Convexity and w^*-compactness in Banach spaces

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ABSTRACT: Suppose $\ell_\infty \hookrightarrow X$. We construct examples of bounded sets $M \subset X$, such that $\overline{M}^{w^*} \subset X + \frac{1}{2}B_{X^*}$, but $\overline{\text{co}M}^{w^*} \not\subset X + \alpha B_{X^{**}}$ for any $\alpha < 1$. These examples show that the previous results of the authors on quantitative versions of Krein's theorem are optimal.

In our previous (independent) papers [FHMZ] and [G] we were concerned with the problem of generalizing the classical Krein's theorem on the weak compactness of closed convex hulls of weakly compact sets. The goal was to replace the compactness assumption by a quantitative notion which measures the failure of a set being weakly compact. The notion that we considered (which was motivated by [FMZ]) is given in Definition 1.

Definition 1
Assume that a bounded subset M of a Banach space X (resp. X^{**}) satisfies $\overline{M}^{w^*} \subset X + \varepsilon B_{X^{**}}$ for some $\varepsilon \geq 0$.We say that M is ε-weakly relatively compact, ε-WRK, with respect to the space X.

Based on this notion, we proved (in fact [G] contained somewhat more general results) the following theorem.

Theorem 2
Suppose a bounded set M in a Banach space X (resp. X^{**}) is ε-WRK. Then the convex hull co$M$ is 2ε-WRK (resp. 5ε-WRK).

Note that the case $\varepsilon = 0$ corresponds to the classical Krein's theorem, and also that the statement is of isometric nature and may depend (for a fixed set M) on the particular renorming of the space. The growth from ε-WRK to 2ε-WRK for convex hulls might at a first sight appear to be just an inefficiency of the proof of Theorem 2. In fact, in our papers [FHMZ] and [G] we isolate classes of Banach spaces (e.g. WCG) for which co$M$ is ε-WRK for every ε-WRK bounded set M, and this holds true for every equivalent norm on X. However, relying on the continuum hypothesis (CH), an example is shown in [G] (which originates in [AMN]) of a bounded subset $M \subset \ell_\infty^c(\omega^+)^{**}$ such that M is $\frac{1}{2}$-WRK and co$M$ is not α-WRK for any $\alpha < 1$. This example can be readily modified to yield a set $M \subset \ell_\infty^c(\omega^+)$ with the same property. This shows that under (CH) the statement of Theorem 2 is optimal for general Banach spaces.

The purpose of the present note is to give ZFC examples of this phenomenon. In particular, we show that there exist renormings $\|\cdot\|_1$ and $\|\cdot\|_0$ of $X = \ell_1([0,1]) \oplus c_0$ such that under $\|\cdot\|_1$ the constant ε is preserved for convex hulls, while under $\|\cdot\|_0$

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passing to 2ε is unavoidable. We also give a renorming of ℓ_∞ and a set $M \subset \ell_\infty^{**}$ for which an increase to 3ε is necessary when passing to co$M$. The optimal value of the increase for a general Banach space X and $M \subset X^{**}$ remains open (it is between 3ε and 5ε).

Our examples will be subspaces and subsets of $(\ell_\infty, \|\cdot\|_\infty)$. Set $I = [0,1]$, λ denote the Lebesgue measure on I. For the standard Banach space facts and notation we refer to [FHHPMZ]. Facts on the Čech-Stone compactification can be found in [W].

