The Day norm and Gruenhage compacta

M. Fabian*, V.Montesinos† and V. Zizler‡

Abstract

A close connection of the strict convexity of the Day norm to the concept of the Gruenhage compacta is shown. As a byproduct we give an elementary characterization of Gul'ko compacta in the sigma-product of lines and a more elementary proof of Mercourakis' renorming result for Vašák spaces.

This note is a result of our effort to classify those Banach spaces with dual ball Corson (in its weak* topology) that would admit a Gâteaux smooth renorming.

Given a non-empty set S, let $\Sigma(S) := \{x \in [-1,1]^S;\ \text{support of } x \text{ is countable}\}$. We shall always assume that $\Sigma(S)$ is endowed with its product topology. A compact space K is called a *Corson compact* if K is homeomorphic to a subset of $\Sigma(S)$.

It was proved in [AM, p. 425] that a Banach space X with dual ball Corson need not in general admit any equivalent Gâteaux differentiable norm. We find here a sufficient condition for a Gâteaux smooth renorming that uses the strict convexity of the Day norm on the dual space and is closely related to the notion of Gruenhage compacta (cf. [Gru], [AM, p. 424], [Ri, Def 2.1]). As a corollary, we prove a renorming theorem which gives, in particular, a result in [Me].

The result in this note is related to the result of Raja that X^* admits a dual norm that is weak* locally uniformly rotund if and only if B_{X^*} in its weak* topology is a descriptive compact [Ra].

As a byproduct of our efforts, we obtain a characterization, in the Sokolov's style [S], of Gul'ko compacta lying in the space $\Sigma(S)$ in the spirit of the characterization of Eberlein compacta given in [Fa].

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Assume that a Banach space X admits a Markushevich basis, i.e. a biorthogonal system $\{x_\alpha, f_\alpha\}_{\alpha\in\Gamma}$ such that $\overline{\text{span}}\{x_\alpha\} = X$ and $\{f_\alpha\}$ separates points in X. Then, it was shown in [FMZ] and [FGMZ] that X admits an equivalent uniformly Gâteaux differentiable norm if and only if the set Γ can be split into $\Gamma = \bigcup \Gamma_n$ such that the formula $|||f|||^2 := \|f\|^2 + \sum D_n^2(f)$ gives a dual weak* uniformly rotund norm on X^*. Here by $\|f\|$ we mean the original sup norm of X^* and $D_n(f)$ is the Day norm applied to the coordinate functionals $\{f_\alpha\}_{\alpha\in\Gamma_n}$. The norm $\|\cdot\|$ of X^* is weak* uniformly rotund if $f_n - g_n \to 0$ in the weak* topology whenever $f_n, g_n \in S_{X^*}$ and $\|f_n + g_n\| \to 2$. This property is known to be the dual property to the uniform Gâteaux differentiability of the norm, [DGZ]. The norm $\|\cdot\|$ on X is uniformly Gâteaux differentiable if $\lim\limits_{t\to 0} \frac{1}{t}(\|x + th\| + \|x - th\| - 2) = 0$ uniformly for $x \in S_X$, the unit sphere of X. This means that if we know that X admits an equivalent Gâteaux differentiable norm, we can use the Day norm on the dual space to construct another uniformly Gâteaux differentiable norm on X. therefore the situation is similar to that of uniformly Fréchet differentiable norms (James-Enflo theorem, see e.g. [DGZ, Ch. 4] or Chapter 9 in[F$^\sim$]).

In [Fa], the Eberlein compacta lying in $\Sigma(S)$ were classified by using infinite combinatorics.

In [FGHZ] and in [FGMZ] we characterized several classes of non-separable Banach spaces X by the existence of a total subset (i.e. linearly dense) subset $\Gamma \subset X$ which can be split in a certain way. Our method uses only projectional resolutions and techniques in Markushevich bases. In particular, it does not use any combinatorics. Instead, we use the method of projections to characterize Gul'ko compacta in $\Sigma(S)$, in the style of Sokolov [S]. A compact space K is a *Gul'ko compact* if $(C(K), p)$ is $\mathcal{K} - cd$, where p denotes the topology of the pointwise convergence. A topological subspace T of a compact space K is $\mathcal{K} - cd$ whenever there exist closed sets $K_n \subset K$, $n \in I\!N$ with the property that for every $t \in T$ and for every $k \in K \setminus T$ there exists $n \in I\!N$ such that $t \in K_n$ and $k \notin K_n$. A Banach space X is *Vašák* (or *weakly countably determined*, in short, WCD), if (X, w) is $\mathcal{K} - cd$. Then, a compact space K is Gul'ko if and only if $(C(K), \|\cdot\|)$ is Vašák, see, e.g., [F, Thm. 7.1.8].

