A quantitative version of Krein's Theorem

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To the memory of Vlastimil Pták

Abstract

A quantitative version of Krein's Theorem on convex hulls of weak compact sets is proved. Some applications to weakly compactly generated Banach spaces are given.

1 Introduction

A natural question related to the classical Theorem of Krein (see, for example, [FHHPMZ, Thm. 3.58]) is the following: assume that a bounded subset M of a Banach space X satisfies $\overline{M}^{w^*} \subset X + \varepsilon B_{X^{**}}$ for some $\varepsilon \geq 0$. Does the same hold for its convex hull? (if $\varepsilon = 0$ the answer is "yes" and this is the statement of Krein's Theorem). To answer in the affirmative this apparently simple question turns to be quite elusive in general.

This question arose when studying the problem of characterizing subspaces of weakly compact generated Banach spaces by countable covers of its closed unit ball (see [FMZ] and Theorem 15 in section 3 below).

The following definition describes the central object in this note.

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Definition 1 *Let X be a Banach space and let M be a bounded subset of X. Given $\varepsilon \geq 0$, we say that M is $\varepsilon-$*weakly relatively compact *($\varepsilon-$WRK, for short) if $\overline{M}^{w^*} \subset X + \varepsilon B_{X^{**}}$.*

The case $\varepsilon = 0$ is the classical weakly relatively compactness.

Using techniques of double limits due to Grothendieck and Pták, we will prove that the answer to the former question for any $\varepsilon \geq 0$ is affirmative for Banach spaces X with ω^*-angelic dual closed unit ball [1] (in particular, separable Banach spaces –a result due to Rosenthal, see the acknowledgements at the end of this note –or, more generally, weakly compactly generated or even weakly Lindelöf determined Banach spaces). Moreover, if a relaxation to 2ε of the constant is allowed, it holds true for any Banach space. The following is the main result of this note.

Theorem 2 *Let $(X, \|\cdot\|)$ be a Banach space. Let $M \subset X$ be a bounded subset of X. Assume that M is $\varepsilon-$WRK for some $\varepsilon > 0$. Then $conv(M)$ is $2\varepsilon-$WRK. If (B_{X^*}, ω^*) is angelic, or if X^* does not contain a copy of ℓ^1, then $conv(M)$ is $\varepsilon-$WRK.*

The following, to our knowledge, is still open[2]:

Problem 3 *Let X be a Banach space. Let M be a $\varepsilon-$weakly relatively compact subset of X. Is $conv(M)$ $\varepsilon-$weakly relatively compact?*

Remark 4 The decisive case for Problem 3 seems to be the space ℓ^∞. The answer should be related to the so-called *boundary problem* (see [DGZ, Problem I.2]): *Let X be a Banach space and B a subset of S_{X^*} such that every $x \in X$ attains its norm at some point of B (B is called a* boundary *of X). Let A be a bounded subset of X that is compact for the topology of the pointwise convergence on B. Is A weakly compact?* (see [DGZ, Chap. I] and the references therein).

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[1] A topological space T is called *angelic* if every relatively countably compact set $A \subset T$ is relatively compact and if every point in $\overline{A}$ is the limit of a sequence in A.

[2] See the remark added in proof at the end of this paper.

2 Proofs

Given a Banach space X and an element $x^{**} \in X^{**}$, the following function on (B_{X^*}, w^*) is introduced in [DGZ, III.2, p.105]. $\hat{x}^{**} : B_{X^*} \to I\!R$ is the infimum of the real continuous functions on (B_{X^*}, w^*) which are greater than or equal to x^{**}. The following proposition gives two alternative descriptions of $\hat{x}^{**}$. The first one is a standard result in general topology. The second one is in [DGZ, III.2.3].

Proposition 5 *Let X be a Banach space. Then, given $x^{**} \in X^{**}$,*
(i)

$$\hat{x}^{**}(x_0^*) = \lim_{N \in \mathcal{N}(x_0^*)} \{\sup\langle x^{**}, N\rangle\}, \quad \forall x_0^* \in B_{X^*}, \tag{1}$$

where $\mathcal{N}(x_0^*)$ denotes the filter of neighborhoods of x_0^* in (B_{X^*}, w^*).
(ii)

$$\hat{x}^{**}(x_0^*) = \inf\{\langle x, x_0^*\rangle + \|x^{**} - x\|;\ x \in X\},\ \forall x_0^* \in B_{X^*}. \tag{2}$$

Remark 6 In particular, it follows from (ii) that if $d := dist(x^{**}, X)$ denotes the distance in the norm from x^{**} to X then $\hat{x}^{**}(0) = d$. From (i) we get then that for every $N \in \mathcal{N}(0)$, $d \leq \sup\langle x^{**}, N\rangle$, and for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathcal{N}(0)$ such that $\sup\langle x^{**}, N_\varepsilon\rangle < d + \varepsilon$.