To start with, fix for every $n \in I\!N$ a system of n^{2n} continuous functions $g_{[i_1,\dots,i_n]} : I \to I$, $i_j \in \{0,\dots,n^2-1\}$, such that:

$$g_{[i_1,\dots,i_n]}\big([\frac{i_j}{n^2}, \frac{i_j+1}{n^2}]\big) = 0 \tag{1}$$

$$g_{[i_1,\dots,i_n]}(t) = 1 \text{ whenever } \min_{j=1,\dots,n} |t - \frac{i_j}{n^2}| \geq \frac{2}{n^2} \tag{2}$$

It is easy to verify that

$$\lambda\big(g^{-1}_{[i_1,\dots,i_n]}(1)\big) \geq 1 - \frac{4}{n} \tag{3}$$

$$\forall t_1,\dots,t_k \in I \text{ distinct, } \forall A \subset \{1,\dots,k\}\ \exists n \in I\!N\ \exists g_{[i_1,\dots,i_n]} \text{ such that}$$

$$g_{[i_1,\dots,i_n]}(t_i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

To see (4), it suffices to take n such that $\frac{4}{n^2} < \min\limits_{i \neq j} |t_i - t_j|$ and apply conditions (1) and (2). For the rest of the proof reindex the collection $\{g_{[i_1,\dots,i_n]}\}_{n \in I\!N,\ i_j \in \{0,\dots,n^2-1\}}$ as $\{h_n\}_{n \in I\!N}$. Thus we have immediately from (3) and (4):

$$\forall \varepsilon > 0\ \exists n_0(\varepsilon) \text{ such that } n > n_0(\varepsilon) \text{ implies } \lambda\big(h_n^{-1}(1)\big) \geq 1 - \varepsilon \tag{5}$$

$$\forall t_1,\dots,t_k \in A \text{ distinct, } \forall A \subset \{1,\dots,k\}\ \exists \text{ infinitely many } n \text{ such that}$$

$$h_n(t_i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Next, put $H : I \to (\ell_\infty, \text{p.w.})$, $H(t) = (h_1(t), h_2(t), \dots)$. Clearly, $K = H(I)$ is compact in $(\ell_\infty, \text{p.w.})$. We define projections onto the first n-coordinates P_n :

$\ell_\infty \to \ell_\infty$ as $P_n\big((a_1, a_2, \ldots)\big) = (a_1, \ldots, a_n, 0, 0, \ldots)$. Set $K_0 = \bigcup_{n \in I\!N} P_n(K)$. It is easy to verify that $\overline{K_0}^{p.w.} = K_0 \cup K$. Indeed, suppose by contradiction that $y \in \overline{K_0}^{p.w.} \setminus (K_0 \cup K)$. So there exist $\delta > 0$, $m \in I\!N$ such that $\max\limits_{i \le m} |x_i - y_i| > \delta$ for every $x \in K$. Therefore $\max\limits_{i \le m} |x_i - y_i| > \delta$ for every $x \in P_n(K), n \ge m$. Consequently, $y \in \overline{P_1(K)_2(K) \cup \ldots_{n-1}(K)}^{p.w.} = S$. However, $P_i(K)$ are p.w. compact, so $S = \bigcup_{i<n} P_i(K)$ and $y \in K_0$. Let us now collect a few facts about the constructed sets.

Fact 3
Set $K \subset \ell_\infty$ is 2-isomorphic to the canonical basis $\{e_\gamma\}_{\gamma \in I}$ of $\ell_1(I)$.

Proof. Given any finite set of distinct $k_i = H(t_i)$ and $\alpha_i \in I\!R$, $1 \le i \le m$, put $A = \{i : \alpha_i \ge 0\}$, $B = \{1, \ldots, m\} \setminus A$.
Suppose WLOG that $\sum\limits_{i \in A} \alpha_i \ge -\sum\limits_{i \in B} \alpha_i$. By (6) there exists an n such that

$$\sum_{i \le m} |\alpha_i| \ge \| \sum_{i \le m} \alpha_i k_i \|_\infty \ge \sum_{i \in A} \alpha_i k_i(n) \ge \frac{1}{2} \sum_{i \le m} |\alpha_i|.$$

$\diamondsuit$

Since the vectors $e_n = (0, \ldots, 0, \underset{n}{1}, 0 \ldots)$ form a canonical basis of $c_0 \hookrightarrow \ell_\infty$, we have the following.

Fact 4
$Z = \overline{\text{span}}\big(K \cup \{e_n\}_{n \in I\!N}\big) \cong \ell_1(I) \oplus c_0$.

