resolution of the identity* (*PRI*) on $(X, \|\cdot\|)$ is a family $(P_\alpha:\ \omega \le \alpha \le \mu)$ of linear projections on X such that $P_\omega \equiv 0$, P_μ is the identity mapping on X, and for all $\omega < \alpha \le \mu$ the following hold: (i) $\|P_\alpha\| = 1$, (ii) the density of the subspace $P_\alpha X$ is not greater than the cardinality of α, (iii) $P_\beta \circ P_\alpha = P_\beta$ whenever $\omega \le \beta \le \alpha$, and (iv) $\bigcup_{\beta<\alpha} P_{\beta+1}X$ is norm dense in $P_\alpha X$. Though there are many references for this concept, we shall rather refer to the book [F]. Behind the construction of a PRI there is a so called *projectional generator* [OV]. It can be defined as a multivalued mapping $\Phi : X^* \to 2^X$ such that for every $x^* \in X^*$ the set $\Phi(x^*)$ is nonempty and at most countable and $\Phi(B)^\perp \cap \overline{B \cap B_{X^*}}^{\,*} = \{0\}$ for every set $\emptyset \ne B \subset X^*$ such that $B = \mathrm{sp}_Q(B)$, where sp_Q mean the linear hull made with only rational coefficients. We recall that such a Φ exists in every Banach space considered in Theorems 1 to 8, see [F, Propositions 7.1.6, 7.2.1, and 8.3.1]. In one particular case we can construct a projectional generator easily, and this goes back to ideas in [JZ]. This is when a Banach space X admits a total set $\Gamma \subset B_X$ which countably supports X^* (see the definition before the statement of Theorem 2); then it is enough to put $\Phi(x^*)$ equal to this set (note that X is then WLD). In order to check that this Φ is a projectional generator, take any $\emptyset \ne B \subset X^*$ such that $\mathrm{sp}_Q(B) = B$, and consider $x^* \in \Phi(B)^\perp \cap \overline{B \cap B_{X^*}}^{\,*}$. If $x^* \ne 0$, find $\gamma \in \Gamma$ so that $\langle \gamma, x^* \rangle \ne 0$. Find $b \in B$ so that $\langle \gamma, b \rangle \ne 0$. Then $\gamma \in \Phi(b) \subset \Phi(B)$ and hence $\langle \gamma, x^* \rangle = 0$, a contradiction. The following proposition will be of frequent use.

Proposition 1. *Let $(Z, \|\cdot\|)$ be a nonseparable Banach space admitting a projectional generator. Let $M_1, M_2, \ldots$ be an at most countable family of bounded closed convex and symmetric subsets in Z. Let $\Gamma \subset B_Z$ be a set which countably supports Z^*. Then there exists a PRI $(P_\alpha:\ \omega \le \alpha \le \mu)$ on Z such that $P_\alpha(M_n) \subset M_n$ and $P_\alpha(\gamma) \in \{\gamma, 0\}$ for every $\alpha \in [\omega, \mu]$, every $n \in \mathbb{N}$, and every $\gamma \in \Gamma$.*

Proof. Denote $M_0 = B_Z$. Let $\Phi_0 : Z^* \to 2^Z$ be a projectional generator on Z. Put

$$\Phi(z^*) = \Phi_0(z^*) \cup \{\gamma \in \Gamma;\ \langle \gamma, z^* \rangle \ne 0\}, \quad z^* \in Z^*.$$

Clearly, Φ is also a projectional generator. For $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$ let $\|\cdot\|_{n,m}$ be the Minkowski functional of the set $M_n + \frac{1}{m} B_Z$; this will be an equivalent norm on Z. We shall use a standard back-and-forth argument, see, e.g., [F, Section 6.1]. For every $z \in Z$ we find a countable set $\Psi(z) \subset Z^*$ such that

$$\|z\|_{n,m} = \sup\big\{\langle z, z^* \rangle;\ z^* \in \Psi(z) \ \text{ and } \ \|z^*\|^*_{n,m} \le 1\big\}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. Thus we defined $\Psi : Z \to 2^{Z^*}$.

For the construction of projections $P_\alpha : Z \to Z$ we shall need the following

`Claim`. *Let $\aleph <$ dens Z be any infinite cardinal and consider two nonempty sets $A_0 \subset Z,\ B_0 \subset Z^*$, with $\#A_0 \le \aleph,\ \#B_0 \le \aleph$. Then there exists sets $A_0 \subset A \subset Z,\ B_0 \subset B \subset Z^*$ such that $\#A \le \aleph,\ \#B \le \aleph,\ \overline{A},\ \overline{B}$ are linear and $\Phi(B) \subset A,\ \Psi(A) \subset B$.*
In order to prove this, put $A = \bigcup_{n=1}^{\infty} A_n,\ B = \bigcup_{n=1}^{\infty} B_n$, where the sets

$$A_n = \mathrm{sp}_Q\big(A_{n-1} \cup \Phi(B_{n-1})\big), \quad B_n = \mathrm{sp}_Q\big(B_{n-1} \cup \Psi(A_n)\big), \quad n = 1, 2, \ldots,$$

are defined inductively. Then it is easy to verify all the proclaimed properties of the sets A and B.

Having the sets A, B constructed, we observe that $A^{\perp} \cap \overline{B \cap B_{Z^*}}^{*} \subset \Phi(B)^{\perp} \cap \overline{B \cap B_{Z^*}}^{*} = \{0\}$. Therefore [F, Lemmas 6.1.1 and 6.1.2] yield a linear projection $P : Z \to Z$, with $PZ = \overline{A},\ P^{-1}(0) = B_{\perp}$, and $P^*Z^* = \overline{B}^{*}$, and such that $\|P\|_{n,m} = 1$ for every $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. Then

$$PM_n \subset \bigcap_{m=1}^{\infty} P\big(M_n + \tfrac{1}{m}B_Z\big) \subset \bigcap_{m=1}^{\infty} \overline{M_n + \tfrac{1}{m}B_Z} \subset \bigcap_{m=1}^{\infty} \big(M_n + \tfrac{2}{m}B_Z\big) = M_n$$

for every $n \in \mathbb{N} \cup \{0\}$, and in particular, $\|P\| = 1$.

Fix any $\gamma \in \Gamma$. It remains to prove that $P\gamma \in \{\gamma, 0\}$. If $\gamma \in PZ$, then, trivially, $P\gamma = \gamma$. Second, assume that $\gamma \notin PZ\ (= \overline{A})$. Then $\gamma \notin \Phi(B)$, which implies that $\langle \gamma, b\rangle = 0$ for every $b \in B$, that is, that $\gamma \in B_{\perp}$. But $B_{\perp} = P^{-1}(0)$. Hence $P\gamma = 0$.

Now, once knowing how to construct one projection $P : Z \to Z$, the construction of the whole PRI is standard, see, e.g., [F, Section 6.1]. ∎

The following Šmulyan like duality will also be frequently used.

Proposition 2. *Let $(X, \|\cdot\|)$ be a Banach space, $M \subset B_X,\ \varepsilon \ge 0$, and consider vectors $x \in X,\ x^* \in X^*$ such that $\langle x, x^*\rangle = 1 = \|x\| = \|x^*\|^*$.*
If the norm $\|\cdot\|$ on X is $\varepsilon - M-$smooth at x, then

$$\limsup_{n\to\infty} \sup \big|\big\langle M, x^* - x_n^*\big\rangle\big| \le \varepsilon \ \ \textit{whenever} \ \ x_n^* \in B_{X^*} \ \ \textit{and} \ \ \langle x, x_n^*\rangle \to 1. \qquad (1)$$

If (1) holds, then the norm $\|\cdot\|$ on X is $2\varepsilon - M-$smooth at x.

Proof. Assume that $\|\cdot\|$ is $\varepsilon - M-$smooth at x and consider $x_n^*,\ n \in \mathbb{N}$, as in (1). Then for every $h \in M$, every $n \in \mathbb{N}$, and every $t > 0$ we have

$$\begin{aligned}\big\langle \pm th, x^* - x_n^*\big\rangle &= \big\langle x \pm th, x^*\big\rangle + \big\langle x \mp th, x_n^*\big\rangle - 2 + \big(2 - \big\langle x, x^* + x_n^*\big\rangle\big) \\ &\le (\|x + th\| + \|x - th\| - 2) + \big(2 - \big\langle x, x^* + x_n^*\big\rangle\big).\end{aligned}$$

Hence, for every $t > 0$ and every $n \in \mathbb{N}$

$$\sup \big|\big\langle M, x^* - x_n^*\big\rangle\big| \le \frac{1}{t} \sup\big\{\|x + th\| + \|x - th\| - 2;\ h \in M\big\} + \frac{1}{t}\big(2 - \big\langle x, x^* + x_n^*\big\rangle\big).$$

Therefore

$$\limsup_{n\to\infty} \sup \big|\big\langle M, x^* - x_n^*\big\rangle\big| \le \frac{1}{t} \sup\big\{\|x + th\| + \|x - th\| - 2;\ h \in M\big\}$$

for every $t > 0$. Thus, taking into account the $\varepsilon - M-$smoothness of $\|\cdot\|$ at x, we conclude that $\limsup_{n\to\infty} \sup \left|\left\langle M, x^* - x_n^*\right\rangle\right| \leq \varepsilon$, and (1) is proved.

Assume that (1) holds. Take an arbitrary $\varepsilon' > \varepsilon$. From (1) find $\delta > 0$ so small that $\sup |\langle M, x^* - y^*\rangle| < \varepsilon'$ whenever $y^* \in B_{X^*}$ and $1 - \langle x, y^*\rangle < 2\delta$. Take any $0 < t < \delta$. Fix for a while any $h \in M$. Find $u^*, v^* \in B_{X^*}$ such that $\|x + th\| = \langle x + th, u^*\rangle$ and $\|x - th\| = \langle x - th, v^*\rangle$. We remark that then

$$1 - \langle x, u^*\rangle \leq 1 - \langle x + th, u^*\rangle + t = \|x\| - \|x + th\| + t \leq 2t < 2\delta,$$

and likewise, $1 - \langle x, v^*\rangle < 2\delta$. Hence $\langle h, u^* - x^*\rangle < \varepsilon'$, $\langle h, x^* - v^*\rangle < \varepsilon'$, and so

$$\|x + th\| + \|x - th\| - 2\|x\| \leq \langle th, u^*\rangle - \langle th, v^*\rangle = t\langle h, u^* - x^*\rangle + t\langle h, x^* - v^*\rangle < 2\varepsilon' t.$$

This holds for every $h \in M$ and every $0 < t < \delta$. Hence $\|\cdot\|$ is $2\varepsilon' - M-$smooth at x. And, as ε' could be taken arbitrarily close to ε, we are done. ∎

The following result exhibits a remarkable link between smoothness and weak compactness. A bounded subset $M \subset X$ of a Banach space X is called $\varepsilon-$*weakly compact* if $\overline{M}^* \subset X + \varepsilon B_{X^{**}}$. It was proved in [FHMZ] that the weak$^*-$closed convex hull in X^{**} of an $\varepsilon-$weakly compact set is $2\varepsilon-$weakly compact, and in [GHM] that 2 is the best possible factor. It is 1 in case of WLD spaces [FHMZ].

Proposition 3. *Let $\emptyset \neq M \subset B_X$, $\varepsilon \geq 0$, and assume that the norm $\|\cdot\|$ on X is such that the double dual norm $\|\cdot\|^{**}$ on X^{**} is $\varepsilon - M-$smooth. Then M is $\varepsilon-$weakly compact.*

Proof. Take an arbitrary $x^{**} \in \overline{M}^*$. Put $d = \operatorname{dist}(x^{**}, X)$. Assume that $d > 0$. By Hahn-Banach theorem find $F \in X^{***}$, with $\|F\|^{***} = 1$, such that it vanishes on X and that $\langle x^{**}, F\rangle = d$. Fix any $\delta > 0$. From the Bishop-Phelps theorem find $G \in X^{***}$ and $x_0^{**} \in X^{**}$ such that $\|G - F\| < \delta$ and $\langle x_0^{**}, G\rangle = 1 = \|x_0^{**}\|^{**} = \|G\|^{***}$. Using Goldstine's theorem we find a sequence (x_k^*) in B_{X^*} so that

$$\langle x_0^{**}, x_k^*\rangle \to \langle x_0^{**}, G\rangle \quad \text{and} \quad \langle x^{**}, x_k^*\rangle \to \langle x^{**}, G\rangle \quad \text{as} \quad k \to \infty.$$

Since the norm $\|\cdot\|^{**}$ is $\varepsilon - M-$smooth, Proposition 2 yields that $\limsup_{k\to\infty} \sup \left\langle M, x_k^* - G\right\rangle \leq \varepsilon$. But F is vanishing on X; so

$$\begin{aligned}
&\limsup_{k\to\infty} \sup \left\langle M, x_k^*\right\rangle = \limsup_{k\to\infty} \sup \left\langle M, x_k^* - F\right\rangle \\
&\leq \limsup_{k\to\infty} \sup \left\langle M, x_k^* - G\right\rangle + \limsup_{k\to\infty} \sup \left\langle M, G - F\right\rangle \leq \varepsilon + \delta.
\end{aligned}$$

Hence $\limsup_{k\to\infty} \langle x^{**}, x_k^*\rangle \leq \varepsilon + \delta$, and so, $\langle x^{**}, G\rangle \leq \varepsilon + \delta$. Now

$$\operatorname{dist}(x^{**}, X) = d = \langle x^{**}, F\rangle = \langle x^{**}, G\rangle + \langle x^{**}, F - G\rangle \leq \varepsilon + \delta + \delta.$$

Then, since $\delta > 0$ was arbitrary, we get $\operatorname{dist}(x^{**}, X) \leq \varepsilon$. We thus proved that $\overline{M}^* \subset X + \varepsilon B_{X^{**}}$. ∎

Note that the assumptions of Proposition 3 are satisfied if the norm on X is *uniformly* $\varepsilon - M-$*smooth*; this is a simple consequence of Goldstine's theorem.

Let Γ be an infinite set. We recall that *Day's norm* $\|\cdot\|_{\mathcal{D}}$ on $\ell_\infty(\Gamma)$ is defined by

$$\|u\|_{\mathcal{D}}{}^2 = \sup\Big\{\sum_{j=1}^n 2^{-j}u(\gamma_j)^2;\ n \in \mathbb{N},\ \gamma_1,\dots,\gamma_n \in \Gamma,\ \gamma_k \neq \gamma_l \text{ if } k \neq l\Big\},\quad u \in \ell_\infty(\Gamma).$$

It is easy to check that $\|\cdot\|_{\mathcal{D}}$ is an equivalent norm on $\ell_\infty(\Gamma)$.

Lemma 1. ([D, page 95]) *Let* $(s_k)_{(k\in\mathbb{N})}$, $(t_k)_{(k\in\mathbb{N})}$ *be two non-increasing sequences of non-negative numbers such that* $s_k = t_k = 0$ *for all large* $k \in \mathbb{N}$. *Let* $\pi : \mathbb{N} \to \mathbb{N}$ *be an injective surjection. Then*

$$\sum_{k=1}^\infty s_k(t_k - t_{\pi(k)}) \geq 0$$

and for every $K \in \mathbb{N}$ *either* $\pi\{1,\dots,K\} = \{1,\dots,K\}$ *or*

$$(s_K - s_{K+1})(t_K - t_{K+1}) \leq \sum_{k=1}^\infty s_k(t_k - t_{\pi(k)}).$$

Proposition 4. *Let* Γ *be an infinite set, let* $u \in \ell_\infty(\Gamma)$, $\varepsilon > 0$, *and assume that the set* $\{\gamma \in \Gamma;\ |u(\gamma)| > \varepsilon\}$ *is finite. Let* $u_n \in \ell_\infty(\Gamma)$, $n \in \mathbb{N}$, *be such that*

$$2\|u\|_{\mathcal{D}}{}^2 + 2\|u_n\|_{\mathcal{D}}{}^2 - \|u+u_n\|_{\mathcal{D}}{}^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Then $\limsup_{n\to\infty}\|u-u_n\|_\infty \leq 3\varepsilon$.