The use of double limits in the study of compactness is implicit in the approach of Eberlein [Eb] and explicit in Grothendieck (see, for example, [Gr]). The following concept relaxes the usual double limit condition.

Definition 7 *Let M be a bounded set of a Banach space X, and let S be a bounded subset of X^*. We say that M $\varepsilon-$*interchanges limits *with S (and in this case we shall write $M\S\varepsilon\S S$) if for any two sequences (x_n) in M and (x_m^*) in S such that the following limits exist,*

$$\lim_n \lim_m \langle x_n, x_m^*\rangle, \quad \lim_m \lim_n \langle x_n, x_m^*\rangle,$$

then

$$|\lim_n \lim_m \langle x_n, x_m^*\rangle - \lim_m \lim_n \langle x_n, x_m^*\rangle| \leq \varepsilon.$$

Proposition 8 *Let M be a bounded set and $\varepsilon \geq 0$ some number. Then we have*
(i) If M is $\varepsilon-WRK$ then $M\S 2\varepsilon\S B_{X^*}$.
(ii) If $M\S\varepsilon\S B_{X^*}$ then M is $\varepsilon-WRK$.

Proof. (i) Let (x_n) and (x_m^*) be sequences in M and B_{X^*}, respectively, such that both limits

$$\lim_n \lim_m \langle x_n, x_m^* \rangle, \quad \lim_m \lim_n \langle x_n, x_m^* \rangle$$

exist. Let $x^{**} \in \overline{M}^{w^*}$ be a w^*−cluster point of (x_n). Then

$$\lim_n \langle x_n, x_m^* \rangle = \langle x^{**}, x_m^* \rangle, \ \forall m.$$

Fix $\delta > 0$. We can find $x \in X$ such that $\|x^{**} - x\| \leq \varepsilon + \delta$. Choose a subsequence of (x_m^*) (denoted again by (x_m^*)) such that $\lim_m \langle x, x_m^* \rangle$ exists. Let $x^* \in X^*$ be a w^*−cluster point of (x_m^*). We get

$$\lim_m \langle x_n, x_m^* \rangle = \langle x_n, x^* \rangle, \ \forall n,$$
$$\lim_n \lim_m \langle x_n, x_m^* \rangle = \lim_n \langle x_n, x^* \rangle = \langle x^{**}, x^* \rangle,$$

and then

$$|\lim_n \lim_m \langle x_n, x_m^* \rangle - \lim_m \lim_n \langle x_n, x_m^* \rangle| = |\lim_n \langle x_n, x^* \rangle - \lim_m \langle x^{**}, x_m^* \rangle| =$$
$$= |\langle x^{**}, x^* \rangle - \lim_m \langle x^{**}, x_m^* \rangle| = |\lim_m \langle x^{**}, x^* - x_m^* \rangle| \leq$$
$$\leq |\lim_m \langle x, x^* - x_m^* \rangle| + 2(\varepsilon + \delta) = 2(\varepsilon + \delta).$$

As $\delta > 0$ is arbitrary, we get the conclusion.

(ii) Assume now $M§\varepsilon§B_{X^*}$. Let $x^{**} \in \overline{M}^{w^*}$ and let $d := d(x^{**}, X)$. We shall define inductively two sequences, (x_n) in M and (x_m^*) in B_{X^*}. To begin with, choose any $x_1 \in M$. Define then $N(x_1; 1) := \{x^* \in B_{X^*};\ |\langle x_1, x^* \rangle| < 1\}$, a neighbourhood of 0 in (B_{X^*}, w^*). By Remark 6 we can find $x_1^* \in N(x_1; 1)$ such that

$$d - 1 \leq \langle x^{**}, x_1^* \rangle < d + 1.$$

Choose $x_2 \in M$ such that $|\langle x^{**} - x_2, x_1^* \rangle| < 1/2$. Define $N(x_1, x_2; 1/2) := \{x^* \in B_{X^*};\ |\langle x_i, x^* \rangle| < 1/2,\ i = 1, 2\}$, a neighborhood of 0 in (B_{X^*}, w^*). Again by Remark 6 we can find $x_2^* \in N(x_1, x_2; 1/2)$ such that $d - 1/2 \leq \langle x^{**}, x_2^* \rangle < d + 1/2$. Continue in this way. We get (x_n) and (x_m^*) such that