We follow the standard notation that can be found, for example, in [F$^\sim$] and in [F]. For all concepts and results not explained here we refer to [F$^\sim$], [DGZ] and [F].

Given a subset $\Gamma \subset X$ and a projectional resolution of the identity (in short, a PRI) $(P_\alpha)_{\omega_0\leq\alpha\leq\mu}$ on X (see, for example, [F]), we will say that they are *subordinated* (to each other) if $P_\alpha(\gamma) \in \{\gamma, 0\}$ for all $\gamma \in \Gamma$ and $\omega_0 \leq \alpha \leq \mu$.
A subset Γ of a Banach space X is said to *countably support* X^* if $\#\{\gamma \in \Gamma : \langle\gamma, x^*\rangle \neq 0\} \leq \aleph_0$, for all $x^* \in X^*$ (here $\#$ denotes the cardinal number of a set). It follows that if a Banach space X has a total subset Γ which countably supports X^*, then there exists a PRI on X subordinated to Γ, as $\Phi(x^*) := \{\gamma \in \Gamma : \langle\gamma, x^*\rangle \neq 0\}$, for all $x^* \in X^*$, is a projectional generator (see, e.g., [F, Def. 6.1.6] and [FMZ]).

The following result follows by transfinite induction on the density character of X. For the concept of separable PRI see, for example, [F, Def. 6.2.6]:

Proposition 1 *Let X be a WCD Banach space. Let $\Gamma \subset X$ be a total subset of X that countably supports X^*. Then there exists a separable PRI on X subordinated to Γ.*

A subset $\Gamma \subset X$ of a Banach space X will be called *weakly $\sigma-$shrinking* if there exists a sequence $(\Gamma_n)_{n=1}^{\infty}$ of subsets of Γ such that $\Gamma = \bigcup_{n\in I\!N} \Gamma_n$ and for each $\varepsilon > 0$, for each $\gamma_0 \in \Gamma$ and for each $x^* \in B_{X^*}$ there is $n \in I\!N$ so that $\gamma_0 \in \Gamma_n$ and $\#\{\gamma \in \Gamma_n; |\langle \gamma, x^* \rangle| \geq \varepsilon\} < \aleph_0$.
The following theorem is a consequence of the previous proposition and the method of proof of [FMZ, Theorem 3].

Proposition 2 *Let X be a WCD Banach space. Then X contains a bounded total weakly $\sigma-$shrinking subset. Moreover, every total and bounded subset Γ of X which countably supports X^* is weakly $\sigma-$shrinking.*

We can now characterize Gul'ko compacta in $\Sigma(S)$. Our approach is similar to that in [FGMZ] and uses ideas in [S].

Theorem 3 *Given a compact subset K of the topological space $\Sigma(S)$, the two following properties are equivalent:*

1. *K is a Gul'ko compact.*
2. *There exists a sequence $(S_n)_{n=1}^{\infty}$ of subsets of S such that $S = \bigcup_{n=1}^{\infty} S_n$ and given $k \in K$, $\epsilon > 0$ and $s_0 \in S$ there exists $n \in I\!N$ such that $s_0 \in S_n$ and $\#\{s \in S_n : \ |k(s)| > \epsilon\} < \aleph_0$.*