Proof. The argument is an elaboration of that due to Rainwater [D, pages 94–100]. Denote $A = \{\gamma \in \Gamma;\ |u(\gamma)| > \varepsilon\}$ and let $\{\alpha_1,\dots,\alpha_K\}$ be an enumeration of A such that $|u(\alpha_1)| \geq |u(\alpha_2)| \geq \dots \geq |u(\alpha_K)|$ $(> \varepsilon)$. Denote

$$\Delta = \big(2^{-K} - 2^{-K-1}\big)\big(u(\alpha_K)^2 - \varepsilon^2\big);$$

this is a positive number. Fix an arbitrary $n \in \mathbb{N}$. We find a set $B_n \neq A$, $A \subset B_n \subset \Gamma$ such that

$$\|u+u_n\|_{\mathcal{D}}{}^2 - \tfrac{1}{n} < \big\|(u+u_n)_{|B_n}\big\|_{\mathcal{D}}{}^2.$$

Enumerate

$$B_n = \big\{\alpha_1^n,\dots,\alpha_{K_n}^n\big\} = \big\{\beta_1^n,\dots,\beta_{K_n}^n\big\}$$

in such a way that

$$|u(\alpha_1^n)| \geq |u(\alpha_2^n)| \geq \dots \geq |u(\alpha_{K_n}^n)|,$$

$$|(u+u_n)(\beta_1^n)| \geq |(u+u_n)(\beta_2^n)| \geq \dots \geq |(u+u_n)(\beta_{K_n}^n)|.$$

Then, of course, $\alpha_1^n = \alpha_1, \dots, \alpha_K^n = \alpha_K$ and $K_n > K$. Note that

$$\sum_{k=1}^{K_n} 2^{-k} u(\alpha_k^n)^2 = \left\| u_{|B_n} \right\|_{\mathcal{D}}^2 \le \|u\|_{\mathcal{D}}^2, \qquad \sum_{k=1}^{K_n} 2^{-k} u_n(\beta_k^n)^2 \le \left\| u_{n|B_n} \right\|_{\mathcal{D}}^2 \le \|u_n\|_{\mathcal{D}}^2,$$

and

$$\|u+u_n\|_{\mathcal{D}}^2 - \tfrac{1}{n} < \left\| (u+u_n)_{|B_n} \right\|_{\mathcal{D}}^2 = \sum_{k=1}^{K_n} 2^{-k} (u+u_n)(\beta_k^n)^2.$$

Let us estimate

$$\begin{aligned}
&2\|u\|_{\mathcal{D}}^2 + 2\|u_n\|_{\mathcal{D}}^2 - \|u+u_n\|_{\mathcal{D}}^2 \\
>&2\|u_{|B_n}\|_{\mathcal{D}}^2 + 2\|u_{n|B_n}\|_{\mathcal{D}}^2 - \|(u+u_n)_{|B_n}\|_{\mathcal{D}}^2 - \tfrac{1}{n} \\
\ge&2\sum_{k=1}^{K_n} 2^{-k} u(\alpha_k^n)^2 + 2\sum_{k=1}^{K_n} 2^{-k} u_n(\beta_k^n)^2 - \sum_{k=1}^{K_n} 2^{-k} (u+u_n)(\beta_k^n)^2 - \tfrac{1}{n} \\
=&2\sum_{k=1}^{K_n} 2^{-k} \big(u(\alpha_k^n)^2 - u(\beta_k^n)^2 \big) + \sum_{k=1}^{K_n} 2^{-k} \big(u(\beta_k^n) - u_n(\beta_k^n) \big)^2 - \tfrac{1}{n} \ge -\tfrac{1}{n}
\end{aligned}$$

(indeed, the first summand is nonnegative by Lemma 1). Hence, letting $n \to \infty$ here, we get

$$\sum_{k=1}^{K_n} 2^{-k} \big(u(\alpha_k^n)^2 - u(\beta_k^n)^2 \big) \to 0 \quad \text{and} \quad u(\beta_k^n) - u_n(\beta_k^n) \to 0 \quad \text{for} \quad k = 1, \dots, K.$$

Find $n_0 \in \mathbb{N}$ so large that for all $n \in \mathbb{N}$ greater than n_0

$$\sum_{k=1}^{K_n} 2^{-k} \big(u(\alpha_k^n)^2 - u(\beta_k^n)^2 \big) < \Delta \quad \text{and} \quad |u(\beta_k^n) - u_n(\beta_k^n)| < 3\varepsilon \quad \text{for} \quad k = 1, \dots, K. \tag{2}$$

Fix for a while any such n. Let $\pi : \mathbb{N} \to \mathbb{N}$ be defined as

$$\pi(k) = \begin{cases} k & \text{if } k \in \mathbb{N} \text{ and } k > K, \\ j & \text{if } k \in \mathbb{N},\ k \le K_n, \text{ and } \beta_k^n = \alpha_j^n. \end{cases}$$

Clearly, π is an injective mapping from $\mathbb{N}$ onto $\mathbb{N}$. We `claim` that $\{\alpha_1^n, \dots, \alpha_K^n\} = \{\beta_1^n, \dots, \beta_K^n\}$, that is, $\pi\{1, \dots, K\} = \{1, \dots, K\}$. Assume that this is not true. Putting $s_k = 2^{-k}$, $t_k = u(\alpha_k^n)^2$ for $k = 1, \dots, K_n$ and $s_k = t_k = 0$ for $k = K_n + 1, K_n + 2, \dots$, we get from Lemma 1 and (2)

$$(0 < \Delta \le)\ \big(2^{-K} - 2^{-K-1}\big)\Big(u(\alpha_K^n)^2 - u(\alpha_{K+1}^n)^2 \Big) \le \sum_{k=1}^{K_n} 2^{-k} \big(u(\alpha_k^n)^2 - u(\beta_k^n)^2 \big)\ (< \Delta),$$

a contradiction. This proves the claim. For all $n > n_0$ we thus have that $\{\beta_1^n, \ldots, \beta_K^n\} = A = \{\alpha_1, \ldots, \alpha_K\}$ and, by (2), that

$$|(u_n - u)(\alpha_1)| < 3\varepsilon, \ldots, \ |(u_n - u)(\alpha_K)| < 3\varepsilon.$$

Now we are ready to prove that $\limsup_{n\to\infty} \|u_n - u\|_\infty \le 3\varepsilon$. Assume the contrary. Then there is an infinite set $N \subset \mathbb{N}$ such that for every $n \in N$ there is $\gamma_n \in \Gamma$ so that $|(u_n - u)(\gamma_n)| > 3\varepsilon$. This immediately implies that $\gamma_n \notin A$ for all $n \in N$ with $n > n_0$. But for these n's we have

$$\sum_{k=1}^{K} 2^{-k} u_n(\alpha_k)^2 + 2^{-K-1} u_n(\gamma_n)^2 \le \|u_n\|_{\mathcal{D}}{}^2,$$

and so

$$2^{-K-1} \limsup_{n\in N,\ n\to\infty} u_n(\gamma_n)^2 \le \lim_{n\to\infty} \|u_n\|_{\mathcal{D}}{}^2 - \lim_{n\to\infty} \sum_{k=1}^{K} 2^{-k} u_n(\alpha_k)^2$$

$$= \|u\|_{\mathcal{D}}{}^2 - \sum_{k=1}^{K} 2^{-k} u(\alpha_k)^2 = \|u_{|\Gamma\setminus A}\|_{\mathcal{D}}{}^2 \le \varepsilon^2 \sum_{k=K+1}^{\infty} 2^{-k} = \varepsilon^2 \cdot 2^{-K},$$

$$\limsup_{n\in N,\ n\to\infty} |u_n(\gamma_n)| \le \sqrt{2}\varepsilon < 2\varepsilon.$$

Thus

$$(3\varepsilon \le) \ \limsup_{n\in N,\ n\to\infty} |(u_n - u)(\gamma_n)| < 2\varepsilon + \varepsilon = 3\varepsilon,$$

a contradiction. ∎

If $u \in c_0(\Gamma)$ in the above Proposition 4, then the Day's norm $\|\cdot\|_{\mathcal{D}}$ is LUR at u. Thus we get a well known result that this norm on $c_0(\Gamma)$ is LUR.

The next proposition shows a property of uniform rotundity of Day's norm. We shall elaborate the proof of [Tr2, Proposition 1] due to Troyanski, see also [FGHZ, Remark 1]. Let Γ be an infinite set. If $\beta \in \Gamma$, we define a canonical projection $\pi_\beta : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ by

$$\pi_\beta u(\gamma) = \begin{cases} u(\beta) & \text{if } \gamma = \beta \\ 0 & \text{if } \gamma \in \Gamma\setminus\{\beta\}, \end{cases} \qquad u \in \ell_\infty(\Gamma).$$

We shall need the following easily provable facts. It should be noted that Troyanski considers elements of $c_0(\Gamma)$. However, a careful look at its proofs reveals that the facts work with elements of $\ell_\infty(\Gamma)$.

Fact 1. ([Tr2]) *Let $u \in \ell_\infty(\Gamma)$ and $\beta \in \Gamma$ be such that $u(\beta) \ne 0$ and assume that $\#\{\gamma \in \Gamma;\ |u(\gamma)| \ge 2^{-1/2}|u(\beta)|\} =: i < +\infty$. Then*

$$\|u\|_{\mathcal{D}}{}^2 \ge \|u - \pi_\beta u\|_{\mathcal{D}}{}^2 + 2^{-i-1} u(\beta)^2.$$

Fact 2. ([Tr2]) *Let* $u, v \in B_{\ell_\infty(\Gamma)}$ *and* $\beta \in \Gamma$ *be such that* $u(\beta) + v(\beta) \neq 0$ *and assume that* $\#\{\gamma \in \Gamma;\ |u(\gamma) + v(\gamma)| \geq |u(\beta) + v(\beta)|\} =: k < +\infty$. *Then*

$$2\|u\|_{\mathcal{D}}{}^2 + 2\|v\|_{\mathcal{D}}{}^2 - \|u + v\|_{\mathcal{D}}{}^2 \geq 2^{-k-1}\big(u(\beta) - v(\beta)\big)^2.$$

Proposition 5. *Let* $\Gamma \neq \emptyset$ *be a set and consider a linear set* $Y \subset \ell_\infty(\Gamma)$. *Assume that there exist* $\varepsilon > 0$, *and* $i, k \in \mathbb{N}$ *such that*

$$\forall u \in Y \cap B_{\ell_\infty(\Gamma)} \quad \#\big\{\gamma \in \Gamma;\ u(\gamma) > \varepsilon\big\} < i \quad \text{and} \quad \#\big\{\gamma \in \Gamma;\ u(\gamma) > 2^{-i-1}\varepsilon\big\} < k.$$

Let $u_n,\ v_n \in Y \cap B_{\ell_\infty(\Gamma)},\ n \in \mathbb{N}$, *be such that* $2\|u_n\|_{\mathcal{D}}{}^2 + 2\|v_n\|_{\mathcal{D}}{}^2 - \|u_n + v_n\|_{\mathcal{D}}{}^2 \to 0$ *as* $n \to \infty$. *Then* $\limsup_{n\to\infty} \big\|u_n - v_n\big\|_{\ell_\infty(\Gamma)} \leq 4\varepsilon$.

Proof. The argument is a refinement of the proof of [Tr1, Proposition 1], see also [FGHZ, Lemma 5]. Assume that the conclusion is false. Then, when replacing the original sequences $(u_n),\ (v_n)$ by suitable subsequences, we may and do assume that $\big\|u_n - v_n\big\| > 4\varepsilon$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ find $\gamma_n \in \Gamma$ so that $|u_n(\gamma_n) - v_n(\gamma_n)\big| > 4\varepsilon$. We shall first observe that $\limsup_{n\to\infty} \big|u_n(\gamma_n) + v_n(\gamma_n)\big| > 2^{-i}\varepsilon$. Assume this is not so. Then for all (large) $n \in \mathbb{N}$ we have $\big|u_n(\gamma_n) + v_n(\gamma_n)\rangle\big| \leq 2^{-i}\varepsilon$ and so

$$2\big|u_n(\gamma_n)\big| \geq \big|u_n(\gamma_n) - v_n(\gamma_n)\big| - \big|u_n(\gamma_n) + v_n(\gamma_n)\big| > 4\varepsilon - 2^{-i}\varepsilon > 2\sqrt{2}\varepsilon,$$

and hence

$$\#\big\{\gamma \in \Gamma;\ \big|u_n(\gamma)\big| > 2^{-1/2}\big|u_n(\gamma_n)\big|\big\} \leq \#\big\{\gamma \in \Gamma;\ \big|u_n(\gamma)\big| > \varepsilon\big\} < 2i,$$

and by the Fact 1,

$$\big\|u_n\big\|_{\mathcal{D}}{}^2 \geq \big\|u_n - \pi_{\gamma_n}(u_n)\big\|_{\mathcal{D}}{}^2 + 2^{-2i-1}u_n(\gamma_n)^2 \geq \big\|u_n - \pi_{\gamma_n}(u_n)\big\|_{\mathcal{D}}{}^2 + 2^{-2i} \cdot \varepsilon^2.$$

Also, for every large $n \in \mathbb{N}$ we have

$$\big\|(u_n + v_n)\big\|_{\mathcal{D}}{}^2 - \big\|(u_n + v_n) - \pi_{\gamma_n}(u_n + v_n)\big\|_{\mathcal{D}}{}^2 \leq \tfrac{1}{2}\big(u_n(\gamma_n) + v_n(\gamma_n)\big)^2 \leq 2^{-2i-1} \cdot \varepsilon^2.$$

Thus, by the above and the convexity

$$\begin{aligned}
&2\big\|u_n\big\|_{\mathcal{D}}{}^2 + 2\big\|v_n\big\|_{\mathcal{D}}{}^2 - \big\|(u_n + v_n)\big\|_{\mathcal{D}}{}^2 \geq 2\big\|u_n - \pi_{\gamma_n}(u_n)\|_{\mathcal{D}}{}^2 + 2^{-2i+1} \cdot \varepsilon^2\\
&+2\big\|v_n - \pi_{\gamma_n}(v_n)\big\|_{\mathcal{D}}{}^2 - \big\|(u_n + v_n) - \pi_{\gamma_n}(u_n + v_n)\big\|_{\mathcal{D}}{}^2\\
&+\big\|(u_n + v_n) - \pi_{\gamma_n}(u_n + v_n)\big\|_{\mathcal{D}}{}^2 - \big\|(u_n + v_n)\big\|_{\mathcal{D}}{}^2\\
&\geq 2^{-2i+1} \cdot \varepsilon^2 - 2^{-2i-1} \cdot \varepsilon^2 > 0
\end{aligned}$$

for all large $n \in \mathbb{N}$. But, for $n \to \infty$ the most left hand side in the above chain of inequalities goes to 0, a contradiction. We thus proved that $\limsup_{n\to\infty} \big|u_n(\gamma_n) + v_n(\gamma_n)\big| > 2^{-i}\varepsilon$.

Then for infinitely many $n \in \mathbb{N}$ we have from the assumptions

$$\#\{\gamma \in \Gamma;\ |u_n(\gamma)+v_n(\gamma)| \geq |u_n(\gamma_n)+v_n(\gamma_n)|\} \leq \#\{\gamma \in \Gamma;\ |u_n(\gamma)+v_n(\gamma)| > 2^{-i}\varepsilon\} < 2k.$$

Hence, by the Fact 2,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left(2\|u_n\|_{\mathcal{D}}{}^2 + 2\|v_n\|_{\mathcal{D}}{}^2 - \|(u_n+v_n)\|_{\mathcal{D}}{}^2\right) \\ &\geq 2^{-2k-1} \limsup_{n\to\infty} \left(u_n(\gamma_n) - v_n(\gamma_n)\right)^2 > 2^{-2k-1} 16\varepsilon^2 \ (> 0), \end{aligned}$$

a contradiction. Therefore $\limsup_{n\to\infty} \|u_n - v_n\|_{\ell_\infty(\Gamma)} \leq 4\varepsilon$. ∎

Section 3 – Proofs of results

Proof of Theorem 1. We shall prove the following chain of implications: (i)⇒(ii)⇒(iii)⇒(iv)⇒(i)

(i)⇒(ii). For completeness we shall prove this. Find a total convex symmetric weakly compact set $K \subset B_X$; it exists by Krein's theorem. We shall find a set $\Gamma \subset K$ satisfying (ii). If X is separable, we can take $\Gamma = \{\frac{1}{n}k_n;\ n \in \mathbb{N}\}$ where $\{k_n;\ n \in \mathbb{N}\}$ is any dense countable set in K. Let $\aleph$ be an uncountable cardinal and assume that we already found a set $\Gamma \subset K$ as in the assertion (ii) whenever the density of X was less than $\aleph$. Now assume that our X has density $\aleph$.

Proposition 1 (where we take $\Gamma = \emptyset$) yields a PRI $(P_\alpha;\ \omega \leq \alpha \leq \mu)$ on X such that $P_\alpha K \subset K$ for every $\alpha \leq \mu$. For $\alpha \in [\omega, \mu)$ denote $Q_\alpha = P_{\alpha+1} - P_\alpha$; observe that then $Q_\alpha X$ has density less than $\aleph$ and is also a WCG space. Indeed, it contains a total weakly compact set $\frac{1}{2}Q_\alpha K$. For every $\alpha \in [\omega, \mu)$ find, by the induction assumption, a total set $\Gamma_\alpha \subset \frac{1}{2}Q_\alpha K$ $(\subset K)$ satisfying the assertion (ii).

Put $\Gamma = \bigcup_{\alpha<\mu} \Gamma_\alpha$. It remains to verify the assertion (ii) for this set. As the set $\bigcup_{\alpha<\mu} Q_\alpha X$ is total in X, so is the set Γ. Fix any $\varepsilon > 0$ and any $x^* \in X^*$. We have to show that the set $\{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle > \varepsilon\}$ is finite. Denote

$$F = \{\alpha \in [\omega, \mu);\ \langle \gamma, x^* \rangle > \varepsilon \ \text{ for some } \ \gamma \in \Gamma_\alpha\}.$$

We shall show that the set F is finite. Assume, by contrary, that F contains an infinite sequence $\alpha_1 < \alpha_2 < \cdots < \mu$. For each $i \in \mathbb{N}$ find $\gamma_i \in \Gamma_{\alpha_i}$ $(\subset K)$. Let $k \in K$ be a weak cluster point of the sequence $(\gamma_i)_{i\in\mathbb{N}}$; it exists as K is a weakly compact set. Then $\langle k, x^* \rangle \geq \varepsilon > 0$. But, for every fixed $\alpha \in [\omega, \mu)$ we have $Q_\alpha \circ Q_{\alpha_i} = 0$ for all $i \in \mathbb{N}$ but, eventually, one; hence $Q_\alpha(k) = 0$. Therefore $k = 0$, a contradiction. This proves that the set F is finite. Now we can estimate

$$\#\{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle > \varepsilon\} \leq \sum_{\alpha \in F} \#\{\gamma \in \Gamma_\alpha;\ \langle \gamma, x^*{}_{|Q_\alpha X} \rangle > \varepsilon\} < \aleph_0$$

and hence the assertion (ii) is verified for our X.