$$x_n \in M, \quad x_m^* \in B_{X^*}, \ \forall n, m,$$
$$|\langle x^{**} - x_n, x_m^* \rangle| < \frac{1}{n},\ m = 1, 2, \dots, n-1,$$
$$|\langle x_n, x_m^* \rangle| < \frac{1}{m},\ n = 1, 2, \dots, m,$$
$$d - \frac{1}{m} \leq \langle x^{**}, x_m^* \rangle < d + \frac{1}{m},\ m = 1, 2, \dots$$

Then

$$\begin{aligned}
\lim_n \langle x_n, x_m^* \rangle &= \langle x^{**}, x_m^* \rangle, \ \forall m, \\
\lim_m \lim_n \langle x_n, x_m^* \rangle &= \lim_m \langle x^{**}, x_m^* \rangle = d, \\
\lim_m \langle x_n, x_m^* \rangle &= 0, \ \forall n, \\
\lim_n \lim_m \langle x_n, x_m^* \rangle &= 0,
\end{aligned}$$

so

$$|\lim_m \lim_n \langle x_n, x_m^* \rangle - \lim_n \lim_m \langle x_n, x_m^* \rangle| = d \leq \varepsilon.$$

∎

Remark 9 The case $\varepsilon = 0$ gives Grothendieck's characterization of relatively weak compactness (see [Gr]).

Remark 10 In Proposition 8, (i) cannot be improved. There are examples where B_X is (obviously) 1-WRK although $B_X \S \varepsilon \S B_{X^*}$ is false for every $0 < \varepsilon < 2$. A simple instance is provided by $X := (\ell^1, \| \cdot \|_1)$: let $\mathcal{N}$ be a non-trivial ultrafilter on $I\!N$ and, for every $u \in \ell^\infty$, let $\langle x^{**}, u \rangle$ be the limit of u along the ultrafilter $\mathcal{N}$. By using (i) in Proposition 5 we get easily that $\langle x^{**}, x^* \rangle = 1$ for all $x^* \in B_{X^*}$, and this implies that, for every $0 < \delta < 1$, $S(x^{**}; \delta)$ is dense in (B_{X^*}, ω^*), where

$$S(x^{**}; \delta) := \{x^* \in B_{X^*};\ \langle x^{**}, x^* \rangle > 1 - \delta\}$$

(see Proposition 11 below). Choose $0 < \delta < (2 - \varepsilon)/2$ and an element $x^* \in S(-x^{**}; \delta)$. We can find then a sequence (x_m^*) (as (B_{X^*}, w^*) is metrizable) in $S(x^{**}; \delta)$ such that $x_m^* \to x^*$ in the w^*-topology. By a diagonal procedure we can choose a sequence (x_n) in B_X such that $x_n \to x^{**}$ on the set $\{x^*, x_m^*;\ m \in I\!N\}$. Then we have

$$\begin{aligned}
&|\lim_n \lim_m \langle x_n, x_m^* \rangle - \lim_m \lim_n \langle x_n, x_m^* \rangle| = \\
&\quad |\lim_n \langle x_n, x^* \rangle - \lim_m \langle x^{**}, x_m^* \rangle| = |\langle x^{**}, x^* \rangle - \lim_m \langle x^{**}, x_m^* \rangle| = \\
&\quad = |\lim_m \langle x^{**}, (x^* - x_m^*) \rangle| > 2 - 2\delta > \varepsilon.
\end{aligned}$$

and the assertion is proved.

The construction in the previous example can be carried over to every separable Banach space X which contains an isomorphic copy of ℓ^1. It follows that an equivalent norm can be found on X such that, in this norm, B_X is (obviously) 1-WRK although $B_X \S \varepsilon \S B_{X^*}$ is false for every $0 < \varepsilon < 2$. The argument depends on the notion of an octahedral norm. A norm $\| \cdot \|$ on X is said to be *octahedral* (see, for example, [DGZ, III.2]) if for every finite dimensional subspace F of X and every $\eta > 0$, there exists $y \in S_X$ such that for every $x \in F$, we have

$$\|x + y\| \geq (1 - \eta)(\|x\| + 1).$$

By [DGZ, Lemma III.2.2], if there exists $x^{**} \in X^{**} \setminus \{0\}$ such that $\|x^{**} + x\| = \|x^{**}\| + \|x\|$ for every $x \in X$, then $\| \cdot \|$ is octahedral. The converse implication is true if X is separable ([GK]). The following proposition characterizes such elements x^{**} in X^{**}.