Proof. Given a compact set K in $\Sigma(S)$, let $\pi : S \to C(K)$ be defined by $\pi(s)(k) := k(s)$, for all $s \in S$, $k \in K$. From the fact that K is in $\Sigma(S)$ it follows that $\#\pi^{-1}(f) \leq \aleph_0$ for all $f \in \pi(S)$, say $\pi^{-1}(f) := \{s_{(f,1)}, s_{(f,2)}, \ldots\}$ (repeating elements if $\pi^{-1}(f)$ is a finite set). Given $m \in I\!N$, let $w_m : \pi(S) \to S$ be the selector for the multivalued mapping $\pi^{-1} : \pi(S) \to 2^S$ given by $w_m(f) := s_{(f,m)}$, $f \in \pi(S)$ and let $S_m := w_m(\pi(S))$. Obviously, $S = \bigcup_{m\in I\!N} S_m$. Now the mapping $R_m : [-1,1]^S \to [-1,1]^{S_m}$ defined by $R_m(x)(s) := x(s)$ for all $s \in S_m$ and $x \in [-1,1]^S$, is continuous and $R_m|_K$ is one to one. Thus $K_m := R_m(K)$ is homeomorphic to K. Let $\pi_m : S_m \to C(K_m)$ be defined by $\pi_m(s)(R_m(k)) := k(s)$, for all $s \in S_m$, $k \in K$. π_m is a one-to-one mapping.
$(1 \Rightarrow 2)$ Fix $m \in I\!N$. Let $X_m := \overline{\text{span}}\{\pi_m(s) : \ s \in S_m\} \subset C(K_m)$. As it was mentioned before, $(C(K_m), \|\cdot\|)$ is WCD, and then $(B_{C(K_m)^*}, w^*)$ is angelic (a compact space is *angelic* if the closure of every subset is attained by sequences), see for example [F$^{\sim}$, Ex. 12.55]. The set $\pi_m(S_m)$ countably supports K_m as K_m is in $\Sigma(S_m)$. Thus $\pi_m(S_m)$ countably supports $\text{conv}(\pm K_m)$ in the dual space and hence $\pi_m(S_m)$ countably supports $B_{C(K_m)^*}$ because of the angelicity of $(B_{C(K_m)^*}, w^*)$. Thus $\pi_m(S_m)$ countably supports $B_{X_m^*}$. X_m is also WCD.

Apply now Proposition 2 to get $S_m = \bigcup_{n=1}^{\infty} S_{m,n}$ from the weakly $\sigma-$shrinking property of $\pi_m(S_m)$. Then the family $\{S_{m,n} : \ m, n \in I\!N\}$ gives the conclusion.
($2 \Rightarrow 1$) By the definition, we need to prove that $(C(K), p)$ is $\mathcal{K} - cd$. Note that $\pi(S)$ separates points of K. From 2 we get a sequence $(S_n)_{n=1}^{\infty}$ of subsets of S such that $\forall k \in K, \ s \in S, \ \epsilon > 0$, there exists $n \in I\!N$ such that $s \in S_n$ and

$$\#\{s' \in S_n : \ |k(s')| \geq \epsilon\} < \aleph_0. \tag{1}$$

It follows that $(\pi(S), p)$ is $\mathcal{K}-$cd: indeed, consider the closure $\overline{\pi(S)}$ of $\pi(S)$ in $([0,1]^K, p)$, a compact space. By adding one coordinate if necessary, we can always assume that there exists $s_0 \in S$ such that $k(s_0) = 0, \ \forall k \in K$. Then $0 \in \pi(S)$. Given $v \in \overline{\pi(S)} \setminus \pi(S)$, there exists $k \in K$ such that $v(k) \neq 0$. Choose $\epsilon > 0$ such that $0 < \epsilon < |v(k)|$. Then, given $s \in S$, we can find, for these s, k, ϵ, some $n \in I\!N$ such that (1) holds. It is then obvious that $v \notin \overline{\pi(S_N)}$ and this proves that $(\pi(S), p)$ is $\mathcal{K}-$cd.
Let W be the algebra generated by $\pi(S)$. By the Stone-Weierstrass Theorem it is $\|\cdot\|-$dense, and by elementary properties of $\mathcal{K} - cd$, it is again $\mathcal{K}-$cd. Define the mapping

$$\Phi : (W, p) \times (B_W, p)^{I\!N} \mapsto (C(K), p)$$

given by

$$\Phi(f, (f_n)) := f + \sum_{n=1}^{\infty} 2^{-n} f_n.$$

This is a continuous mapping from the $\mathcal{K}-$cd topological space $(W, p) \times (B_W, p)^{I\!N}$ onto $(C(K), p)$, hence the latter space is also $\mathcal{K}-$cd. ∎

Remark. A similar proof gives Farmaki's characterization [Fa] of Eberlein compacta in $\Sigma(S)$ (see [FGMZ]).