(ii)$\Rightarrow$(iii). Define $Tx^* = (\langle \gamma, x^* \rangle;\ \gamma \in \Gamma),\ x^* \in X^*$. By (ii), T is a linear bounded weak* to weak continuous injection from X^* into $c_0(\Gamma)$. Then the adjoint operator T^* maps $\ell_1(\Gamma)$ into X. Let $\|\cdot\|_{\mathcal{D}}$ denote Day's norm on $\ell_\infty(\Gamma)$. (Actually, any equivalent LUR norm on $c_0(\Gamma)$ would work.) Denote by U the dual unit ball in $\ell_1(\Gamma)$ with respect to this norm. Put then

$$D = \mathrm{co}_2\big(B_X, T^*U\big) =: \big\{\alpha u + \beta v;\ u \in B_X,\ v \in T^*U,\ \alpha \geq 0,\ \beta \geq 0,\ \alpha^2 + \beta^2 \leq 1\big\}.$$

Then D is a convex symmetric bounded set with the origin in its interior. Using the weak compactness of T^*U, it is not difficult to show that the set D is weakly closed, and hence closed. Let $|\cdot|$ be the Minkowski functional of D; this will be an equivalent norm on X and $B_{(X,|\cdot|)} = D$. We observe that, owing to the weak compactness of T^*U, the unit ball in $(X^{**}, |\cdot|^{**})$ will be $\mathrm{co}_2\big(B_{X^{**}}, T^*U\big)$. Thus

$$\big(|F|^{***}\big)^2 = \big(\|F\|^{***}\big)^2 + \sup\langle F, T^*U\rangle^2 \quad \text{for} \quad F \in X^{***}.$$

It remains to verify that $|\cdot|^{***}$ is $\Gamma-$LUR. So consider $F,\ F_n \in X^{***},\ n \in \mathbb{N}$, satisfying

$$2\big(|F|^{***}\big)^2 + 2\big(|F_n|^{***}\big)^2 - \big(|F + F_n|^{***}\big)^2 \to 0 \quad \text{as} \quad n \to \infty.$$

The convexity yields that

$$2\sup\langle T^*U, F\big\rangle^2 + 2\sup\langle T^*U, F_n\big\rangle^2 - \sup\langle T^*U, F + F_n\big\rangle^2 \to 0 \quad \text{as} \quad n \to \infty$$

and hence

$$2\big\|T(F_{|X})\big\|_{\mathcal{D}}^{\,2} + 2\big\|T(F_{n|X})\big\|_{\mathcal{D}}^{\,2} - \big\|T(F_{|X} + F_{n|X})\big\|_{\mathcal{D}}^{\,2} \to 0 \quad \text{as} \quad n \to \infty.$$

But $\|\cdot\|_{\mathcal{D}}$ restricted to $c_0(\Gamma)$ is LUR; this well known fact easily follows from Proposition 4. Therefore

$$\big\|(T(F_{|X}) - T(F_{n|X})\big\|_{\mathcal{D}} \to 0, \quad \text{i.e.,} \quad \sup\big|\big\langle \Gamma, F - F_n\big\rangle\big| \to 0 \quad \text{as} \quad n \to \infty$$

since the Day's norm is equivalent with the canonical norm $\|\cdot\|_\infty$ on $c_0(\Gamma)$. The parenthetic part of (iii) is thus proved. From this, using a Šmulyan duality argument, we can easily deduce that the norm $|\cdot|^{**}$ is $\Gamma-$smooth, see Proposition 2 or [DGZ, Proposition II.1.5].

(iii)$\Rightarrow$(iv). By Proposition 3, (iii) implies that the set M is weakly relatively compact. Thus X is WCG, and hence weakly Lindelöf determined, see, e.g., the implication (i)$\Rightarrow$(ii). The second part of (iv) follows trivially.

(iv)$\Rightarrow$(ii). Clearly, we may and do assume that the set M in (iv) is convex symmetric and closed. We shall find the set Γ satisfying the assertion (ii) by a transfinite induction. If X is separable, then we can take $\Gamma = \big\{\frac{1}{n}x_n;\ n \in \mathbb{N}\big\}$ where $\big\{x_n;\ n \in \mathbb{N}\big\}$ is any dense countable set in M. Let $\aleph$ be an uncountable cardinal and assume that we already found a set $\Gamma \subset M$ as in the assertion (ii) whenever the density of X was less than $\aleph$. Now assume that our X has density $\aleph$.

Proposition 1 (where we take $\Gamma = \emptyset$) yields a PRI $(P_\alpha;\ \omega \le \alpha \le \mu)$ on $(X, |\cdot|)$ such that $P_\alpha M \subset M$ for every $\alpha \in [\omega, \mu)$. For $\alpha \in [\omega, \mu)$ denote $Q_\alpha = P_{\alpha+1} - P_\alpha$; observe that then $Q_\alpha X$ has density less than $\aleph$ and the norm $|\cdot|$ restricted to this subspace is $Q_\alpha M-$smooth. For every $\alpha \in [\omega, \mu)$ find, by the induction assumption, a total set $\Gamma_\alpha \subset \frac{1}{2} Q_\alpha M\ (\subset M)$ satisfying (ii).

Put $\Gamma = \bigcup_{\alpha<\mu} \Gamma_\alpha$. It remains to verify the assertion (ii) for this set. As the set $\bigcup_{\alpha<\mu} Q_\alpha X$ is total in X, so is the set Γ. Fix any $\varepsilon > 0$ and any $x^* \in B_{X^*}$. We have to show that the set $\big\{\gamma \in \Gamma;\ \langle \gamma, x^*\rangle > \varepsilon\big\}$ is finite. In order to do so we shall be proving the following statement:

$$\#\big\{\alpha \in [\omega, \mu);\ \langle \gamma, P_\beta^* x^*\rangle > \varepsilon \ \text{ for some } \ \gamma \in \Gamma_\alpha\big\} < \aleph_0 \tag{β}$$

for all $\beta \in [\omega, \mu]$. Clearly, (ω) is valid. Also, since $P_{\beta+1} \circ Q_{\beta+1} = 0$, we have that (β) implies $(\beta + 1)$ for every $\beta < \mu$. Now let $\lambda \le \mu$ be any limit ordinal and assume that we verified (β) for every $\beta < \lambda$. Find $\beta < \lambda$ so that $\sup\langle M, P_\lambda^* x^* - P_\beta^* x^*\rangle < \varepsilon$. This follows from the $M-$smoothness of $|\cdot|$ via Proposition 2 (actually, if $P_\lambda^* x^*$ does not attain its norm at an element of $P_\lambda X$, some extra work is needed here). We observe that if $\langle \gamma, P_\lambda^* x^*\rangle > \varepsilon$ for some $\gamma \in \Gamma_\alpha$, where $\alpha \in [\omega, \mu)$, then, as $\gamma \in M$,

$$\langle \gamma, P_\beta^* x^*\rangle = \langle \gamma, P_\lambda^* x^*\rangle - \langle \gamma, P_\lambda^* x^* - P_\beta^* x^*\rangle > \varepsilon - \varepsilon = 0,$$

and, so we must have $\alpha < \beta$. Thus

$$\begin{aligned} &\#\big\{\alpha \in [\omega, \mu);\ \langle \gamma, P_\lambda^* x^*\rangle > \varepsilon \ \text{ for some } \ \gamma \in \Gamma_\alpha\big\} \\ =&\#\big\{\alpha \in [\omega, \mu);\ \langle \gamma, P_\beta^* x^*\rangle > \varepsilon \ \text{ for some } \ \gamma \in \Gamma_\alpha\big\} < \aleph_0 \end{aligned}$$

and hence (λ) holds. We thus proved (β) for every $\beta \le \mu$. In particular, (μ) holds, that is, given any $\varepsilon > 0$, the set

$$F = \big\{\alpha \in [\omega, \mu);\ \langle \gamma, x^*\rangle > \varepsilon \ \text{ for some } \ \gamma \in \Gamma_\alpha\big\}$$

is finite and, so

$$\#\big\{\gamma \in \Gamma;\ \langle \gamma, x^*\rangle > \varepsilon\big\} = \sum_{\alpha \in F} \#\big\{\gamma \in \Gamma_\alpha;\ \langle \gamma, x^*\rangle > \varepsilon\big\} < \aleph_0$$

by the induction assumption. The assertion (ii) is thus verified for our X.

(ii)⇒(i) is obvious since the set $\Gamma \cup \{0\}$ is then weakly compact. ∎

Proof of Theorem 2. We shall be proving the following chain of implications:
(i)⇒(ii)⇒(iii)⇒(i)⇒(iv)⇒(v)⇒(ii).

(i)⇒(ii). If X is separable, then, clearly, every total countable set $\Gamma \subset B_X$ satisfies (ii) (indeed, the sets Γ_n^ε can be singletons). Let $\aleph$ be an uncountable cardinal and assume that the implication has already been verified for every space of density less than $\aleph$. Now assume that a Banach space X, of density $\aleph$, is a subspace of a WCG space Z. Find a total convex symmetric weakly compact set $K \subset Z$; it exists by Krein's theorem.

Proposition 1 (where we take $\Gamma = \emptyset$) yields a PRI $(P_\alpha;\ \omega \le \alpha \le \mu)$ on Z such that $P_\alpha X \subset X$ and $P_\alpha K \subset K$ for every $\alpha \in [\omega, \mu)$. For $\alpha \in [\omega, \mu)$ denote $Q_\alpha = P_{\alpha+1} - P_\alpha$; observe that then $Q_\alpha X$ has density less than $\aleph$ and is a subspace of the WCG space $Q_\alpha Z$. For every $\alpha \in [\omega, \mu)$ find, by the induction assumption, a total set $\Gamma_\alpha \subset B_{Q_\alpha X}$ with the properties listed in the assertion (ii).

Put $\Gamma = \bigcup_{\alpha<\mu} \Gamma_\alpha$. It remains to verify the assertion (ii) for this set. As the set $\bigcup_{\alpha<\mu} Q_\alpha X$ is total in X, so is the set Γ. Further fix any $\varepsilon > 0$. For $\alpha \in [\omega, \mu)$ let $\Gamma_\alpha = \bigcup_{n=1}^\infty \Gamma^\varepsilon_{\alpha,n}$ be the decomposition from the assertion (ii) in the subspace $Q_\alpha X$. For $n, m \in \mathbb{N}$ put then

$$\Gamma^\varepsilon_{n,m} = \big(mK + \tfrac{\varepsilon}{4} B_Z\big) \cap \bigcup_{\alpha<\mu} \Gamma^\varepsilon_{\alpha,n} \backslash \big(\Gamma^\varepsilon_{n,m-1} \cup \cdots \cup \Gamma^\varepsilon_{n,1} \cup \{0\}\big).$$

Clearly, this is a countable family of mutually disjoint sets and $\Gamma = \bigcup_{n,m=1}^\infty \Gamma^\varepsilon_{n,m}$.

Fix any $n, m \in \mathbb{N}$ and any $x^* \in X^*$. It remains to show that

$$\#\big\{\gamma \in \Gamma^\varepsilon_{n,m};\ \langle \gamma, x^* \rangle > \varepsilon\big\} < \aleph_0.$$

Denote

$$F = \big\{\alpha \in [\omega, \mu);\ \langle \gamma, x^* \rangle > \varepsilon \ \text{ for some } \ \gamma \in \Gamma^\varepsilon_{n,m} \cap \Gamma_\alpha\big\}.$$

Fix for a while any $\alpha \in F$. Then, as $\Gamma_\alpha \cap \Gamma_\beta \subset \{0\}$ for $\beta \ne \alpha$, we have $\Gamma^\varepsilon_{n,m} \cap \Gamma_\alpha \subset \Gamma^\varepsilon_{\alpha,n} \cup \{0\}$, and so

$$\big\{\gamma \in \Gamma^\varepsilon_{n,m} \cap \Gamma_\alpha;\ \langle \gamma, x^* \rangle > \varepsilon\big\} \subset \big\{\gamma \in \Gamma^\varepsilon_{\alpha,n};\ \langle \gamma, x^* \rangle > \varepsilon\big\} = \big\{\gamma \in \Gamma^\varepsilon_{\alpha,n};\ \big\langle \gamma, {x^*}_{|Q_\alpha X} \big\rangle > \varepsilon\big\},$$

where the last set is, by the induction assumption, finite.

So, it remains to prove that the set F is finite. Assume, by contrary, that F contains an infinite sequence $\alpha_1 < \alpha_2 < \cdots < \mu$. Find $z^* \in Z^*$ such that ${z^*}_{|X} = x^*$. For each $i \in \mathbb{N}$ find $\gamma_i \in \Gamma^\varepsilon_{n,m} \cap \Gamma_{\alpha_i}$ so that $\langle \gamma_i, x^* \rangle > \varepsilon$. Write $\gamma_i = mk_i + z_i$, where $k_i \in K$ and $z_i \in \frac{\varepsilon}{4} B_Z$; then $\gamma_i = Q_{\alpha_i}(\gamma_i) = mQ_{\alpha_i}(k_i) + Q_{\alpha_i}(z_i)$, as $\gamma_i \in \Gamma_{\alpha_i} \subset Q_{\alpha_i} X$. We have

$$\varepsilon < \langle \gamma_i, x^* \rangle = \langle \gamma_i, z^* \rangle = m\langle Q_{\alpha_i}(k_i), z^* \rangle + \langle Q_{\alpha_i}(z_i), z^* \rangle \le m\langle Q_{\alpha_i}(k_i), z^* \rangle + \tfrac{\varepsilon}{2},$$

and so $\langle Q_{\alpha_i}(k_i), z^* \rangle > \frac{\varepsilon}{2m}$ for every $i \in \mathbb{N}$. Let $2k \in 2K$ be a weak cluster point of the sequence $\big(Q_{\alpha_i}(k_i)\big)_{i\in\mathbb{N}}$; it exists as $Q_\alpha(K) \subset 2K$ for every $\alpha \in [\omega, \mu)$. Then $\langle 2k, z^* \rangle \ge \frac{\varepsilon}{2m} > 0$. But, for every fixed $\alpha \in [\omega, \mu)$ we have $Q_\alpha \circ Q_{\alpha_i} = 0$ for all $i \in \mathbb{N}$ but, eventually, one; so $Q_\alpha(2k) = 0$. Therefore $2k = 0$, a contradiction. This proves that the set F is finite and hence the assertion (ii) is verified for our X.

Now assume that we have already given a total set $\Gamma \subset B_X$ which countably supports X^* and let (i) be satisfied. Then, of course, Γ countably supports Z^*. Proposition 1 yields a PRI $(P_\alpha;\ \omega \le \alpha \le \mu)$ on Z as above, with the additional property that $P_\alpha(\gamma) \in \{\gamma, 0\}$ for every $\alpha \in [\omega, \mu)$ and every $\gamma \in \Gamma$. For every $\alpha \in [\omega, \mu)$ put $\Gamma_\alpha = \Gamma \cap Q_\alpha X$. This is a total set in $Q_\alpha X$ which countably supports $(Q_\alpha X)^*$. Clearly $\Gamma = \bigcup_{\alpha<\mu} \Gamma_\alpha$. The rest of the proof is as above.

(ii)⇒(iii). Assume the assertion (ii) is satisfied. We realize that this is actually a Talagrand-Argyros-Farmaki condition from [Fa, Theorem 2.9]. Therefore (B_{X^*}, w^*) is an Eberlein compact. However, there is a direct way how to get (iii), see [FMZ4]. We shall repeat here this method. For $i \in \mathbb{N}$ define a function $\tau_i : \mathbb{R} \to \mathbb{R}$ by

$$\tau_i(y) = \begin{cases} t + \frac{1}{i} & \text{if } t \leq -\frac{1}{i}, \\ 0 & \text{if } -\frac{1}{i} \leq t \leq \frac{1}{i}, \\ t - \frac{1}{i} & \text{if } t \geq \frac{1}{i}. \end{cases}$$

Define then $\Phi : B_{X^*} \to \mathbb{R}^{\Gamma \times \mathbb{N}}$ by

$$\Phi(x^*)(\gamma, i) = \frac{1}{ni} \tau_i(\langle \gamma, x^* \rangle) \quad \text{if} \quad \gamma \in \Gamma_n^{1/i}, \quad n \in \mathbb{N}, \quad \text{and} \quad i \in \mathbb{N}.$$

Clearly, Φ is weak* to pointwise continuous. It is also injective. Indeed, take any two distinct elements x_1^*, x_2^* in B_{X^*}. Since the set Γ is total, there is $\gamma \in \Gamma$ so that $\langle \gamma, x_1^* \rangle \neq \langle \gamma, x_2^* \rangle$. Find $i \in \mathbb{N}$ so that $\frac{1}{i} < \max \big\{ |\langle \gamma, x_1^* \rangle|, |\langle \gamma, x_2^* \rangle| \big\}$. Then, surely, $\tau_i(\langle \gamma, x_1^* \rangle) \neq \tau_i(\langle \gamma, x_2^* \rangle)$, and hence $\Phi(x_1^*) \neq \Phi(x_2^*)$.