Proof. Let $z = \sum\limits_{i=1}^{m} \alpha_i k_i + \sum\limits_{i=1}^{l} \beta_i e_i$ where $k_i \in K, \alpha_i, \beta_i \in I\!R$. If $\max\limits_{i \le l} |\beta_i| > 2 \sum\limits_{i=1}^{m} |\alpha_i|$ then clearly $\|z\|_\infty \ge \frac{1}{2} \max\limits_{i \le l} |\beta_i|$. Otherwise by (6) there exists $n > l$ such that $|z(n)| > \frac{1}{2} \sum\limits_{i=1}^{m} |\alpha_i|$. Altogether we obtain

$$\sum_{i=1}^{m} |\alpha_i| + \max |\beta_i| \ge \|z\|_\infty \ge \frac{1}{2} \max\{\sum_{i=1}^{m} |\alpha_i|, \max_{i \le l} |\beta_i|\}.$$

$\diamondsuit$

The following easy lemma is the key to our results.

Lemma 5
$\forall \delta > 0$ $\exists n_0$ $\forall m > n_0$ $\exists y \in \text{co} K_0$ such that $\forall i$ satisfying $m \ge i \ge n_0$ we have $y(i) \ge 1 - \delta$.

Proof. Given any $\delta > 0$, it is enough to choose (using condition (5)) $n_0 > n_0(\frac{\delta}{2})$. Set $L = P_m(K) = P_m \circ H(I)$, $\mu = P_m \circ H(\lambda)$ be the image of the Lebesgue measure λ on I under the mapping $P_m \circ H$, mapped on L. We have for $n_0 \le i \le m$

$$\int\limits_L x(i) d\mu = \int\limits_I (P_m \circ H(t))(i) dt = \int\limits_I h_i(t) dt \ge 1 - \frac{\delta}{2}.$$

Consequently, the element $y = \int_L x d\mu \in \overline{\text{co}L} = \text{co}L$ (L lies in m-dimensional space) satisfies the requirements.

$\diamondsuit$

For $x \in \ell_\infty$ put $x^{-1}(0) = \{i \in I\!N : x(i) = 0\}$. From (6) we have immediately

Fact 6
The system $\{k^{-1}(0) : k \in K\} \subset \mathcal{P}(I\!N)$ generates a filter $\mathcal{T}$ on $I\!N$, extendible to a nonprincipal ultrafilter $\mathcal{F} \in \beta I\!N$.

Let us denote $\kappa : (\ell_\infty, \|\cdot\|_\infty) \to (C(\beta I\!N), \|\cdot\|_\infty)$ the canonical identification of ℓ_∞ with the space of continuous functions on the Čech-Stone compactification $\beta I\!N$ of $I\!N$ (we identify $\beta I\!N$ with the space of ultrafilters on $I\!N$). More precisely, $\kappa(f)(p) = \lim_p f(i)$ for $p \in \beta I\!N$. Since $\mathcal{F} \in \beta I\!N \setminus I\!N$ we can set $C_0(\beta I\!N) = \{f \in C(\beta I\!N) : f(\mathcal{F}) = 0\}$. For the rest of the proof, we will work with the following system of subspaces of $C(\beta I\!N)$

$$\mathcal{S} = \{X \hookrightarrow C(\beta I\!N) : \kappa(Z) \subset X \subset C_0(\beta I\!N)\}$$