Proposition 11 *Let X be a Banach space and let $x^{**} \in S_{X^{**}}$. The following assertions are equivalent:*
(i) $\|x^{**} + x\| = \|x^{**}\| + \|x\|$ for every $x \in X$.
(ii) $\hat{x}^{**}(x^*) = 1$, for every $x^* \in B_{X^*}$.
(iii) For every $0 < \delta < 1$, $S(x^{**}; \delta)$ is dense in (B_{X^*}, w^*), where

$$S(x^{**}; \delta) := \{x^* \in B_{X^*};\ \langle x^{**}, x^* \rangle > 1 - \delta\}.$$

Proof. The equivalence between (i) and (ii) is proved in [DGZ, III.2.4].

$(ii) \Rightarrow (iii)$. Let $x_0^* \in B_{X^*}$. Let $N_1(x_0^*)$ be a neighborhood of x_0^* in (B_{X^*}, w^*). By Proposition 5, given $0 < \delta < 1$ we can find $N_2(x_0^*) \subset N_1(x_0^*)$, a neighborhood of x_0^* in (B_{X^*}, w^*), such that $\sup \langle x^{**}, N_2(x_0^*) \rangle \geq 1$. Choose $x^* \in N_2(x_0^*)$ such that $\langle x^{**}, x^* \rangle > 1 - \delta$. Then $x^* \in S(x^{**}; \delta) \cap N_1(x_0^*)$. It follows that $S(x^{**}; \delta)$ is dense in (B_{X^*}, w^*).

$(iii) \Rightarrow (ii)$ follows from Proposition 5. ∎

Now, in any separable Banach space X containing an isomorphic copy of ℓ^1 there exists an octahedral equivalent norm $||| \cdot |||$, and according to [GK], there exists $x^{**} \in S_{X^{**}}$ such that $|||x^{**} + x||| = |||x^{**}||| + |||x|||$ for every $x \in X$. The rest of the argument follows from Proposition 11 as in the example.

The proof of the following theorem is a quantitative modification of the proof of Krein's Theorem due to Pták, in which he used his combinatorial lemma together with Grothendieck's double limit criterion (see, for example, [Pt], [Ko, §24.5] or [BHO]).

We need the following definitions.

$$C(I\!N) := \{\lambda : I\!N \to [0,1] : \ supp\ \lambda \text{ finite },\ \lambda(I\!N) = 1\},$$

where *supp* λ denotes the support of λ , i.e., the set $\{n \in I\!N : \ \lambda(n) \neq 0\}$, and $\lambda(B) := \sum_{n\in B} \lambda(n)$ for any $B \subset I\!N$. Let $\mathcal{G}$ be a family of finite subsets of $I\!N$. Given $B \subset I\!N$, let

$$C(B) := \{\lambda \in C(I\!N) : \ supp\ \lambda \subset B\}.$$

Given $\gamma > 0$, let $C(B, \mathcal{G}, \gamma) := \{\lambda \in C(B) : \ \lambda(G) < \gamma,\ \forall G \in \mathcal{G}\}$. Pták's Combinatorial Lemma reads

Lemma 12 (Pták[Pt]) *The two following conditions on $\mathcal{G}$ are equivalent:*

1. *There exists a strictly increasing sequence $A_1 \subset A_2 \subset \ldots$ of finite subsets of $I\!N$ and a sequence (G_n) in $\mathcal{G}$ with $A_n \subset G_n$ for all n.*
2. *There exists an infinite subset $B \subset I\!N$ and an $\gamma > 0$ such that*

$$C(B, \mathcal{G}, \gamma) = \emptyset.$$

Theorem 13 *Let $(X, \|\cdot\|)$ be a Banach space. Let $M \subset X$ be a bounded subset of X. Assume that $M\S\varepsilon\S B_{X^*}$ for some $\varepsilon \geq 0$. Then $conv(M)\S\varepsilon\S B_{X^*}$.*