Fix any $x^* \in B_{X^*}$. It remains to show that $\Phi(x^*) \in c_0(\Gamma \times \mathbb{N})$. So, fix any $\varepsilon > 0$. Clearly, if $n, i \in \mathbb{N}$, and $n > \frac{1}{\varepsilon}$ or $i > \frac{1}{\varepsilon}$, then $|\Phi(x^*)(\gamma, i)| < \varepsilon$ for every $\gamma \in \Gamma_n^{1/i}$. In what follows let us fix any $n, i \in \mathbb{N}$, with $n, i \leq \frac{1}{\varepsilon}$ (This is a finite set of couples.) Then

$$\begin{aligned} \big\{ \gamma \in \Gamma_n^{1/i};\ |\Phi(x^*)(\gamma, i)| > \varepsilon \big\} &\subset \big\{ \gamma \in \Gamma_n^{1/i};\ \tau_i(\langle \gamma, x^* \rangle) \neq 0 \big\} \\ &= \big\{ \gamma \in \Gamma_n^{1/i};\ \langle \gamma, x^* \rangle > \tfrac{1}{i} \big\} \cup \big\{ \gamma \in \Gamma_n^{1/i};\ \langle \gamma, -x^* \rangle > \tfrac{1}{i} \big\}; \end{aligned}$$

the set in the last line being finite according to the assertion (ii). Therefore $\Phi(x^*) \in c_0(\Gamma \times \mathbb{N})$. We thus showed that $\Phi(B_{X^*}) \subset c_0(\Gamma \times \mathbb{N})$ and hence (iii) is proved.

(iii)⇒(i) is standard [AL], [Di, pp. 146,147].

(i)⇒(iv). Assume that X is a subspace of a WCG space Z. Let $j : X \to Z$ be the canonical injection. Find an equivalent norm $|\cdot|$ on Z and a total set $M \subset B_Z$ as is stated in Theorem 1 (iii). We may and do assume that M is convex and symmetric; then $\bigcup_{n=1}^\infty nM$ is dense in Z. Put

$$M_n^\varepsilon = j^{-1}\big(nM + \tfrac{\varepsilon}{2} B_Z\big) \cap B_X, \quad \varepsilon > 0, \quad n \in \mathbb{N}.$$

Then $\bigcup_{n=1}^\infty M_n^\varepsilon = B_X$ for every $\varepsilon > 0$. We shall show that these sets together with the restriction of the norm $|\cdot|$ to X, denoted by the same symbol, have the properties quoted in (iv). So fix any $\varepsilon > 0$ and any $n \in \mathbb{N}$. Let $F,\ F_m \in X^{***},\ m \in \mathbb{N}$, satisfy

$$2(|F|^{***})^2 + 2(|F_m|^{***})^2 - (|F + F_m|^{***})^2 \to 0 \ \text{ as } \ m \to \infty.$$

By Hahn-Banach theorem we find $G,\ G_m \in Z^{***}$ such that $G_{|j^{**}(X^{**})} = F,\ G_{m|j^{**}(X^{**})} = F_m$ and $|G|^{***} = |F|^{***},\ |G_m|^{***} = |F_m|^{***},\ m \in \mathbb{N}$. Here $j^{**} : X^{**} \to Z^{**}$ is the second conjugate to j. Then, since $|G + G_m|^{***} \geq |F + F_m|^{***}$, we have

$$2(|G|^{***})^2 + 2(|G_m|^{***})^2 - (|G + G_m|^{***})^2 \to 0 \ \text{ as } \ m \to \infty.$$

Then, since $|\cdot|^{***}$ on Z^{***} is $M-$LUR, we get that $\sup\big|\langle M, G_m - G\rangle\big| \to 0$ as $m \to \infty$. Thus, for every $\varepsilon > 0$ and every $n \in \mathbb{N}$,

$$\begin{aligned}\limsup_{m\to\infty} \sup \big|\big\langle M_n^\varepsilon, F_m - F\big\rangle\big| \le & \limsup_{m\to\infty} \sup \big|\big\langle nM + \tfrac{\varepsilon}{2} B_Z, G_m - G\big\rangle\big| \\ \le & \tfrac{\varepsilon}{2} \limsup_{m\to\infty} |G_m - G|^{***} \le \varepsilon |G|^{***},\end{aligned}$$

so $|\cdot|^{***}$ is $\varepsilon - M_n^\varepsilon-$LUR. Having this, Proposition 2 guarantees that the norm $|\cdot|^{**}$ is $2\varepsilon - M_n^\varepsilon-$smooth.

(iv)⇒(v). By Proposition 3, the situation described in (iv) implies that for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ the set M_n^ε is $\varepsilon-$weakly compact, i.e., that $\overline{M_n^\varepsilon}^* \subset X + \varepsilon B_{X^{**}}$. Thus $M \subset \bigcap_{i=1}^\infty \bigcup_{n=1}^\infty \overline{M_n^{1/i}}^* \subset X$. We observe that the middle set here is $\mathcal{K}-$analytic in $(B_{X^{**}}, w^{**})$. Now [T, Théorème 3.4 (iii)] guarantees that the whole B_X is $\mathcal{K}$-analytic in $(B_{X^{**}}, w^*)$, that is, that X is weakly $\mathcal{K}$-analytic space. Thus X is weakly Lindelöf determined, see, e.g. [F, Theorem 7.2.7]. The second part of (v) holds trivially.

(v)⇒(ii). Clearly, we may and do assume that each set M_n^ε in (v) is convex symmetric and closed. We shall find a set $\Gamma \subset M$ satisfying the assertion (ii) by a transfinite induction. If X is separable, then we can take for Γ any countable dense subset in M. Let $\aleph$ be an uncountable cardinal and assume that we already found a set $\Gamma \subset M$ as in the assertion (ii) whenever the density of X was less than $\aleph$. Now assume that our X has density $\aleph$.

Proposition 1 yields a PRI $(P_\alpha;\ \omega \le \alpha \le \mu)$ on $(X, |\cdot|)$ such that $P_\alpha(M_n^{1/i}) \subset M_n^{1/i}$ for every $\alpha \in [\omega, \mu)$ and every $n, i \in \mathbb{N}$. For $\alpha \in [\omega, \mu)$ denote $Q_\alpha = P_{\alpha+1} - P_\alpha$; observe that then $Q_\alpha X$ is weakly Lindelöf determined, has density less than $\aleph$, we have $\bigcup_{n=1}^\infty \frac{1}{2} Q_\alpha M_n^{1/i} = \frac{1}{2} Q_\alpha M$ for every $i \in \mathbb{N}$, the set $\frac{1}{2} Q_\alpha M$ is total in $Q_\alpha X$, and the norm $|\cdot|$ restricted to $Q_\alpha X$ is $\frac{1}{i} - \frac{1}{2} Q_\alpha M_n^{1/i}-$smooth for every $n, i \in \mathbb{N}$. We thus verified the condition (v) for every subspace $Q_\alpha X$. For every $\alpha \in [\omega, \mu)$ we then find, by the induction assumption, a total set $\Gamma_\alpha \subset \frac{1}{2} Q_\alpha M$ $(\subset M)$ satisfying (ii) (in the subspace $Q_\alpha X$).

Further fix any $\varepsilon > 0$. Find $i \in \mathbb{N}$ so that $\frac{1}{i} < \varepsilon$. For $\alpha \in [\omega, \mu)$ let $\Gamma_\alpha = \bigcup_{n=1}^\infty \Gamma_{\alpha,n}^{1/i}$ be the decomposition from the assertion (ii) in the subspace $Q_\alpha X$. For $n, m \in \mathbb{N}$ put then

$$\Gamma_{n,m}^\varepsilon = M_m^{1/i} \cap \bigcup_{\alpha<\mu} \Gamma_{\alpha,n}^{1/i} \backslash \big(\Gamma_{n,m-1}^\varepsilon \cup \cdots \cup \Gamma_{n,1}^\varepsilon \cup \{0\}\big).$$

Clearly, this is a countable family of mutually disjoint sets and $\Gamma = \bigcup_{n,m=1}^\infty \Gamma_{n,m}^\varepsilon$.

Fix any $n, m \in \mathbb{N}$ and any $x^* \in X^*$. We have to show that the set $\big\{\gamma \in \Gamma_{n,m}^\varepsilon;\ \langle \gamma, x^*\rangle > \varepsilon\big\}$ is finite. Instead of this, we will be proving by a transfinite induction, the following subtler statement:

$$\big\{\gamma \in \Gamma_{n,m}^\varepsilon;\ \langle \gamma, P_\beta^* x^*\rangle > \varepsilon\big\} < \aleph_0 \qquad (\beta)$$

for every $\beta \in [\omega, \mu]$; then (μ) will be what we need. Clearly, (ω) is true. Also, for every $\beta < \mu$, if (β) is valid, so is $(\beta + 1)$ since

$$\begin{aligned}&\#\big\{\gamma \in \Gamma_{n,m}^\varepsilon;\ \langle \gamma, P_{\beta+1}^* x^*\rangle > \varepsilon\big\} \\ \le &\#\big\{\gamma \in \Gamma_{n,m}^\varepsilon;\ \langle \gamma, P_\beta^* x^*\rangle > \varepsilon\big\} + \#\big\{\gamma \in \Gamma_{\beta,n}^\varepsilon;\ \langle \gamma, P_{\beta+1}^* x^*\rangle > \varepsilon\big\} < \aleph_0;\end{aligned}$$

here we used the property of the set $\Gamma^\varepsilon_{\alpha,n}$ and the fact that $Q_\alpha \circ P_{\beta+1} = 0$ whenever $\beta < \alpha < \mu$.

Let $\lambda \in (\omega, \mu]$ be now a limit ordinal and assume that we verified (β) for each $\beta < \lambda$. Find $\beta < \lambda$ so that $\sup \left\langle M_n^{1/i}, P^*_\lambda x^* - P^*_\beta x^* \right\rangle < \frac{1}{i}$. This follows from the $\frac{1}{i} - M^\varepsilon_n$−smoothness of $|\cdot|$ via Proposition 2 (actually, if $P^*_\lambda x^*$ does not attain its norm at an element of $P_\lambda X$, some extra work is needed here). Take any $\gamma \in \Gamma^\varepsilon_{n,m}$ satisfying $\langle \gamma, P^*_\lambda x^* \rangle > \varepsilon$. Then, as $\gamma \in M_n^{1/i}$, we have

$$\langle \gamma, P^*_\beta x^* \rangle = \langle \gamma, P^*_\lambda x^* \rangle - \langle \gamma, P^*_\lambda x^* - P^*_\beta x^* \rangle > \varepsilon - \tfrac{1}{i} > 0$$

and so $\gamma \in Q_\alpha X$ for some $\alpha < \beta$. But then $\langle \gamma, P^*_\lambda x^* \rangle = \langle \gamma, P^*_\beta x^* \rangle$. Hence

$$\left\{\gamma \in \Gamma^\varepsilon_{n,m};\ \langle \gamma, P^*_\lambda x^* \rangle > \varepsilon\right\} = \left\{\gamma \in \Gamma^\varepsilon_{n,m};\ \langle \gamma, P^*_\beta x^* \rangle > \varepsilon\right\}$$

where the latter set is finite by the induction assumption. We thus proved (λ). And, taking $\lambda = \mu$ we get (ii). ∎

Remark. We confess that we did not succeed to prove (iv) directly from (ii) via Day's norm, and using Proposition 4.

Proof of Theorem 3. (i)⇒(ii). Let $K_m \subset B_{X^{**}}$, $m \in \mathbb{N}$, be the weak* closed sets witnessing that X is Vašák, i.e., for every $x \in B_X$ there is $N \subset \mathbb{N}$ so that $x \in \bigcap_{m\in N} K_m \subset X$. We may and do assume that for all $m, n \in \mathbb{N}$, if $K_m \cap K_n \neq \emptyset$, then there is $l \in \mathbb{N}$ so that $K_m \cap K_n = K_l$. Let $(P_\alpha;\ \omega \le \alpha \le \mu)$ be a *separable PRI* on X, see Proposition 1, [F, Proposition 6.2.7 and Definition 6.2.6]. We recall that one of the features of such a PRI is that the range of the projection $Q_\alpha := P_{\alpha+1} - P_\alpha$ is separable for every $\alpha \in [\omega, \mu)$. For each such α we find a dense subset $\left\{v^\alpha_n;\ n \in \mathbb{N}\right\}$ in $B_{Q_\alpha X}$. Put then $\Gamma = \bigcup_{n,m=1}^\infty \Gamma_{m,n}$, where

$$\Gamma_{m,n} = \left\{v^\alpha_n;\ \alpha \in [\omega, \mu)\right\} \cap K_m, \quad m, n \in \mathbb{N}.$$

Clearly, Γ is total in X.

Now, fix any $\varepsilon > 0$, any $x^* \in X^*$, and any $\gamma \in \Gamma$. Find a set $N \subset \mathbb{N}$ so that $\gamma \in \bigcap_{m\in N} K_m \subset X$. We can then choose a sequence $(m_i)_{i\in\mathbb{N}}$ in N (not necessarily injective) such that $K_{m_1} \supset K_{m_2} \supset \cdots$ and $\bigcap_{i=1}^\infty K_{m_i} \subset X$. Find $n \in \mathbb{N}$ and a (unique) $\alpha \in [\omega, \mu)$ so that $\gamma = v^\alpha_n$. We `claim` that there is $j \in \mathbb{N}$ so that

$$\#\left\{\gamma' \in \Gamma_{m_j,n};\ \langle \gamma', x^* \rangle > \varepsilon\right\} < \aleph_0.$$

Once having this, (ii) will be proved since clearly $\gamma \in \Gamma_{m_i,n}$ for every $i \in \mathbb{N}$.

Assume that the claim is false. Pick then subsequently $\gamma_1 \in \Gamma_{m_1,n}$, with $\langle \gamma_1, x^* \rangle > \varepsilon$, $\gamma_2 \in \Gamma_{m_2,n} \backslash \{\gamma_1\}$ with $\langle \gamma_2, x^* \rangle > \varepsilon, \ldots, \gamma_{i+1} \in \Gamma_{m_{i+1},n} \backslash \{\gamma_1, \ldots, \gamma_i\}$, with $\langle \gamma_{i+1}, x^* \rangle > \varepsilon, \ldots$ For every $i \in \mathbb{N}$ find a (unique) $\alpha_i < \mu$ so that $\gamma_i = v^{\alpha_i}_n$. Let x^{**} be a weak* cluster point of the sequence $(\gamma_i)_{i\in\mathbb{N}}$. Then, necessarily, $x^{**} \in \bigcap_{i=1}^\infty K_{m_i} \subset X$. Fix for a while any $\beta < \mu$. We recall that the sequence $(\gamma_i)_{i\in\mathbb{N}}$ is injective. Hence so is the sequence $(\alpha_i)_{i\in\mathbb{N}}$. Then we have $Q_\beta \circ Q_{\alpha_i} = 0$ for all large $i \in \mathbb{N}$. Hence $Q_\beta x^{**} = 0$. This holds for every $\beta \in [\omega, \mu)$. Therefore $x^{**} = 0$. However, $\langle \gamma_i, x^* \rangle > \varepsilon$ for every $i \in \mathbb{N}$, and so $(0 =) \langle x^{**}, x^* \rangle \geq \varepsilon > 0$, a contradiction.

Now assume that we have already given a total set $\Gamma \subset B_X$ which countably supports X^* and let (i) be satisfied. Proposition 1 and the proof of [F, Proposition 6.2.7] yield a separable PRI $(P_\alpha;\ \omega \leq \alpha \leq \mu)$ on X as above, with the additional property that $P_\alpha(\gamma) \in \{\gamma, 0\}$ for every $\alpha \in [\omega, \mu)$ and every $\gamma \in \Gamma$. For every $\alpha \in [\omega, \mu)$ the set $\Gamma \cap Q_\alpha X$ is countable; this can be seen as follows: $Q_\alpha X$ is separable, hence there exists a w^*-dense subset $\{x_n^* :\ n \in \mathbb{N}\}$ of $(Q_\alpha(X))^*$. Let $S_n = \{\gamma \in \Gamma \cap Q_\alpha X :\ \langle \gamma, x_n^* \rangle \neq 0\}$. Then S_n is countable for all $n \in \mathbb{N}$. If $\gamma \in (\Gamma \cap Q_\alpha X) \setminus \bigcup_{n=1}^\infty S_n$ we have $\langle \gamma, x_n^* \rangle = 0$ for all $n \in \mathbb{N}$, hence $\gamma = 0$. Enumerate $\Gamma \cap Q_\alpha X$ as $\{v_n^\alpha;\ n \in \mathbb{N}\}$. The rest of the proof is as above.

(ii)$\Rightarrow$(iii). Define $T : X^* \to \mathbb{R}^\Gamma$ by

$$Tx^* = \big(\langle \gamma, x^* \rangle;\ \gamma \in \Gamma\big), \quad x^* \in X^*;$$

this is a linear bounded weak* to pointwise continuous injection. Avoid from the family $\big\{ \bigcap_{n \in F} \Gamma_n;\ F \subset \mathbb{N} \text{ finite}\big\}$ empty sets and enumerate it as $M_1, M_2, \ldots$ Put $M = \bigcup_{n=1}^\infty M_n (= \Gamma)$. Let $\|\cdot\|_{\mathcal{D}}$ be the Day's norm and define

$$|x^*|^{*2} = \|x^*\|^{*2} + \sum_{n=1}^\infty 2^{-n} \big\|Tx^*{}_{|M_n}\big\|_{\mathcal{D}}^{\ 2}, \quad x^* \in X^*;$$

this is clearly an equivalent dual norm on X^*. Let $|\cdot|$ denote the predual norm corresponding to $|\cdot|^*$. It easy to check that the set Γ from the condition (ii) countably supports X^*. Proposition 1 thus yields a PRI $(P_\alpha;\ \omega \leq \alpha \leq \mu)$ on $(X, |\cdot|)$ such that $P_\alpha(\gamma) \in \{\gamma, 0\}$ for every $\alpha \in [\omega, \mu]$ and every $\gamma \in \Gamma$.