Spaces from $\mathcal{S}$ are taken together with the supremum norm inherited from $C(\beta I\!N)$, and we denote by $\pi : X \to C(\beta I\!N)$ the canonical imbedding. To simplify notation, put $L_0 = \kappa(K_0), L = \kappa(K)$. Recall that by the Riesz representation theorem $C(\beta I\!N)^* = M(\beta I\!N) = \ell_1(I\!N) \oplus M(\beta I\!N \setminus I\!N)$. Denote by $B(\beta I\!N)$ the space of all Borel measurable functions on $\beta I\!N$. We will identify elements $f \in B(\beta I\!N)$ with elements from $C(\beta I\!N)^{**} = M(\beta I\!N)^*$ using the natural interpretation $\langle f, \mu \rangle = \int_{\beta I\!N} f d\mu$, for every $\mu \in M(\beta I\!N)$. This gives rise to natural isometric imbeddings

$$\big(C(\beta I\!N), \|\cdot\|_\infty\big) \hookrightarrow \big(B(\beta I\!N), \|\cdot\|_\infty\big) \hookrightarrow M(\beta I\!N)^*$$

Denote also $i : C(\beta I\!N) \to C(\beta I\!N)^{**}$ and $j : X \to X^{**}$ the canonical imbeddings. We will also need an isometric imbedding $\rho : (\ell_\infty, \|\cdot\|_\infty) \to (B(\beta I\!N), \|\cdot\|_\infty)$ defined as

$$\rho(f)(p) = \begin{cases} f(p) & \text{for } p \in I\!N \\ 0 & \text{for } p \in \beta I\!N \setminus I\!N. \end{cases}$$

Theorem 7
Let $X \in S$. Then $L_0 \subset X$, L_0 is $\frac{1}{2}$-WRK, and coL_0 is not α-WRK for any $\alpha < 1$.

Proof. Since $K_0 \subset Z$ and $\kappa(Z) \subset X$ we have $L_0 = \kappa(K_0) \subset X$. Recall that $\pi : X \to C(\beta I\!N)$ is an imbedding, and so we have commuting isometries:

$$i \circ \pi = \pi^{**} \circ j \tag{7}$$

$$\pi^{**}(\overline{L_0}^{w^*}) = \overline{\pi(L_0)}^{w^*} \tag{8}$$

As $x \in c_0(I\!N)$ for every $x \in K_0$, we have $\lim_p x(i) = 0$ for every $p \in \beta I\!N \setminus I\!N$ and so supp $\kappa(x) \subset I\!N$. Therefore for every $y \in \overline{\pi(L_0)}^{w^*}$ and $\mu \in M(\beta I\!N \setminus I\!N)$ we have $\langle y, \mu \rangle = 0$. Consequently, y can be represented by a function from $B(\beta I\!N)$, supp$(y) \subset I\!N$. Since the pointwise topology on ℓ_∞ and the duality topology (ℓ_∞, ℓ_1) coincides for bounded sets in ℓ_∞ we have that the set $\overline{\pi(L_0)}^{w^*}$ coincides with $\rho(K_0) \cup \rho(K) \subset B(\beta I\!N)$. Choose any $y \in \overline{\pi(L_0)}^{w^*}$, $y = \rho(x)$ where $x \in K_0 \cup K$. Clearly, $\kappa(\frac{1}{2}x) \in i \circ \pi(X)$. Now since both y and $\kappa(\frac{1}{2}x)$ correspond to functions from $B(\beta I\!N)$, we obtain

$$\|y - \kappa(\frac{1}{2}x)\|_\infty = \sup_{\beta I\!N} |y - \kappa(\frac{1}{2}x)| = \max\{\sup_{i \in I\!N} |\frac{1}{2}x(i)|, \sup_{p \in \beta I\!N \setminus I\!N} |\kappa(\frac{1}{2}x)(p)|\} \leq \frac{1}{2}$$