Proof. Assume $\|x\| \leq \mu$ for all $x \in M$ and some $\mu > 0$. Choose $\varepsilon > 0$ and $0 < \beta < \varepsilon$. Select now $\delta > 0$ and $\gamma > 0$ such that $\beta + 2\gamma\mu < \varepsilon - \delta$. Suppose that there exists a sequence (x_n) in $conv(M)$ and a sequence (x^*_m) in B_{X^*} such that

$$|\lim_n \lim_m \langle x_n, x^*_m\rangle - \lim_m \lim_n \langle x_n, x^*_m\rangle| = \varepsilon > 0.$$

Let $x^*_0 \in B_{X^*}$ be a cluster point of (x^*_m) in (B_{X^*}, w^*). Let $T \subset M$ be a countable set such that $\{x_n : \ n \in I\!N\} \subset conv(T)$ and choose a subsequence (denoted again by (x^*_m)) such that $x^*_m \to x^*_0$ on the set T. Then, for some $\sigma \in \{-1, 1\}$,

$$\sigma(\lim_n \langle x_n, x^*_0\rangle - \lim_m \lim_n \langle x_n, x^*_m\rangle) = \varepsilon.$$

By suppressing a finite number of indices, we may assume

$$\sigma(\lim_n \langle x_n, x^*_0\rangle - \lim_n \langle x_n, x^*_m\rangle) = \sigma \lim_n \langle x_n, x^*_0 - x^*_m\rangle > \varepsilon - \delta,\ \forall m.$$

Define

$$\Gamma(t) := \{m \in I\!N : \ |\langle t, x_0^* - x_m^* \rangle| \geq \beta\}, \ t \in T.$$

Those are finite subsets of $I\!N$. Let

$$\mathcal{G} := \{\Gamma(t) : \ t \in T\}.$$

Assume $C(I\!N, \mathcal{G}, \gamma) \neq \emptyset$ and choose $\lambda \in C(I\!N, \mathcal{G}, \gamma)$. It follows that

$$\lambda(\Gamma(t)) < \gamma, \ \forall t \in T.$$

Form

$$x^* := \sum_{k \in I\!N} \lambda(k)(x_0^* - x_k^*) \in 2B_{X^*}.$$

Given $t \in T$,

$$|\langle t, x^* \rangle| = \left| \sum_{k \in I\!N} \lambda(k) \langle t, x_0^* - x_k^* \rangle \right| \leq$$
$$\leq \textstyle\sum_{\Gamma(t)} \lambda(k) |\langle t, x_0^* - x_k^* \rangle| + \sum_{I\!N \setminus \Gamma(t)} \lambda(k) |\langle t, x_0^* - x_k^* \rangle| < 2\gamma\mu + \beta.$$

It follows that $|\langle x_n, x^* \rangle| \leq 2\gamma\mu + \beta, \ \forall n$. Then

$$2\gamma\mu + \beta \geq \lim_n |\langle x_n, x^* \rangle| =$$
$$= |\textstyle\sum_{k \in I\!N} \lambda(k) \lim_n \langle x_n, x_0^* - x_k^* \rangle| = \sigma \sum_{k \in I\!N} \lambda(k) \lim_n \langle x_n, x_0^* - x_k^* \rangle > \varepsilon - \delta,$$

a contradiction.

Assume then $C(I\!N, \mathcal{G}, \gamma) = \emptyset$. Then, by Lemma 12 we can find $A_p := \{m_1, m_2, \ldots, m_p\} \subset I\!N$ and $t_p \in T$ such that

$$A_p \subset \Gamma(t_p), \ \forall p \in I\!N,$$

i.e., $|\langle t_p, x_0^* - x_{m_k}^* \rangle| \geq \beta, \ k = 1, 2, \ldots, p$. Choose a subsequence of (t_n) (denoted again by (t_n)) such that there exists $\lim_n \langle t_n, x_0^* - x_{m_k}^* \rangle$, for any k. Then we get

$$\lim_n \lim_k \langle t_n, x_{m_k}^* \rangle = \lim_n \langle t_n, x_0^* \rangle,$$
$$|\lim_n \langle t_n, x_0^* \rangle - \lim_k \lim_n \langle t_n, x_{m_k}^* \rangle| = \lim_k \lim_n |\langle t_n, x_0^* - x_{m_k}^* \rangle| \geq \beta,$$

so

$$|\lim_n \lim_k \langle t_n, x_{m_k}^* \rangle - \lim_k \lim_n \langle t_n, x_{m_k}^* \rangle| \geq \beta. \tag{3}$$