Fix any $\varepsilon > 0$ and any $0 \neq x^* \in X^*$. By (ii), for every $\gamma \in \Gamma$ we find $n_\gamma \in \mathbb{N}$ so that $\gamma \in M_{n_\gamma}$ and $\#\{\gamma' \in M_{n_\gamma};\ |\langle \gamma', x^* \rangle| > \frac{\varepsilon}{3}\} < \aleph_0$. Put $N = \{n_\gamma;\ \gamma \in \Gamma\}$; then, clearly, $\bigcup_{n \in N} M_n = M$. Now, fix any $n \in N$ and any $\alpha \in [\omega, \mu)$. We have to show that $|\cdot|^*$ is $\varepsilon / |P_\alpha^* x^*|^* - M_n-$LUR at $P_\alpha^* x^*$. So let $x_i^* \in X^*$, $i \in \mathbb{N}$, be such that $2|P_\alpha^* x^*|^{*2} + 2|x_i^*|^{*2} - |P_\alpha^* x^* + x_i^*|^{*2} \to 0$ as $i \to 0$. Since

$$\#\big\{\gamma \in M_n;\ |\langle \gamma, P_\alpha^* x^* \rangle| > \tfrac{\varepsilon}{3}\big\} = \#\big\{\gamma \in M_n;\ |\langle \gamma, x^* \rangle| > \tfrac{\varepsilon}{3}\big\} < \aleph_0,$$

(here we used the fact that, once $P_\alpha(\gamma) \neq 0$, then necessarily $P_\alpha(\gamma) = \gamma$). Proposition 4 yields

$$\limsup_{i \to \infty} \sup \big|\big\langle M_n, x_i^* - P_\alpha^* x^* \big\rangle\big| \leq 3 \tfrac{\varepsilon}{3} = \varepsilon = \frac{\varepsilon}{|P_\alpha^* x^*|^*} |P_\alpha^* x^*|^*.$$

(iii)$\Rightarrow$(ii). Assume that $\operatorname{dens} X = \aleph_1$. For $\alpha < \omega_1$ put $Q_\alpha = P_{\alpha+1} - P_\alpha$ and let $\{v_n^\alpha;\ n \in \mathbb{N}\}$ be a dense set in $Q_\alpha(M) \cap B_X$. Put

$$\Gamma = \bigcup_{n,m=1}^\infty \Gamma_{n,m}, \quad \text{where} \quad \Gamma_{n,m} = \big\{v_n^\alpha;\ \alpha < \omega_1\big\} \cap M_m, \quad n, m \in \mathbb{N}.$$

We shall show that these sets satisfy (ii). Clearly, Γ is total in X. Fix any $\varepsilon > 0$, $0 \neq x^* \in X^*$, and $\gamma \in \Gamma$. For ε and x^* find by (iii) a corresponding set $N \subset \mathbb{N}$. Find $m \in N$

so that $\gamma \in M_m$. Find a (unique) $\alpha < \omega_1$ and $n \in \mathbb{N}$ so that $\gamma = v_n^\alpha$. Thus $\gamma \in \Gamma_{n,m}$. It remains to verify that

$$\#\big\{\gamma' \in \Gamma_{n,m};\ \langle \gamma', x^*\rangle > \varepsilon\big\} < \aleph_0. \tag{3}$$

Instead of showing this, we shall be proving the following subtler `claim`:

$$\#\big\{\gamma' \in \Gamma_{n,m};\ \langle \gamma', P_\beta^* x^*\rangle > \varepsilon\big\} < \aleph_0. \tag{β}$$

for every $\beta \le \omega_1$. Clearly, (ω) holds. Assume that (β) holds for some $\beta < \omega_1$. Then

$$\#\big\{\gamma' \in \Gamma_{n,m};\ \langle \gamma', P_{\beta+1}^* x^*\rangle > \varepsilon\big\} \le \#\big\{\gamma' \in \Gamma_{n,m};\ \langle \gamma', P_\beta^* x^*\rangle > \varepsilon\big\} + 1 < \aleph_0$$

and hence $(\beta+1)$ holds as well. Now, let a limit ordinal $\lambda \le \omega_1$ be given and assume that (β) was verified for every $\beta < \lambda$. As $\beta \uparrow \lambda$ implies $|P_\beta^* x^*|^* \le |P_\lambda^* x^*|^*$ and $|P_\beta^* x^* + P_\lambda^* x^*|^* \to 2|P_\lambda^* x^*|^*$, we have from (iii) that

$$\sup \big\langle M_m, P_\beta^* x^* - P_\lambda^* x^*\big\rangle < \frac{\varepsilon}{|P_\lambda^* x^*|^*} \cdot |P_\lambda^* x^*|^* = \varepsilon$$

for some $\beta < \lambda$. We observe that if $\gamma' \in \Gamma_{n,m}$ satisfies $\langle \gamma', P_\lambda^* x^*\rangle > \varepsilon$, then

$$\big\langle \gamma', P_\beta^* x^*\big\rangle = \big\langle \gamma', P_\lambda^* x^*\big\rangle - \big\langle \gamma', P_\lambda^* x^* - P_\beta^* x^*\big\rangle > \varepsilon - \varepsilon = 0,$$

and so $\gamma' \in Q_\alpha X$ with some $\alpha < \beta$; thus $\langle \gamma', P_\beta^* x^*\rangle = \langle \gamma', P_\lambda^* x^*\rangle$. Therefore

$$\#\big\{\gamma' \in \Gamma_{n,m};\ \langle \gamma', P_\lambda^* x^*\rangle > \varepsilon\big\} = \#\big\{\gamma' \in \Gamma_{n,m};\ \langle \gamma', P_\beta^* x^*\rangle > \varepsilon\big\} < \aleph_0.$$

The claim is thus verified for every $\beta \le \omega_1$. In particular, (ω_1) is just (3) and (ii) is proved.

(ii)⇒(i). Assume that (ii) holds. Clearly, it will remain valid if we add $0 \in X$ to each of the sets $\Gamma,\ \Gamma_n,\ n \in \mathbb{N}$. We shall show that Γ is $\mathcal{K}$-countably determined in $(B_{X^{**}}, w^*)$. And since Γ is a total set in X, [T, Théorème 3.4 (iii)] will guarantee that B_X is $\mathcal{K}$-countably determined, that is, that X is a Vašák space. The $\mathcal{K}$-countable determinacy of Γ is guaranteed by the family of the sets $\overline{\Gamma_n}^{\,*} \subset B_{X^{**}},\ n \in \mathbb{N}$. Indeed, take any $\gamma \in \Gamma$ and any $x^{**} \in B_{X^{**}} \backslash \Gamma$. Find $x^* \in X^*$ such that $\langle x^{**}, x^*\rangle > 0$. For our x^*, our γ, and for $\varepsilon := \frac{1}{2}\langle x^{**}, x^*\rangle$ find, from (ii), $n \in \mathbb{N}$ such that $\gamma \in \Gamma_n$ $(\subset \overline{\Gamma_n}^{\,*})$ and that the set $\big\{\gamma' \in \Gamma_n;\ \langle \gamma', x^*\rangle > \varepsilon\big\}$ is finite. Then for sure $x^{**} \notin \overline{\Gamma_n}^{\,*}$. ∎

Proof of Theorem 4. (i)⇒(ii). Assume that X is weakly $\mathcal{K}-$analytic. If X is separable, we can take for Γ any countable total set in B_X; then the sets $\Gamma_{(n)},\ n \in \mathbb{N}$, can be singletons consisting from all elements of Γ, and we can put $\Gamma_s = \Gamma_{(s(1))}$ for every $s \in \mathbb{N}^{<\mathbb{N}}$. Let $\aleph$ be an uncountable cardinal and assume that we have proved the necessity for every space whose density is less than $\aleph$. Now let X be a weakly $\mathcal{K}-$analytic space of density $\aleph$ and assume that the sets $K_s \subset X^{**},\ s \in \mathbb{N}^{<\mathbb{N}}$, witness for this. We may and do assume that $K_s \supset K_t$ whenever $s, t \in \mathbb{N}^{<\mathbb{N}}$ and t is an "extension" of s. Let $(P_\alpha;\ \omega \le \alpha \le \mu)$ be a PRI on X, see Proposition 1. For $\alpha \in [\omega, \mu)$ put $Q_\alpha = P_{\alpha+1} - P_\alpha$ and, by the induction assumption, find a total set $\Gamma_{\alpha,\emptyset} \subset B_{Q_\alpha X}$, and subsets $\Gamma_{\alpha,s} \subset \Gamma_{\alpha,\emptyset},\ s \in \mathbb{N}^{<\mathbb{N}}$, as in the condition (ii), for the subspace $Q_\alpha X$ and moreover with the property that that $\Gamma_{\alpha,s} \supset \Gamma_{\alpha,t}$ whenever $s,t \in \mathbb{N}^{<\mathbb{N}}$ and t is an "extension" of s; this is possible since such a space is clearly weakly $\mathcal{K}-$analytic and its density is less than $\aleph$. Put $\Gamma = \Gamma_\emptyset = \bigcup_{\alpha<\mu} \Gamma_{\alpha,\emptyset}$ and for $k \in \mathbb{N}$ and $(n_1, n_2, \ldots, n_{2k}) \in \mathbb{N}^{<\mathbb{N}}$

$$\Gamma_{(n_1,n_2,\ldots,n_{2k-1})} = \bigcup_{\alpha<\mu} \Gamma_{\alpha,(n_1,n_3,\ldots,n_{2k-1})} \cap K_{(n_2,n_4,\ldots,n_{2k-2})},$$

$$\Gamma_{(n_1,n_2,\ldots,n_{2k})} = \Gamma_{(n_1,n_2,\ldots,n_{2k-1})} \cap K_{(n_2,n_4,\ldots,n_{2k})},$$

where we take $K_\emptyset = B_{X^{**}}$. From these definitions and from the induction assumption we easily get that $\Gamma_\emptyset$ is total in X and that $\Gamma = \bigcup_{\sigma\in\mathbb{N}^\mathbb{N}} \bigcap_{i=1}^\infty \Gamma_{\sigma|i}$.

Fix now any $x^* \in X^*$, any $\sigma \in \mathbb{N}^\mathbb{N}$, and any $\varepsilon > 0$. It remains to prove that

$$\#\big\{\gamma \in \Gamma_{\sigma|i};\ \langle\gamma, x^*\rangle > \varepsilon\big\} < \aleph_0$$

for some $i \in \mathbb{N}$. Assume that this is not the case. Find $\tau, \rho \in \mathbb{N}^\mathbb{N}$ so that $\sigma = (\tau(1), \rho(1), \tau(2), \rho(2), \ldots)$. Pick then subsequently $\gamma_1 \in \Gamma_{\sigma|2}$, with $\langle\gamma_1, x^*\rangle > \varepsilon$, $\gamma_2 \in \Gamma_{\sigma|4}\backslash\{\gamma_1\}$, with $\langle\gamma_2, x^*\rangle > \varepsilon, \ldots,\ \gamma_{i+1} \in \Gamma_{\sigma|(2i+2)}\backslash\{\gamma_1, \ldots, \gamma_i\}$, with $\langle\gamma_{i+1}, x^*\rangle > \varepsilon, \ldots$ For every $i \in \mathbb{N}$ find a (unique) $\alpha_i < \mu$ so that $\gamma_i \in \Gamma_{\alpha_i,\tau|i}$. Let x^{**} be a weak* cluster point of the sequence $(\gamma_i)_{i\in\mathbb{N}}$. Then, necessarily, $x^{**} \in \bigcap_{i=1}^\infty K_{\rho|i} \subset X$. (Here we used that $K_{\rho|1} \supset K_{\rho|2} \supset \cdots$) Fix for a while any $\beta < \mu$. We realize that the set $\{i \in \mathbb{N};\ \alpha_i = \beta\}$ is finite. Indeed, otherwise $\gamma_i \in \Gamma_{\beta,\tau|i}$ for infinitely many $i \in \mathbb{N}$ (as $\Gamma_{\beta,\tau|1} \supset \Gamma_{\beta,\tau|2} \supset \cdots$) and hence the sets $\big\{\gamma' \in \Gamma_{\beta,\tau|i};\ \big\langle\gamma', {x^*}_{|Q_\beta X}\big\rangle > \varepsilon\big\}$ would be infinite for all large $i \in \mathbb{N}$, a contradiction. Thus $Q_\beta \circ Q_{\alpha_i} = 0$ for all large $i \in \mathbb{N}$, and so $Q_\beta x^{**} = 0$. This holds for every $\beta < \mu$; hence $x^{**} = 0$. However, $\langle\gamma_i, x^*\rangle > \varepsilon$ for every $i \in \mathbb{N}$, and so $(0 =)\ \langle x^{**}, x^*\rangle \geq \varepsilon > 0$, a contradiction.

Now assume that we have already given a total set $\Gamma \subset B_X$ which countably supports X^* and let (i) be satisfied. Proposition 1 yields a PRI $(P_\alpha;\ \omega \leq \alpha \leq \mu)$ on X as above, with the additional property that $P_\alpha(\gamma) \in \{\gamma, 0\}$ for every $\alpha \in [\omega, \mu)$ and every $\gamma \in \Gamma$. For every $\alpha \in [\omega, \mu)$ put $\Gamma_{\alpha,\emptyset} = \Gamma \cap Q_\alpha X$; this is a total set in $Q_\alpha X$ which countably supports $(Q_\alpha X)^*$. The rest of the proof is as above.

(ii)$\Rightarrow$(iii). Define

$$\varphi(\sigma) = \bigcap_{n=1}^\infty \overline{\Gamma_{\sigma|n}}^*, \quad \sigma \in \mathbb{N}^\mathbb{N}.$$

By joining 0 to Γ as well as to each $\Gamma_s,\ s \in \mathbb{N}^{<\mathbb{N}}$, we do not violate the condition (ii) and will guarantee that $\varphi(\sigma) \neq \emptyset$ for every $\sigma \in \mathbb{N}^\mathbb{N}$. Fix any $\sigma \in \mathbb{N}^\mathbb{N}$. We shall show that $\varphi(\sigma)$ is a subset of Γ $(\subset X)$. So take any $x^{**} \in \varphi(\sigma)$. If $x^{**} = 0$, we are done. Otherwise find $x^* \in X^*$ so that $\langle x^{**}, x^*\rangle =: \varepsilon > 0$. From (ii) find $i \in \mathbb{N}$ so that the set $\big\{\gamma \in \Gamma_{\sigma|i};\ \langle\gamma, x^*\rangle > \varepsilon\big\}$ is finite. Then, as $x^{**} \in \overline{\Gamma_{\sigma|i}}^*$, we get that, necessarily, $x^{**} \in \Gamma_{\sigma|i}$ $(\subset \Gamma)$. Therefore $\varphi(\sigma)$ is a weakly compact set. Consider now any weakly open set $\varphi(\sigma) \subset U \subset X$. Find a weak* open set $\widetilde{U} \subset X^{**}$ so that $U = \widetilde{U} \cap X$. A simple compactness argument yields $N \in \mathbb{N}$ so large that $\bigcap_{n=1}^{N} \overline{\Gamma_{\sigma|n}}^{*} \subset \widetilde{U}$. Then $\varphi(\tau) \subset \widetilde{U} \cap X = U$ whenever $\tau \in \mathbb{N}^{\mathbb{N}}$ and τ "begins" by $\sigma|N$. We have proved that φ is upper semicontinuous at σ.

Define $T : X^* \to \mathbb{R}^\Gamma$ by

$$Tx^* = \big(\langle \gamma, x^* \rangle;\ \gamma \in \Gamma\big), \quad x^* \in X^*;$$

this is a linear bounded weak* to pointwise continuous injection. Let $\{s_1, s_2, \ldots\}$ be an enumeration of $\mathbb{N}^{<\mathbb{N}}$ and define

$$|x^*|^{*2} = \|x^*\|^{*2} + \sum_{j=1}^{\infty} 2^{-j} \big\| Tx^*{}_{|\Gamma_{s_j}} \big\|_{\mathcal{D}}^{2}, \quad x^* \in X^*;$$

this will be an equivalent dual norm on X^*. Let $|\cdot|$ denote the predual norm corresponding to $|\cdot|^*$. It remains to prove the geometrical property of $|\cdot|^*$. So fix again any $\sigma \in \mathbb{N}^{\mathbb{N}}$ and consider any $0 \neq x^* \in X^*$, $x_i^* \in X^*$, $i \in \mathbb{N}$, such that $2|x^*|^{*2} + 2|x_i^*|^{*2} - |x^* + x_i^*|^{*2} \to 0$ as $i \to \infty$. Fix any $\varepsilon > 0$. From (ii) find $n \in \mathbb{N}$ such that the set $\big\{\gamma \in \Gamma_{\sigma|n};\ \langle \gamma, x^* \rangle > \varepsilon\big\}$ is finite. Then the set $\big\{x^{**} \in \overline{\Gamma_{\sigma|n}}^{*};\ \langle x^{**}, x^* \rangle > \varepsilon\big\}$ is also finite (actually it coincides with the latter). Hence, by Proposition 4, $\limsup_{i\to\infty} \sup \big|\big\langle \overline{\Gamma_{\sigma|n}}^{*}, x^* - x_i^* \big\rangle\big| \le 3\varepsilon$, and a fortiori, $\limsup_{i\to\infty} \sup \big|\big\langle \varphi(\sigma), x^* - x_i^* \big\rangle\big| \le 3\varepsilon$. But $\varepsilon > 0$ was here arbitrary. We thus proved that the norm $|\cdot|^*$ is $\varphi(\sigma)-$LUR. Then the norm $|\cdot|$ is $\varphi(\sigma)-$smooth by Proposition 2.