We will now define a property of a Banach space that is related to Gruenhage compacta. Note that in this definition we do not explicitly assume that the unit ball of the dual space is a Corson compact.

Definition 4 *We will say that a Banach space X has* property G *if X contains a bounded total set Γ so that Γ can be split into $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ in such a way that given $f, g \in B_{X^*}$, $f \neq g$, there are $\gamma \in \Gamma$ and $n \in I\!N$ such that $|(f-g)(\gamma)| > 0$, $\gamma \in \Gamma_n$ and either $\#\{\gamma \in \Gamma_n \,; |f(\gamma)| > \frac{|(f-g)(\gamma)|}{4}\} < \infty$ or $\#\{\gamma \in \Gamma_n \,; |g(\gamma)| > \frac{|(f-g)(\gamma)|}{4}\} < \infty$.*

Remark. Clearly, every weakly $\sigma-$shrinking total bounded set in a Banach space X makes X have property G. Thus it follows from Proposition 2 that Vašák spaces have property G. We conjecture that the non Vašák space $C(\Omega)$ constructed in [AM, p. 421] (see also [F, Sec. 7.3]) has property G.

Theorem 5 *Assume that a Banach space X has property G. Then there is an equivalent norm on X the dual of which is strictly convex.*

Proof. Let $\{\Gamma_n\}$ be the collection of all $\Gamma_n's$ as in Definition 4. For $n \in I\!N$, let $\|\cdot\|_n$ be a seminorm on X^* defined for $f \in X^*$ by $\|f\|_n := \|(\mathrm{Re})_n f\|_D$, where $(\mathrm{Re})_n$ is the operator of the restriction to Γ_n and $\|\cdot\|_D$ is the Day norm on $\ell_\infty(\Gamma_n)$. Let the dual equivalent norm $\|\cdot\|$ on X^* be defined for $f \in X^*$ by

$$\|f\|^2 := \sum_{n=0}^{\infty} \frac{1}{2^n} \|f\|_n^2,$$

where $\|\cdot\|_0$ is the original dual norm on X^*. We will show that $\|\cdot\|$ is a strictly convex norm on X^*. Let $f, g \in X^*$ be such that $2\|f\|^2 + 2\|g\|^2 - \|f+g\|^2 = 0$. Then a similar equality holds for each n. We may and do assume that $f, g \in B_{X^*}$. If $f \neq g$, choose $\gamma \in \Gamma$ so that $\varepsilon := |(f-g)(\gamma)| > 0$ and then choose Γ_n so that $\gamma \in \Gamma_n$ and either $\#\{\gamma' \in \Gamma_n; |f(\gamma')| > \frac{\epsilon}{4}\} < \aleph_0$ or $\#\{\gamma' \in \Gamma_n; |g(\gamma')| > \frac{\epsilon}{4}\} < \aleph_0$. Then by [FGMZ, Proposition 4], $\sup_{\gamma' \in \Gamma_n} |(f-g)(\gamma')| \leq \frac{3\epsilon}{4}$, which is a contradiction with $|(f-g)(\gamma)| = \epsilon$ and $\gamma \in \Gamma_n$.

∎

Theorem 5 reminds of a result in [Ta] which says that a very smooth Banach space X has a dual X^* with an equivalent (not necessarily dual) strictly convex norm. However, the space $C[0, \omega_1]$ has an equivalent Fréchet smooth norm and yet does not have property G, since does not have norm whose dual is strictly convex (see, e.g., [DGZ]).
From the remark before Theorem 5 we obtain the following

Corollary 6 ([Me]) *Every Vašák space has an equivalent norm the dual of which is strictly convex.*

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Mailing Addresses

Mathematical Institute of the Czech Academy of Sciences. Žitná 25, 11567, Prague 1. Czech Republic. e-mail: fabian@math.cas.cz (M. Fabian)

Departamento de Matemática Aplicada, E.T.S.I. Telecomunicación, Universidad Politécnica de Valencia. C/Vera, s/n. 46071 Valencia, Spain. e-mail: vmontesinos@mat.upv.es (V. Montesinos)

Department of Mathematical Sciences. University of Alberta, 632 Central Academic Building. Edmonton, Alberta T6G 2G1, Canada.
e-mail: vzizler@math.ualberta.ca (V. Zizler)