We have proved that for every $y \in \overline{\pi(L_0)}^{w^*}$ there exists $u \in i \circ \pi(X)$ such that $\|y - u\|_\infty \leq \frac{1}{2}$. Due to (7) and (8)

$$\pi^{**-1}(y - u) = y_1 - \pi^{**-1} \circ \pi^{**} \circ j(u_1) = y_1 - j(u_1)$$

and $\|y_1 - j(u_1)\| \leq \frac{1}{2}$. Since $y_1 \in \overline{L_0}^{w^*}$ can be arbitrary and $u_1 \in X$, we get that L_0 is $\frac{1}{2}$-WRK. To finish the proof, we rely on Lemma 5. It says, in particular, that there exists $z \in \ell_\infty$, $\lim_{i \to \infty} z(i) = 1$, $z \in \overline{\text{co}K_0}^{p.w.}$. By the same consideration as before, we obtain that $\rho(z) \in B(\beta I\!N)$ belongs to $\overline{\text{co}\, \pi(L_0)}^{w^*}$. However, for every $x \in X$ we have $i \circ \pi(x) \in C_0(\beta I\!N)$, and since $\lim_{\mathcal{F}} z = 1$ we obtain

$$\sup_{I\!N} |\rho(z) - i \circ \pi(x)| = 1 \ \text{ for every } \ x \in X.$$

Similarly to above, $\inf_{x \in X} \|\pi^{**-1} \circ \rho(z) - j(x)\| = 1$ and coL_0 is not α-WRK for any $\alpha < 1$.

$\diamondsuit$

Example 8
Letting $X = C_0(\beta I\!N) \cong \ell_\infty$ we obtain that there exists a space isomorphic to ℓ_∞ and a bounded $M \subset X$ such that M is $\frac{1}{2}$-WRK but co$M$ is not α-WRK for any $\alpha < 1$. It is not clear to us whether such a set M exists in $(\ell_\infty, \|\cdot\|_\infty)$.

Example 9
Let $X_1 = (\ell_1(I) \oplus_\infty c_0, \|\cdot\|_1)$ where $\|(f,g)\|_1 = \max(\|f\|_{\ell_1}, \|g\|_\infty)$. Then for every bounded $M \subset X$, if M is ε-WRK then co$M$ is also ε-WRK. Indeed, by [G] the same property holds true for both direct summands $(\ell_1(I), \|\cdot\|_{\ell_1})$ and $(c_0, \|\cdot\|_\infty)$, and will therefore be preserved under the direct sum $\oplus_\infty$. On the other hand, let $X_0 = \kappa(Z) \hookrightarrow C(\beta I\!N)$, together with the norm $\|\cdot\|_0$ inherited from the imbedding. By Fact 4, $\ell_1(I) \oplus c_0 \cong X_0 \in \mathcal{S}$. By Theorem 7 there exists $\frac{1}{2}$-WRK bounded set $M \subset X$ such that co$M$ is not α-WRK for any $\alpha < 1$. Moreover these two norms can be combined to get the following. Consider the space $X_\tau \cong \ell_1(I) \oplus c_0$ with the norm $\|\cdot\|_\tau = \tau\|\cdot\|_1 + (1-\tau)\|\cdot\|_0$ where $0 \leq \tau \leq 1$. For a fixed bounded set $M \subset (\ell_1(I) \oplus c_0, \|\cdot\|_\tau)$ the value $\min\{\varepsilon : M \text{ is } \varepsilon - \text{WRK}\}$ depends continuously on τ. Consequently, for any given $1 \leq \beta \leq 2$ there exists τ such that for every

bounded ε-WRK set $M \subset X_\tau$, co$M$ is $\beta\varepsilon$-WRK. At the same time, there exists a bounded set $M \subset X_\tau$ which is $\frac{1}{2}$-WRK and co$M$ is not $\frac{\gamma}{2}$-WRK for any $\gamma < \beta$.

Proposition 10
Let $X = C_0(\beta I\!N)$. There exists a set $M \subset X^{**}$ such that M is $\frac{1}{3}$-WRK (with respect to X), but co$M$ is not α-WRK for any $\alpha < 1$.