(iii)$\Rightarrow$(i). The properties of the mapping φ exactly mean that the set $\varphi\big(\mathbb{N}^{\mathbb{N}}\big)$ is $\mathcal{K}-$analytic in $(B_{X^{**}}, w^*)$. Hence, [T, Théorème 3.4 (iii)] guarantees that the whole B_X is $\mathcal{K}$-analytic in $(B_{X^{**}}, w^*)$, that is, that X is weakly $\mathcal{K}$-analytic space. ∎

Remark. We do not know if (iii) can be strengthened so that ($|\cdot|^{***}$ is $\varphi(\sigma)-$LUR) $|\cdot|^{**}$ is $\varphi(\sigma)-$smooth for every $\sigma \in \mathbb{N}^{\mathbb{N}}$.

Proof of Theorem 5. That (iii)$\Rightarrow$(i) is trivial. Further, (ii) implies that the assignment $x^* \mapsto (\langle \gamma, x^* \rangle;\ \gamma \in \Gamma)$ injects (B_{X^*}, w^*) continuously into $\Sigma(\Gamma)$. Hence, then (iii) is satisfied.

(i)$\Rightarrow$(ii). We shall proceed by a transfinite induction over the density of X. If X has density equal to $\aleph_0$, then we can take for $\Gamma \subset B_X$ any total and countable set and so we are done. Let an uncountable cardinal $\aleph$ be given and assume that the implication was verified for any X with density less than $\aleph$. Now, let X be a Banach space with density $\aleph$, and assume that X is a subspace of a weakly Lindelöf determined space $(Z, \|\cdot\|)$. Then Z has a projectional generator, see, e.g., [F, Proposition 8.3.1]. From the proof of Proposition 1, we can see that there exists a complemented subspace $X \subset Z_1 \subset Z$ with density equal to the density of X. It is easy to check that this Z_1 will also be a weakly Lindelöf determined space. Hence, in what follows, we may and do assume that the density of Z equals the density of X $(= \aleph)$. By Proposition 1, we find a PRI $(P_\alpha;\ \omega \le \alpha \le \mu)$ on Z such that $P_\alpha X \subset X$ for every $\alpha \in [\omega, \mu)$.

Consider any $\omega \le \alpha < \mu$. Put $Q_\alpha = P_{\alpha+1} - P_\alpha$. Note that $Q_\alpha X$ is a subspace of $Q_\alpha Z$, the latter space being also weakly Lindelöf determined. Moreover, the density of

$Q_\alpha X$ is at most $\#\alpha < \#\mu = \aleph$. Hence, by the induction assumption, there is a total set $\Gamma_\alpha \subset B_{Q_\alpha X}$ such that

$$\forall y^* \in (Q_\alpha X)^* \quad \#\big\{\gamma \in \Gamma_\alpha;\ \langle \gamma, y^*\rangle \neq 0\big\} \leq \aleph_0.$$

Performing this for every α, put $\Gamma = \bigcup_{\alpha<\mu} \Gamma_\alpha$. Since $\bigcup_{\alpha<\mu} Q_\alpha X$ is total in X, so is Γ.

Fix any $0 \neq x^* \in X^*$. It remains to prove that the set $\big\{\gamma \in \Gamma;\ \langle \gamma, x^*\rangle \neq 0\big\}$ is at most countable. Find $z^* \in Z^*$ such that $z^*{}_{|X} = x^*$. We `claim` that $z^* \in \overline{\mathrm{sp} \bigcup_{\alpha\in C} Q_\alpha^* z^*}^*$ for a suitable countable set $C \subset [\omega, \mu)$. Assume for a while that this was already proved. Using this, we can see that, whenever $\alpha \in [\omega, \mu)$ and $\gamma \in \Gamma_\alpha$ satisfies $\langle \gamma, z^*\rangle \neq 0$, then $\alpha \in C$. Therefore

$$\#\big\{\gamma \in \Gamma;\ \langle \gamma, z^*\rangle \neq 0\big\} \leq \sum_{\alpha\in C} \#\big\{\gamma \in \Gamma_\alpha;\ \langle \gamma, z^*\rangle \neq 0\big\} \leq \aleph_0 \cdot \aleph_0 = \aleph_0.$$

It remains to prove the claim. To do so, we shall prove the subclaim:

$$P_\beta^* z^* \in \overline{\mathrm{sp} \bigcup_{\alpha\in C_\beta} Q_\alpha^* z^*}^* \quad \text{for a suitable countable set } C_\beta \subset [\omega, \mu) \qquad (\beta)$$

for every $\beta \in [\omega, \mu]$. Clearly, (ω) holds. Also, clearly, the validity of (β) implies $(\beta+1)$ for every $\beta \in [\omega, \mu)$. Now, let $\lambda \in (\omega, \mu]$ be a limit ordinal, and assume that (β) holds for all $\beta < \lambda$. We remark that $P_\lambda^* z^*$ belongs to the weak* closure of the set $\big\{P_\beta^* z^*;\ \beta \in (\omega, \lambda)\big\}$. And since (B_{Z^*}, w^*) is a Corson compact, a standard exhausting argument yields countably many ordinals $\omega < \beta_1, \beta_2, \ldots < \lambda$ such that $P_\lambda^* z^*$ is equal to the weak* limit of the sequence $\big(P_{\beta_i}^* z^*\big)_{i\in\mathbb{N}}$. Thus, by the induction assumption, $P_\lambda^* z^* \in \overline{\mathrm{sp} \bigcup_{\alpha\in C} Q_\alpha^* Z^*}^*$, where $C := C_{\beta_1} \cup C_{\beta_2} \cup \cdots$ is a countable set. This proves the subclaim (λ). And, in particular, we have (μ), which is nothing else than our claim. ∎

Proof of Theorem 6. We shall prove the following chain of implications: (i)⇒(iv)⇒(ii)⇒(iii)⇒(i). Moreover, we shall prove directly (ii)⇒(iv), thus showing another strong feature of Day's norm.

(i)⇒(iv). Assume that a Banach space $(X, \|\cdot\|)$ is a subspace of a Hilbert generated space $(Z, \|\cdot\|)$. Find a Hilbert space H and a bounded linear mapping $T : H \to Z$ with dense range. Put

$$|z^*|^{*2} = \|z^*\|^{*2} + \|T^* z^*\|^{*2}, \quad z^* \in Z^*;$$

this is an equivalent dual norm on Z^*. A convexity argument guarantees that this norm is uniformly $T(B_H)$−rotund. Hence, by a Šmulyan duality argument, the predual norm $|\cdot|$ on Z is uniformly $T(B_H)$−smooth, that is

$$\sup\big\{|z + tTh| + |z - tTh| - 2;\ z \in Z,\ |z| = 1,\ h \in B_H\big\} = o(t) \quad \text{as} \quad t \downarrow 0,$$

see Proposition 2. Now, since $T(H)$ is dense in Z and the norm $|\cdot|$ (as any norm) is Lipschitzian, we get that this norm is uniformly Gâteaux smooth. Then the restriction of $|\cdot|$ to the subspace X gives (iv).

(iv)⇒(ii). We shall elaborate ideas from [FHZ] and [FGZ]. If X is separable, then clearly every total countable set $\Gamma \subset B_X$ satisfies (ii). Let $\aleph$ be an uncountable cardinal and assume that the implication has already been verified for every space of density less than $\aleph$. Now assume that a Banach space X, of density $\aleph$, has an equivalent uniformly Gâteaux smooth norm, say $\|\cdot\|$. A scheme of the proof will be as follows. We shall show that (iv) implies that B_X admits a countable cover by "almost" weakly compact sets. This will imply that the space X is weakly $\mathcal{K}$-analytic, even that it admits a projectional generator, see [F, Definition 6.1.6 and Remark 6.1.8]. We shall then construct a PRI on X. Further, we shall define the set Γ similarly as in the previous proofs. Finally, using the uniform Gâteaux smoothness again, together with the induction assumption, we shall construct the decomposition of Γ with the properties required in the assertion (ii).

For $\varepsilon > 0$ and $m \in \mathbb{N}$ we put

$$B_m^\varepsilon = \big\{h \in B_X;\ \big\|x + \tfrac{1}{m}h\big\| + \big\|x - \tfrac{1}{m}h\big\| - 2 < \tfrac{\varepsilon}{m} \quad \text{whenever} \quad x \in X \ \text{ and } \ \|x\| = 1\big\}.$$

The uniform Gâteaux smoothness of $\|\cdot\|$ guarantees that $B_X = \bigcup_{m=1}^\infty B_m^\varepsilon$. Also, it is clear that the norm $\|\cdot\|$ is uniformly $\varepsilon - B_m^\varepsilon$−smooth, i.e.,

$$\lim_{t\downarrow 0} \frac{1}{t} \sup \big\{\|x + th\| + \|x - th\| - 2\|x\|;\ x \in X,\ \|x\| = 1,\ h \in B_m^\varepsilon\big\} \le \varepsilon.$$

Using Goldstine's theorem, we can easily check that for every $\varepsilon > 0$ and every $m \in \mathbb{N}$ the norm $\|\cdot\|^{**}$ on X^{**} is (uniformly) $\varepsilon - \overline{B_m^\varepsilon}$−smooth. Hence, by Proposition 3, $\overline{B_m^\varepsilon}^* \subset X + \varepsilon B_{X^{**}}$. Thus $B_X = \bigcap_{p=1}^\infty \bigcup_{m=1}^\infty \overline{B_m^{1/p}}^*$, and the weak $\mathcal{K}$-analyticity of X is proved. Indeed, it is enough to put $K_s = \overline{B_{s_1}^1}^* \cap \cdots \cap \overline{B_{s_k}^{1/k}}^*$ for $s = (s_1, \ldots, s_k) \in \mathbb{N}^{<\mathbb{N}}$. (How to construct a projectional generator directly from the sets B_m^ε can be found in [FGZ, Remark 9].) We recall that weakly $\mathcal{K}$-analytic spaces are weakly Lindelöf determined. Then Proposition 1 yields a PRI on X, say $(P_\alpha;\ \omega \le \alpha \le \mu)$. Consider any $\alpha \in [\omega, \mu)$ and put $Q_\alpha = P_{\alpha+1} - P_\alpha$. We realize that the subspace $Q_\alpha X$ has density less than $\aleph$, and is a subspace of the (Hilbert generated) space $Q_\alpha Z$. By the induction assumption, find a set $\Gamma_\alpha \subset B_{Q_\alpha X}$ with the property from the assertion (ii). Put then $\Gamma = \bigcup_{\alpha<\mu} \Gamma_\alpha$. We shall show that this set Γ satisfies the assertion (ii).

So, fix any $0 < \varepsilon < 1$. For every $\alpha \in [\omega, \mu)$ and every $n \in \mathbb{N}$ find the set $\Gamma_{\alpha,n}^\varepsilon \subset \Gamma_\alpha$ as it is stated in the assertion (ii). For $n, m \in \mathbb{N}$ put

$$\Gamma_{n,m}^\varepsilon = \bigcup_{\alpha \in [\omega,\mu)} \Gamma_{\alpha,n}^\varepsilon \cap B_m^{\varepsilon/2} \backslash \big(\Gamma_{n,m-1}^\varepsilon \cup \cdots \cup \Gamma_{n,1}^\varepsilon \cup \{0\}\big);$$

this is a countable family of mutually disjoint sets since $\Gamma_\alpha \cap \Gamma_\beta \subset \{0\}$ if $\alpha \neq \beta$. Also we can easily verify that $\bigcup_{n,m=1}^\infty \Gamma_{n,m}^\varepsilon = \Gamma$.

Fix any $n, m \in \mathbb{N}$ and any $x^* \in B_{X^*}$. We shall show that

$$\#\big\{\gamma \in \Gamma_{n,m}^\varepsilon;\ \langle \gamma, x^*\rangle > \varepsilon\big\} < \frac{4mn}{\varepsilon^2},$$

and thus the assertion (ii) will be almost proved. Define

$$F = \big\{\alpha \in [\omega, \mu);\ \langle \gamma, x^* \rangle > \varepsilon \ \text{ for some } \ \gamma \in \Gamma^\varepsilon_{n,m} \cap \Gamma_\alpha \big\}.$$

We `claim` that $\#F < \frac{4m}{\varepsilon^2}$; then we easily get, by the induction assumption,

$$\#\big\{\gamma \in \Gamma^\varepsilon_{n,m};\ \langle \gamma, x^* \rangle > \varepsilon\big\} \le \sum_{\alpha \in F} \#\big\{\gamma \in \Gamma^\varepsilon_{\alpha,n};\ \langle \gamma, {x^*}_{|Q_\alpha X} \rangle > \varepsilon\big\} < \#F \cdot n < \frac{4mn}{\varepsilon^2}.$$

Let us prove the claim. Here we shall imitate the proof of [FHZ, Lemma 5] which in turn elaborates ideas of Troyanski [Tr2]. If the set F is infinite, let N be any fixed positive integer. Otherwise, denote $N = \#F$. Find $\alpha_1 < \alpha_2 < \cdots < \alpha_N < \mu$ such that $F \supset \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$. For $j = 1, \ldots, N$ find $\gamma_j \in \Gamma^\varepsilon_{n,m} \cap \Gamma_{\alpha_j}$, with $\langle \gamma_j, x^* \rangle > \varepsilon$, and write $v_j = \gamma_1 + \cdots + \gamma_j$. Find $i \in \mathbb{N}$ so that $\frac{m}{\varepsilon} \le i \le \frac{2m}{\varepsilon}$. If $i \ge N$, then $N \le \frac{2m}{\varepsilon} < \frac{4m}{\varepsilon^2}$. Further assume that $i < N$. Since $\|v_i\| \ge \langle v_i, x^* \rangle > i\varepsilon \ge m$, $P_{\alpha_i+1} \circ Q_{\alpha_{i+1}} = 0$, and $\gamma_{i+1} \in B^{\varepsilon/2}_m$, the convexity of $\|\cdot\|$ yields

$$\begin{aligned}
\|v_{i+1}\| &= \|v_i\| \Big(\Big\| \frac{v_i}{\|v_i\|} + \frac{\gamma_{i+1}}{\|v_i\|} \Big\| - 1 \Big) + \|v_i\| \\
&\le m \Big(\Big\| \frac{v_i}{\|v_i\|} + \frac{\gamma_{i+1}}{m} \Big\| - 1 \Big) + \|v_i\| \\
&\le m \Big(\Big\| \frac{v_i}{\|v_i\|} + \frac{\gamma_{i+1}}{m} \Big\| + \Big\| \frac{v_i}{\|v_i\|} - \frac{\gamma_{i+1}}{m} \Big\| - 2 \Big) + \|v_i\| < \frac{\varepsilon}{2} + \|v_i\|.
\end{aligned}$$

Similarly, we get

$$\|v_{i+2}\| < \frac{\varepsilon}{2} + \|v_{i+1}\| < 2\frac{\varepsilon}{2} + \|v_i\|, \ \ldots, \quad \|v_N\| < (N-i)\frac{\varepsilon}{2} + \|v_i\| < N\frac{\varepsilon}{2} + i.$$

Thus

$$N\varepsilon < \langle v_N, x^* \rangle \le \|v_N\| < N\frac{\varepsilon}{2} + i,$$

and so $N < \frac{2}{\varepsilon} i \le \frac{4m}{\varepsilon^2}$. This also shows that the set F cannot be infinite. Hence $\#F = N < \frac{4m}{\varepsilon^2}$ and the claim is proved.

Now, it remains to enumerate the (countable) family $\Gamma^\varepsilon_{n,m}$, $n, m \in \mathbb{N}$, by one index running throughout $\mathbb{N}$, and to insert eventually "a few" empty sets. This will yield (ii).

Now assume that we have already given a total set $\Gamma \subset B_X$ which countably supports X^* and let (iv) be satisfied. Proposition 1 yields a PRI $(P_\alpha;\ \omega \le \alpha \le \mu)$ on X as above, with the additional property that $P_\alpha(\gamma) \in \{\gamma, 0\}$ for every $\alpha \in [\omega, \mu)$ and every $\gamma \in \Gamma$. For every $\alpha \in [\omega, \mu)$ put $\Gamma_\alpha = \Gamma \cap Q_\alpha X$; this is a total set in $Q_\alpha X$ which countably supports $(Q_\alpha X)^*$. The rest of the proof is as above.

(ii)⇒(iii). Assume the assertion (ii) is satisfied. We realize that this is actually a Talagrand-Argyros-Farmaki condition from [Fa, Theorem 2.10]. Therefore (B_{X^*}, w^*) is a uniform Eberlein compact. However, likewise in the proof of (ii)⇒(iii) in Theorem 2, a direct proof exists. Let $\tau_i : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{N}$, be the same functions as there. Define $\Phi : B_{X^*} \to \mathbb{R}^{\Gamma\times\mathbb{N}}$ by

$$\Phi(x^*)(\gamma,i) = \frac{1}{2^n 2^i \sqrt{n}}\tau_i(\langle\gamma,x^*\rangle) \quad \text{if} \quad \gamma\in\Gamma_n^{1/i}, \quad n\in\mathbb{N}, \quad \text{and} \quad i\in\mathbb{N}.$$

Clearly, Φ is weak* to pointwise continuous. The injectivity of Φ can be checked exactly as that time.

It remains to prove that $\Phi(B_{X^*}) \subset \ell_2(\Gamma\times\mathbb{N})$. Fix an arbitrary $x^* \in B_{X^*}$. We observe that for every $n, i \in \mathbb{N}$

$$\#\big\{\gamma\in\Gamma_n^{1/i};\ \Phi(x^*)(\gamma,i)\neq 0\big\} \le \#\big\{\gamma\in\Gamma_n^{1/i};\ |\langle\gamma,x^*\rangle| > \tfrac{1}{i}\big\} < 2n.$$

Therefore

$$\sum\big\{\big(\Phi(x^*)(\gamma,i)\big)^2;\ (\gamma,i)\in\Gamma\times\mathbb{N}\big\} = \sum_{i,n=1}^{\infty}\sum\big\{\big(\Phi(x^*)(\gamma,i)\big)^2;\ \gamma\in\Gamma_n^{1/i}\big\}$$

$$\le \sum_{i,n=1}^{\infty}\frac{1}{4^n4^in}\cdot\#\big\{\gamma\in\Gamma_n^{1/i};\ \Phi(x^*)(\gamma,i)\neq 0\big\} < \sum_{i,n=1}^{\infty}\frac{2}{4^n4^i} = \frac{2}{9} < +\infty,$$

and hence $\Phi(x^*) \in \ell_2(\Gamma\times\mathbb{N})$.

(iii)$\Rightarrow$(i). Assume that (B_{X^*}, w^*) is a uniform Eberlein compact. A result of Benyamini, M.E. Rudin and Wage says that the space of continuous functions on this compact, endowed with the supremum norm, is Hilbert generated, see, e.g., [F$^\frown$, Theorem 12.17]. But X is isomorphic to a subspace of this space. Thus we get (i).

This proves Theorem 6. Though a combination of the above implications gives (ii)$\Rightarrow$(iv), we shall present a direct proof of this by using Day's norm $\|\cdot\|_{\mathcal{D}}$ on $\ell_\infty(\Gamma)$ and Proposition 5. Assume that (ii) holds; thus we have at hand the sets Γ, Γ_n^ε, $\varepsilon>0$, $n\in\mathbb{N}$. For $x^*\in X^*$ define $Tx^* = (\langle\gamma,x^*\rangle;\ \gamma\in\Gamma)$. Put $M=\Gamma$ and

$$M_{i,k}^\varepsilon = \Gamma_i^{\varepsilon/4}\cap\Gamma_k^{2^{-i}\varepsilon/4}, \quad \varepsilon>0,\ i,k\in\mathbb{N};$$

this is a countable family and $\bigcup_{i,k=1}^\infty M_{i,k}^\varepsilon = M$. For $\varepsilon>0$ define

$$|x^*|_\varepsilon^{*2} = \|x^*\|^{*2} + \sum_{i,k=1}^{\infty}2^{-i-k}\big\|Tx^*{}_{|M_{i,k}^\varepsilon}\big\|_{\mathcal{D}}^{\,2}, \quad x^*\in X^*;$$

this is an equivalent dual norm on X^*. Fix any $\varepsilon>0$ and any $i,k\in\mathbb{N}$. Then

$$\forall x^*\in B_{X^*} \quad \#\big\{\gamma\in M_{i,k}^\varepsilon;\ \langle\gamma,x^*\rangle > \tfrac{\varepsilon}{4}\big\} \le \#\big\{\gamma\in\Gamma_i^{\varepsilon/4};\ \langle\gamma,x^*\rangle > \tfrac{\varepsilon}{4}\big\} < i,$$

$$\forall x^*\in B_{X^*} \quad \#\big\{\gamma\in M_{i,k}^\varepsilon;\ \langle\gamma,x^*\rangle > 2^{-i}\tfrac{\varepsilon}{4}\big\} \le \#\big\{\gamma\in\Gamma_k^{2^{-i}\varepsilon/4};\ \langle\gamma,x^*\rangle > 2^{-i}\tfrac{\varepsilon}{4}\big\} < k.$$

Now assume that we have $x_n^*, y_n^* \in B_{X^*}$, $n \in \mathbb{N}$, satisfying $2|x_n^*|_\varepsilon^{*2} + 2|y_n^*|_\varepsilon^{*2} - |x_n^* + y_n^*|_\varepsilon^{*2} \to 0$ as $n \to \infty$. Then, by convexity,

$$2\big\|Tx_n^*{}_{|M_{i,k}^\varepsilon}\big\|_{\mathcal{D}}^2 + 2\big\|Ty_n^*{}_{|M_{i,k}^\varepsilon}\big\|_{\mathcal{D}}^2 - \big\|T(x_n^* + y_n^*)_{|M_{i,k}^\varepsilon}\big\|_{\mathcal{D}}^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, by Proposition 5, $\limsup_{n\to\infty} \big|\big\langle M_{i,k}^\varepsilon, x_n^* - y_n^*\big\rangle\big| \le 4 \cdot \frac{\varepsilon}{4} = \varepsilon$. We thus proved that the dual norm $|\cdot|_\varepsilon^*$ on X^* is uniformly $\varepsilon - M_{i,k}^\varepsilon -$rotund, and hence, by a "uniform" variant of Proposition 2, the corresponding predual norm $|\cdot|_\varepsilon$ on X is uniformly $2\varepsilon - M_{i,k}^\varepsilon -$smooth for every $\varepsilon > 0$ and every $i, k \in \mathbb{N}$. Thus the norm $|\cdot|_\varepsilon$ is uniformly $2\varepsilon - \{h\}-$smooth for every $h \in M$. And since M is total in X, the convexity and Lipschitz property of $|\cdot|_\varepsilon$ guarantee that this norm is uniformly $2\varepsilon - \{h\}-$smooth for every $h \in X$. Now defining $|\cdot|^{*2} = \sum_{j=1}^\infty 2^{-j} |\cdot|_{1/j}^{*\,2}$, a Šmulyan duality argument yields that the corresponding predual norm $|\cdot|$ on X is uniformly Gâteaux smooth. And this is what (iv) asserts. ∎

Proof of Theorem 7. (i)⇒(ii). We shall elaborate the proof of (i)⇒(ii) in Theorem 6. We require now that the PRI $(P_\alpha;\ \omega \le \alpha \le \mu)$ constructed there has an additional property that $P_\alpha(K) \subset K$ for every $\alpha \in [\omega, \mu)$ where $K \subset B_X$ is a total convex symmetric and weakly compact set; this can be done using Proposition 1. Further, the sets Γ_α are not taken in $B_{Q_\alpha X}$ but in $\frac{1}{2} Q_\alpha(K)$. This extra requirement does not damage this argument at all. Moreover, we are then in the setting of the proof of (i)⇒(ii) in Theorem 1. Therefore the set Γ so constructed satisfies the assertion (ii) in Theorem 6 as well as (ii) in Theorem 1.

(ii)⇒(iii). Assume that the assertion (ii) holds. By Theorem 1, X admits a total set $M \subset B_X$ and an equivalent norm $|\cdot|$ on X such that the dual norm $|\cdot|^*$ is $M-$LUR. By Theorem 6, X admits another equivalent norm, $\|\cdot\|$ say, which is uniformly Gâteaux smooth; note that then by a Šmulyan duality argument, the dual norm $\|\cdot\|^*$ on X^* is weak* uniformly rotund, see [DGZ, Theorem II.6.7]. Finally, put

$$|||x^*|||^{*2} = |x^*|^{*2} + \|x^*\|^{*2}, \quad x^* \in X^*;$$

this is an equivalent dual norm. Then a convexity argument together with a Šmulyan duality argument show that the corresponding predual norm $|||\cdot|||$ on X is both uniformly Gâteaux smooth and $M-$smooth.

(iii)⇒(i). This follows from Theorem 6 (iv)⇒(i) and from Theorem 1 (iv)⇒(i).

(i)⇒(iii). This can be also proved directly, using Theorem 6, and an interpolation technique and locally uniformly rotund renorming of reflexive space, see [FGHZ]. ∎

Proof of Theorem 8. (ii)⇒(i). This immediately follows from Proposition 5, see also [FGHZ, Theorem 4].

(i)⇒(ii). This implication was proved in [FGHZ] for the density $\operatorname{dens} X = \aleph_1$. Here we do so for X with density less than $\aleph_{\omega_1}$. If $M \subset B_X$ and the norm $\|\cdot\|$ on X is uniformly $M-$smooth, then a family $m(\varepsilon) \in \mathbb{N}$, $\varepsilon > 0$, is called a *modulus of uniform $M-$smoothness* if

$$\big\|x + \tfrac{1}{m(\varepsilon)} h\big\| + \big\|x - \tfrac{1}{m(\varepsilon)} h\big\| - 2 < \tfrac{\varepsilon}{2m(\varepsilon)} \quad \text{whenever} \quad x \in X, \quad \|x\| = 1, \quad \text{and} \quad h \in M$$

for every $\varepsilon > 0$. We shall need the following

Lemma 2. *Let β be an ordinal. Let Z be a Banach space, with density* $\operatorname{dens} Z = \aleph_{\beta+1}$, *whose norm $\|\cdot\|$ is uniformly $N-$smooth for some total convex symmetric and closed set $N \subset B_Z$, with a modulus of uniform $N-$smoothness $m(\varepsilon) \in \mathbb{N}$, $\varepsilon > 0$. Finally, assume that there exist $\kappa(\varepsilon) \in \mathbb{N}$, $\varepsilon > 0$, such that for every linear bounded projection $R : Z \to Z$, with* $\operatorname{dens} R(Z) \le \aleph_\beta$, *and with $R(N) \subset 2N$, there exists a set $\Gamma_R \subset \frac{1}{2}R(N)$, total in $R(Z)$, such that*

$$\forall \varepsilon > 0 \ \ \forall z^* \in B_{Z^*} \quad \#\big\{\gamma \in \Gamma_R;\ \langle \gamma, z^* \rangle > \varepsilon\big\} < \kappa(\varepsilon).$$

Then there exists a set $\Gamma \subset N$, total in Z, such that

$$\forall \varepsilon > 0 \ \ \forall z^* \in B_{Z^*} \quad \#\big\{\gamma \in \Gamma_R;\ \langle \gamma, z^* \rangle > \varepsilon\big\} < \frac{4m(\varepsilon)}{\varepsilon^2} \cdot \kappa(\varepsilon).$$

Proof. We observe that the set N is weakly compact. Indeed, using Goldstine's theorem, we easily get that the norm $\|\cdot\|^{**}$ on X^{**} is (uniformly) $N-$smooth. Hence, by Proposition 3, $\overline{N}^* \subset Z$, and so the (closed convex bounded) set N must be weakly compact. Therefore Z is WCG and hence it admits a PRI $\big(P_\alpha;\ \omega \le \alpha \le \mu\big)$, with $P_\alpha N \subset N$ for every $\alpha \in [\omega, \mu)$, see Proposition 1. Then for every $\alpha \in [\omega, \mu)$, when denoting $Q_\alpha = P_{\alpha+1} - P_\alpha$, we have $Q_\alpha N \subset 2N$ and $\operatorname{dens} Q_\alpha Z \le \aleph_\beta$. From the assumptions, for every $\alpha \in [\omega, \mu)$ we find a set $\Gamma_{Q_\alpha} \subset \frac{1}{2} Q_\alpha N$, total in $Q_\alpha Z$, such that

$$\forall \varepsilon > 0 \ \ \forall z^* \in B_{Z^*} \quad \#\big\{\gamma \in \Gamma_{Q_\alpha};\ \langle \gamma, z^* \rangle > \varepsilon\big\} < \kappa(\varepsilon).$$

Put then $\Gamma = \bigcup_{\alpha<\mu} \Gamma_{Q_\alpha}$; this is surely a total set. We shall show that Γ satisfies the conclusion.

So fix any $\varepsilon > 0$ and any $z^* \in B_{Z^*}$. Define

$$F = \big\{\alpha \in [\omega, \mu);\ \langle \gamma, z^* \rangle > \varepsilon \ \text{ for some } \ \gamma \in \Gamma_{Q_\alpha}\big\}.$$

We `claim` that $\#F < \frac{4m(\varepsilon)}{\varepsilon^2}$; then we shall get that

$$\#\big\{\gamma \in \Gamma;\ \langle \gamma, z^* \rangle > \varepsilon\big\} = \sum_{\alpha \in F} \#\big\{\gamma \in \Gamma_{Q_\alpha};\ \langle \gamma, z^* \rangle > \varepsilon\big\} < \#F \cdot \kappa(\varepsilon) < \frac{4m(\varepsilon)}{\varepsilon^2} \cdot \kappa(\varepsilon).$$

and we shall be done.

It remains to prove the claim. We shall use ideas from the end of the proof of (iv)$\Rightarrow$(ii) in Theorem 6. If the set F is infinite, let k be any fixed positive integer. Otherwise, denote $k = \#F$. Find $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \mu$ such that $F \supset \{\alpha_1, \alpha_2, \dots, \alpha_k\}$. For $j = 1, \dots, k$ find $\gamma_j \in \Gamma_{\alpha_j}$, with $\langle \gamma_j, z^* \rangle > \varepsilon$, and write $v_j = \gamma_1 + \cdots + \gamma_j$. Find $i \in \mathbb{N}$ so that $\frac{m(\varepsilon)}{\varepsilon} \le i \le \frac{2m(\varepsilon)}{\varepsilon}$. (We forgot to assume that $\varepsilon \le 1$, hence, we do so now.) If $i \ge k$, then $k \le \frac{2m(\varepsilon)}{\varepsilon} < \frac{4m(\varepsilon)}{\varepsilon^2}$. Further assume that $i < k$. Since $\|v_i\| \ge \langle v_i, z^* \rangle > i\varepsilon \ge m(\varepsilon)$, $P_{\alpha_{i+1}} \circ Q_{\alpha_{i+1}} = 0$, and $\gamma_{i+1} \in \Gamma_{Q_{\alpha_{i+1}}} \subset \frac{1}{2} Q_{\alpha_{i+1}} N \subset N$, the convexity of $\|\cdot\|$ yields

$$\begin{aligned} \|v_{i+1}\| &= \|v_i\| \Big(\Big\| \frac{v_i}{\|v_i\|} + \frac{\gamma_{i+1}}{\|v_i\|} \Big\| - 1 \Big) + \|v_i\| \\ &\le m(\varepsilon) \Big(\Big\| \frac{v_i}{\|v_i\|} + \frac{\gamma_{i+1}}{m(\varepsilon)} \Big\| + \Big\| \frac{v_i}{\|v_i\|} - \frac{\gamma_{i+1}}{m(\varepsilon)} \Big\| - 2 \Big) + \|v_i\| < \frac{\varepsilon}{2} + \|v_i\|. \end{aligned}$$

Similarly, we get

$$\|v_{i+2}\| < \frac{\varepsilon}{2} + \|v_{i+1}\| < 2\frac{\varepsilon}{2} + \|v_i\|, \ \ldots, \quad \|v_k\| < (k-i)\frac{\varepsilon}{2} + \|v_i\| < k\frac{\varepsilon}{2} + i.$$

Thus

$$k\varepsilon < \langle v_k, z^* \rangle \le \|v_k\| < k\frac{\varepsilon}{2} + i,$$

and so $k < \frac{2}{\varepsilon}i < \frac{4m(\varepsilon)}{\varepsilon^2}$. This also shows that the set F cannot be infinite. Hence $\#F = k < \frac{4m(\varepsilon)}{\varepsilon^2}$. ∎

Now we continue in proving (i)⇒(ii) in Theorem 8. This implication is contained in the following subtler statement.