Proof. First fix a system of open neighbourhoods $\{O_\lambda\}_{\lambda\in\Lambda}$ of the point $\mathcal{F} \in \beta I\!N$, partially ordered by inclusion. By the complete regularity of $\beta I\!N$ there exists a corresponding system $\{f_\lambda\}_{\lambda\in\Lambda} \subset C_0(\beta I\!N)$ such that $f_\lambda(\beta I\!N \setminus O_\lambda) = 1$. Together with the partial ordering comming from Λ, the system $\{f_\lambda\}_{\lambda\in\Lambda}$ forms a net such that for every $\mu \in M^+(\beta I\!N \setminus \{\mathcal{F}\})$

$$\lim_\lambda \langle f_\lambda, \mu\rangle = \|\mu\|.$$

Thus $w^*\!-\!\lim_\lambda f_\lambda = \chi_{\beta I\!N\setminus\{\mathcal{F}\}}$. Thus $g = \chi_{\beta I\!N\setminus I\!N\cup\{\mathcal{F}\}} \in B(\beta I\!N)$ belongs to $C_0(\beta I\!N)^{**}$. As we know from before, $\rho(K) \subset B(\beta I\!N)$ corresponds to a w^*-compact set in $C_0(\beta I\!N)^{**}$. Set $A = \rho(K) + \frac{1}{3}g$. Clearly, $A \subset B(\beta I\!N)$ is w^*-compact. For any $y \in A$, $y = \rho(x) + \frac{1}{3}g$, where $x \in K$ we estimate $\|y - \kappa(\frac{2}{3}x)\| = \sup_{\beta I\!N} y - \kappa(\frac{2}{3}x) = \max\{\sup_{I\!N} |\frac{1}{3}x|, \sup_{\beta I\!N\setminus I\!N\cup\{\mathcal{F}\}} |\frac{1}{3}(1 - 2\kappa(x)|\} \le \frac{1}{3}$. Thus A is $\frac{1}{3}$-WRK for $C_0(\beta I\!N)$. On the other hand, as we know from above $\rho(z) + \frac{1}{3}g \in \overline{\mathrm{co}A}^{w^*}$, where $\lim_{i\to\infty} z(i) = 1$. Again, $\sup_{I\!N} |\rho(z) + \frac{1}{3}g - x| = 1$ for every $x \in C_0(\beta I\!N)$. The proof is finished.

$\diamondsuit$

Let us finally sketch a proof of the following.

Theorem 11

Let X be a Banach space, $\ell_\infty \hookrightarrow X$. Then there exists a renorming $\|\cdot\|$ of X under which there exists a bounded $\frac{1}{2}$-WRK set $M \subset X$, such that co$M$ is not α-WRK for any $\alpha < 1$. Similarly, there exists a bounded $\frac{1}{3}$-WRK $M \subset X^{**}$ such that co$M$ is not α-WRK for any $\alpha < 1$.

Proof. It is enough to recall that since ℓ_∞ is an injective space (i.e. complemented in every superspace) and $\ell_\infty \hookrightarrow X$, we have $X \cong \ell_\infty \oplus Y$. We may write $X = C_0(\beta I\!N) \oplus Y$ and choose $\|\cdot\| = \max\{\|\cdot\|_\infty, \|\cdot\|_Y\}$. It is standard to check that the sets $M \subset C_0(\beta I\!N)$ constructed in Example 8 and Proposition 10 work in this context as well.

$\diamondsuit$

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References

[AMN] S. Argyros, S. Mercourakis and S. Negrepontis, "Functional-analytic properties of Corson compact spaces", Studia Math. 89 (1988), 197–229.

[FHHMPZ] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant and V. Zizler, "Functional Analysis and Infinite Dimensional Geometry", CMS Books in Mathematics, Springer Verlag 2001.

[FHMZ] M. Fabian, P. Hájek, V. Montesinos and V. Zizler, "A quantitative version of Krein's theorem", to appear in Rev. Mat. Iberoamericana.

[FMZ] M. Fabian, V. Montesinos and V. Zizler, "A characterization of subspaces of weakly compactly generated Banach spaces", to appear.

[G] A. S. Granero, "An extension of the Krein-Šmulian theorem", preprint.

[W] R. C. Walker, "The Stone-Čech Compactification", Springer Verlag 1974.

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