Proposition 6. *Let $\beta_0 < \omega_1$ be an ordinal. Let X be a Banach space, with* dens $X = \aleph_{\beta_0}$, *whose norm $\|\cdot\|$ is uniformly $M-$smooth for some total convex symmetric and closed set $M \subset B_X$. Then for every $\beta \in [0, \beta_0]$ there exist $\kappa_\beta(\varepsilon) \in \mathbb{N}$, $\varepsilon > 0$, such that for every linear bounded projection $Q : X \to X$, with* dens $QX \le \aleph_\beta$, *and with $QM \subset cM$ for some $c > 0$, there exists a set $\Gamma_{\beta,Q} \subset \frac{1}{c}QM$, total in QX, such that*

$$\forall \varepsilon > 0 \ \ \forall x^* \in B_{X^*} \quad \#\big\{\gamma \in \Gamma_{\beta,Q};\ \langle \gamma, x^* \rangle > \varepsilon\big\} < \kappa_\beta(\varepsilon).$$

Hence, in particular, there are $\kappa(\varepsilon) \in \mathbb{N}$, $\varepsilon > 0$, and a set $\Gamma \subset M$, total in X, so that

$$\forall \varepsilon > 0 \ \ \forall x^* \in B_{X^*} \quad \#\big\{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle > \varepsilon\big\} < \kappa(\varepsilon).$$

Proof. If $\beta_0 = 0$, there is almost nothing to prove. So further assume that $\beta_0 > 0$. Let $m(\varepsilon) \in \mathbb{N}$, $\varepsilon > 0$, be the modulus of uniform smoothness of $\|\cdot\|$ with respect to M. We shall proceed by transfinite induction over β. First, fix any $\beta \in [0, \beta_0)$ and assume that the conclusion was already verified for this β. Now, let $Q : X \to X$ be any fixed linear bounded projection, with dens $QX \le \aleph_{\beta+1}$, and such that $QM \subset cM$ for some $c > 0$. Denote $Z = QX$ and $N = \frac{1}{c}QM$. Then N will be a total set in Z and the restriction of $\|\cdot\|$ to Z will be uniformly $N-$smooth, with the same modulus of uniform $N-$smoothness $m(\varepsilon)$, $\varepsilon > 0$. We shall verify the assumptions of Lemma 2. So let $R : Z \to Z$ be any linear bounded projection, with dens $RZ \le \aleph_\beta$ and with $RN \subset 2N$. Then $R \circ Q : X \to X$ is a linear bounded projection on X, dens $(R \circ Q)X \le \aleph_\beta$, and $(R \circ Q)M \subset 2cM$. Hence, by the induction assumption, there is a set $\Gamma_{\beta,R\circ Q} \subset \frac{1}{2c}(R \circ Q)M$ $(= \frac{1}{2}RN)$, total in $(R \circ Q)X$ $(= RZ)$ such that

$$\forall \varepsilon > 0 \ \ \forall x^* \in B_{X^*} \quad \#\big\{\gamma \in \Gamma_{\beta,R\circ Q};\ \langle \gamma, x^* \rangle > \varepsilon\big\} < \kappa_\beta(\varepsilon),$$

that is,

$$\forall \varepsilon > 0 \ \ \forall z^* \in B_{Z^*} \quad \#\big\{\gamma \in \Gamma_R;\ \langle \gamma, z^* \rangle > \varepsilon\big\} < \kappa_\beta(\varepsilon),$$

where we put $\Gamma_R = \Gamma_{\beta,R\circ Q}$. Thus we verified all the assumptions of Lemma 2, and hence there exists a set $\Gamma_{\beta+1,Q} \subset N$ $(= \frac{1}{c}QM)$, total in Z $(= QX)$, such that

$$\forall \varepsilon > 0 \ \ \forall z^* \in B_{Z^*} \quad \#\big\{\gamma \in \Gamma_{\beta+1,Q};\ \langle \gamma, z^* \rangle > \varepsilon\big\} < \frac{4m(\varepsilon)}{\varepsilon^2} \cdot \kappa_\beta(\varepsilon).$$

Then

$$\forall \varepsilon > 0 \ \ \forall x^* \in B_{X^*} \quad \#\big\{\gamma \in \Gamma_{\beta+1,Q};\ \langle \gamma, x^*\rangle > \varepsilon\big\} < \frac{4m(\varepsilon)}{\varepsilon^2} \cdot \kappa_\beta(\varepsilon).$$

Therefore we may put $\kappa_{\beta+1}(\varepsilon) = \frac{4m(\varepsilon)}{\varepsilon^2} \cdot \kappa_\beta(\varepsilon),\ \varepsilon > 0$.

Second, fix any limit ordinal $\lambda \in [\omega, \beta_0]$, and assume that we verified the conclusion of our proposition for every $\beta \in [0, \lambda)$. We enumerate the (countable) interval $[0, \lambda)$ by $\{\beta_1, \beta_2, \ldots\}$. We shall show that

$$\kappa_\lambda(\varepsilon) = \sum_{i=1}^{1/\varepsilon} \kappa_{\beta_i}(\varepsilon), \quad \varepsilon > 0$$

satisfies our needs. So, consider any linear bounded projection $Q : X \to X$, with $\operatorname{dens} QX \le \aleph_\lambda$ and with $QM \subset cM$ for some $c > 0$. Assume first that $\operatorname{dens} QX < \aleph_\lambda$; then $\operatorname{dens} QX = \aleph_{\beta_i}$ for a suitable $i \in \mathbb{N}$ and $\beta_i < \lambda$. From the induction assumption we find the set $\Gamma_{\beta_i,Q}$ and put $\Gamma_{\lambda,Q} = \frac{1}{i}\Gamma_{\beta_i,Q}$. Fix any $\varepsilon > 0$ and any $x^* \in B_{X^*}$. If $i \ge \frac{1}{\varepsilon}$, then $\#\big\{\gamma \in \Gamma_{\lambda,Q};\ \langle \gamma, x^*\rangle > \varepsilon\big\} = 0 < \kappa_\lambda(\varepsilon)$. If $i < \frac{1}{i}$, then $\#\big\{\gamma \in \Gamma_{\lambda,Q};\ \langle \gamma, x^*\rangle > \varepsilon\big\} \le \#\big\{\gamma \in \Gamma_{\beta_i,Q};\ \langle \gamma, x^*\rangle > \varepsilon\big\} < \kappa_{\beta_i,Q}(\varepsilon) \le \kappa_\lambda(\varepsilon)$. Assume further that $\operatorname{dens} QX = \aleph_\lambda$. By Proposition 1, find a PRI $(P_\alpha;\ \omega \le \alpha \le \mu)$ on QX such that $P_\alpha(QM) \subset QM$ for every $\alpha \in [0, \mu)$. For every $i \in \mathbb{N}$ find $\alpha_i \in [\omega, \mu)$ so that $\operatorname{dens} P_{\alpha_i}(QX) = \aleph_{\beta_i}$. Then, surely, the set $\{\alpha_i;\ i \in \mathbb{N}\}$ is cofinal in $[\omega, \mu)$, and hence $\bigcup_{i\in\mathbb{N}} P_{\alpha_i}(QX)$ is dense in QX. By the induction assumption, for every $i \in \mathbb{N}$ find a set $\Gamma_{\beta_i, P_{\alpha_i}\circ Q} \subset \frac{1}{c}(P_{\alpha_i} \circ Q)M$, total in $(P_{\alpha_i} \circ Q)X$, and such that

$$\forall \varepsilon > 0 \ \ \forall x^* \in B_{X^*} \quad \#\big\{\gamma \in \Gamma_{\beta_i, P_{\alpha_i}\circ Q};\ \langle \gamma, x^*\rangle > \varepsilon\big\} < \kappa_{\beta_i}(\varepsilon).$$

Then, putting $\Gamma_{\lambda,Q} = \bigcup_{i=1}^\infty \frac{1}{i}\Gamma_{\beta_i, P_{\alpha_i}\circ Q}$, this will be a subset of $\frac{1}{c}QM$, total in QX, and

$$\forall \varepsilon > 0 \ \ \forall x^* \in B_{X^*} \quad \#\big\{\gamma \in \Gamma_{\lambda,Q};\ \langle \gamma, x^*\rangle > \varepsilon\big\} < \sum_{i=1}^{1/\varepsilon} \kappa_{\beta_i}(\varepsilon) = \kappa_\lambda(\varepsilon).$$

∎

Proof of Theorem 9. (i)⇒(ii). Let $T : \ell_p(\Delta) \to X$ be a bounded linear mapping with dense range. We may assume that $\|T\| = 1$. Let $e_\delta,\ \delta \in \Delta$, denote the canonical basis in $\ell_p(\Delta)$. For any $x^* \in B_{X^*}$ and any $\sum_{\delta\in\Delta} a_\delta e_\delta \in \ell_p(\Delta)$, with finite support, we have

$$\begin{aligned}\sum_{\delta\in\Delta} a_\delta \langle Te_\delta, x^*\rangle &= \Big\langle T\Big(\sum_{\delta\in\Delta} a_\delta e_\delta\Big), x^*\Big\rangle \\ &\le \|T\| \Big\| \sum_{\delta\in\Delta} a_\delta e_\delta \Big\|_{\ell_p(\Delta)} \|x^*\| \le \Big(\sum_{\delta\in\Delta} |a_\delta|^p\Big)^{1/p},\end{aligned}$$

and hence $\sum_{\delta\in\Delta} |\langle Te_\delta, x^*\rangle|^q \le 1$. Now, it is clear that we can take $\Gamma = \big\{Te_\delta;\ \delta \in \Delta\big\}$ in the case when $\#\Delta = \operatorname{dens} X$ (we always have $\#\Delta \ge \operatorname{dens} X$). And, if $\#\Delta > \operatorname{dens} X$, then a simple gymnastics finds a suitable $\Gamma \subset \big\{Te_\delta;\ \delta \in \Delta\big\}$ satisfying (ii).

(ii)⇒(i). For $\sum_{\gamma\in\Gamma} a_\gamma e_\gamma \in \ell_p(\Gamma)$, with finite support, we put $T\big(\sum_{\gamma\in\Gamma} a_\gamma e_\gamma\big) = \sum_{\gamma\in\Gamma} a_\gamma \gamma$ $(\in X)$. Then for every $x^*\in B_{X^*}$ we have from (ii)

$$\Big\langle T\Big(\sum_{\gamma\in\Gamma} a_\gamma e_\gamma\Big), x^*\Big\rangle = \sum_{\gamma\in\Gamma} a_\gamma\langle\gamma, x^*\rangle \le \Big(\sum_{\gamma\in\Gamma}|a_\gamma|^p\Big)^{1/p}\Big(\sum_{\gamma\in\Gamma}|\langle\gamma,x^*\rangle|^q\Big)^{1/q} \le \Big(\sum_{\gamma\in\Gamma}|a_\gamma|^p\Big)^{1/p}.$$

Thus T can be extended to the whole space $\ell_p(\Gamma)$, mapping it linearly and continuously onto a dense subset of X.

The equivalence (i)⇔(iii) for $\operatorname{dens} X = \aleph_1$ can be found in [FGHZ, Theorem 2].

Proof of Theorem 10. `Necessity.` Denote $\Gamma_0 = \{\gamma\in\Gamma;\ k(\gamma) = 0 \text{ for every } k\in K\}$. For $\gamma,\gamma'\in\Gamma$ we write $\gamma\sim\gamma'$ if $k(\gamma) = k(\gamma')$ for every $k\in K$; this is a relation of equivalence. For $\gamma\in\Gamma\backslash\Gamma_0$ let $[\gamma] = \{\gamma'\in\Gamma\backslash\Gamma_0;\ \gamma'\sim\gamma\}$. Denote $\Lambda = \{[\gamma];\ \gamma\in\Gamma\backslash\Gamma_0\}$. Since $K\subset\Sigma(\Gamma)$, we get that every $\lambda\in\Lambda$ consists of at most countably many elements; let us enumerate it as $\lambda = \big\{\gamma_1^\lambda, \gamma_2^\lambda, \dots\big\}$. (The enumeration may not be injective.) For $i\in\mathbb{N}$ put then $\Gamma_i = \big\{\gamma_i^\lambda;\ \lambda\in\Lambda\big\}$. Clearly $\Gamma = \Gamma_0\cup\Gamma_1\cup\Gamma_2\cup\cdots$ For $\gamma\in\Gamma$ we define $\pi_\gamma(k) = k(\gamma),\ k\in K$; then, clearly, $\pi_\gamma\in C(K)$. (It may happen that the correspondence $\gamma\mapsto\pi_\gamma$ is not injective. This is why we have to work with the Γ_i's.)

Fix for a while any $i\in\mathbb{N}$. Put $\widetilde\Gamma_i = \{\pi_\gamma,\ \gamma\in\Gamma_i\}$ and let X_i denote the closed subspace of $C(K)$ generated by $\widetilde\Gamma_i$. Fix any $x^*\in B_{X_i^*}$. We shall observe that

$$\#\big\{\tilde\gamma\in\widetilde\Gamma_i;\ \langle\tilde\gamma, x^*\rangle\ne 0\big\}\le\aleph_0. \tag{4}$$

Find $y^*\in B_{C(K)^*}$ such that ${y^*}_{|X} = x^*$. If $y^* = \delta_k$, the point mass at some $k\in K$, then (4) holds trivially. Also, (4) holds if y^* is equal to a finite linear combination of point masses. Further, K being a (uniform) Eberlein, Gul'ko, or Talagrand compact, the space $C(K)$ is WCG (according to Amir and Lindenstrauss), Vašák, or weakly $\mathcal{K}-$analytic respectively. Thus Theorems 1, 3, and 4 guarantee that $(B_{C(K)^*}, w^*)$ is a Corson compact. Hence every element of $B_{C(K)^*}$ lies in the weak* closure of a *countable* subset of the linear span of $\{\delta_k;\ k\in K\}$. Therefore (4) holds for any $x^*\in X_i^*$. We have thus proved that the set $\widetilde\Gamma_i$, total in X_i, countably supports X_i^*.

Now we are ready to prove the necessary conditions of Theorem 10. Consider first the case of (uniform) Eberlein compacta. So fix any $\varepsilon > 0$. Let $\widetilde\Gamma_i = \bigcup_{n=1}^\infty\widetilde\Gamma_{i,n}^\varepsilon$ be the decomposition provided by the condition (ii) in Theorem 2 (6). Put then

$$\Gamma_{i,n}^\varepsilon = \big\{\gamma\in\Gamma_i;\ \pi_\gamma\in\widetilde\Gamma_{i,n}^\varepsilon\big\},\quad \varepsilon>0,\quad n\in\mathbb{N}.$$

Clearly, $\Gamma_i = \bigcup_{n=1}^\infty\Gamma_{i,n}^\varepsilon$. Now fix any $n\in\mathbb{N}$ and any $k\in K$. Then, profiting from the injectivity of the mapping $\gamma\mapsto\pi_\gamma$ between Γ_i and $\widetilde\Gamma_i$, we have

$$\begin{aligned}\#\big\{\gamma\in\Gamma_{i,n}^\varepsilon;\ |k(\gamma)|>\varepsilon\big\} &= \#\big\{\gamma\in\Gamma_{i,n}^\varepsilon;\ \langle\pi_\gamma,\delta_k\rangle>\varepsilon\big\} + \#\big\{\gamma\in\Gamma_{i,n}^\varepsilon;\ \langle\pi_\gamma,-\delta_k\rangle>\varepsilon\big\}\\ &= \#\big\{\tilde\gamma\in\widetilde\Gamma_{i,n}^\varepsilon;\ \langle\tilde\gamma,\delta_k\rangle>\varepsilon\big\} + \#\big\{\tilde\gamma\in\widetilde\Gamma_{i,n}^\varepsilon;\ \langle\tilde\gamma,-\delta_k\rangle>\varepsilon\big\}\\ &< \aleph_0\quad(<2n).\end{aligned}$$

This holds for every $i \in \mathbb{N}$. We note that $\Gamma = \bigcup_{i=0}^{\infty} \Gamma_i = \bigcup_{i,n=1}^{\infty} \Gamma_{i,n}^{\varepsilon} \cup \Gamma_0$. It remains to enumerate the family $\Gamma_0,\ \Gamma_{i,n}^{\varepsilon},\ i,n \in \mathbb{N}$, by elements of $\mathbb{N}$, to "make" it pairwise disjoint, and in the uniform case, to insert "a few" empty sets. We thus proved the necessity in (a) of Theorem 10.

In the case of Gul'ko compacta, we find for every $i \in \mathbb{N}$ sets $\widetilde{\Gamma}_{i,n} \subset \widetilde{\Gamma},\ n \in \mathbb{N}$, as is stated in (ii) of Theorem 3. Putting then $\Gamma_{i,n} = \{\gamma \in \Gamma_i;\ \pi_\gamma \in \widetilde{\Gamma}_{i,n}\},\ i,n \in \mathbb{N}$, we get a countable family which, together with the set Γ_0, obviously satisfies the required necessary condition in (b) of Theorem 10.

Finally, assume that K is a Talagrand compact. Applying (ii) in Theorem 4 for every $i \in \mathbb{N}$, we get sets $(\widetilde{\Gamma}_i)_s \subset \widetilde{\Gamma}_i,\ s \in \mathbb{N}^{<\mathbb{N}}$. Putting then $\Gamma_{s_1,\dots,s_m} = \{\gamma \in \Gamma_{s_1};\ \pi_\gamma \in (\widetilde{\Gamma}_{s_1})_{s_2,\dots,s_m}\},\ (s_1,\dots,s_m) \in \mathbb{N}^m,\ m = 2,3,\dots$, it is easy to verify the necessary condition in (c) of Theorem 10.

`Sufficiency.` Let $\Gamma_n^{\varepsilon},\ \varepsilon > 0,\ n \in \mathbb{N}$, be as in (a) of Theorem 9. Let $\tau_i,\ i \in \mathbb{N}$, be the functions defined in the proof of Theorem 2 (6). Define then $\Phi : K \to \mathbb{R}^{\Gamma \times \mathbb{N}}$ by

$$\Phi(k)(\gamma, i) = \frac{1}{2^n 2^i \sqrt{n}} \tau_i(k(\gamma)) \quad \text{if} \quad \gamma \in \Gamma_n^{1/i}, \quad n \in \mathbb{N}, \quad \text{and} \quad i \in \mathbb{N}$$

for $k \in K$. Clearly, Φ is continuous. It is also injective. And $\Phi(K)$ is a subset of $c_0(\Gamma \times \mathbb{N})$ (of $\ell_2(\Gamma \times \mathbb{N})$). For more details see the proof of (ii)$\Rightarrow$(iii) in Theorem 2 (6). Therefore K is a (uniform) Eberlein compact.

Now let the condition in (b) or (c) be satisfied. Let X be the subspace of $C(K)$ generated by the set $\widetilde{\Gamma} := \{\pi_\gamma;\ \gamma \in \Gamma\}$. Then we can easily check that this set, provided by the weak topology of X, is $\mathcal{K}-$countably determined or $\mathcal{K}-$analytic in $(B_{X^{**}}, w^*)$, see the proof of (ii)$\Rightarrow$(i) in Theorem 3. And since the set $\widetilde{\Gamma}$ separates the points of K, [T, Théorème 3.4 (iii)] guarantees that the whole $C(K)$ is Vašák or weakly $\mathcal{K}-$analytic. ∎

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