As β satisfies $0 < \beta < \varepsilon$ and it is otherwise arbitrary, we get the conclusion. ∎

Proof of Theorem 2. The general case follows from Proposition 8 and Theorem 13. In order to prove the case when (B_{X^*}, ω^*) is angelic, the following modification of Proposition 8 is needed, together with the fact that, according to the proof of Theorem 13, if some sequence in the convex hull of a set "fails" the double limit condition against a sequence $(x_m^*)_{m\in I\!N}$ in the dual, the same is true for the set and a certain subsequence of $(x_m^*)_{m\in I\!N}$:

Proposition 14 *Let M be a bounded set and let $\varepsilon > 0$. Then*
(i) If M is $\varepsilon-WRK$ then $M\S\varepsilon\S(x_n^*)$, where (x_n^*) is any w^*-null sequence in B_{X^*}.
(ii) If (B_{X^*}, ω^*) is angelic and $M\S\varepsilon\S(x_n^*)$ for any w^*-null sequence in B_{X^*} then M is $\varepsilon-WRK$.

Proof. (i) follows directly from the proof of (i) in Proposition 8. In order to prove (ii), let $x_0^{**} \in \overline{M}^{(X^{**},\omega^*)}$, and let $d := dist(x_0^{**}, X)$. There exists $x_0^{***} \in S_{X^{***}}$ such that

$$\langle x_0^{**}, x_0^{***}\rangle = d, \quad \langle x, x_0^{***}\rangle = 0, \text{ for all } x \in X.$$

Fix $\delta > 0$. Let

$$C := B_{X^*} \cap \{x^{***} \in X^{***};\ \langle x_0^{**}, x^{***}\rangle \in (d-\delta, d+\delta)\}.$$

Then $x_0^{***} \in \overline{C}^{(X^{***},\omega^*)}$ and then $0 \in \overline{C}^{(X^*,\omega^*)}$. As (B_{X^*}, ω^*) is angelic, there exists a sequence $(x_m^*)_{m\in I\!N}$ in C which converges to 0 in (X^*, ω^*). By passing to a subsequence (denoted by the same symbol) we may and do assume that $(\langle x_0^{**}, x_m^*\rangle)_{m\in I\!N}$ converges.

Let $(x_n)_{n\in I\!N}$ be a sequence in M which converges to x_0^{**} in X^{**} on the set $\{x_m^*;\ m \in I\!N\}$. Then

$$\lim_n \lim_m \langle x_n, x_m^*\rangle = 0, \quad \text{and}$$
$$\lim_m \lim_n \langle x_n, x_m^*\rangle = \lim_m \langle x_0^{**}, x_m^*\rangle \in [d-\delta, d+\delta].$$

As $\delta > 0$ was arbitrary, we are done. ∎

To finish the proof of Theorem 2 it remains to show the case when X^* does not contains a copy of ℓ^1: let $C := \overline{conv(M)}^{w^*}$, a w^*-compact convex subset of X^{**}. It is well known (see, for example, [Di, p.215]) that

$$C = \overline{conv(Ext\ C)}^{\|\cdot\|},$$

where $Ext\ C$ denotes the set of extreme points of C. By Milman's Theorem (see, for example, [Ko, §25.1.7]), $Ext\ C \subset \overline{M}^{w^*}$. As $\{x^{**} \in X^{**} :\ d(x^{**}, X) \leq \varepsilon\}$ is $\|\cdot\|-$closed, where d denotes the distance in the norm, this proves that $\overline{conv(M)}$ is $\varepsilon-$WRK. ∎

3 Applications

In [FMZ] a characterization of subspaces of weakly compact generated Banach spaces was provided in terms of countable coverings of the closed unit ball by absolutely convex subsets $(M_{n,p})_{n,p\in \mathbb{N}}$. Using Theorem 2, the convexity requirement on those sets can be removed, and the following is true:

Theorem 15 ([FMZ]) *A Banach space X is a subspace of a weakly compact generated Banach space if and only if it admits a family $\{M_{n,p};\ n,p \in \mathbb{N}\}$ of subsets of B_X such that $\bigcup_{n=1}^{\infty} M_{n,p} = B_X$ for every $p \in \mathbb{N}$, and $\overline{M_{n,p}}^{w^*} \subset X + \frac{1}{p}B_{X^{**}}$ for every $n, p \in \mathbb{N}$.*

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Added in Proof: The problem mentioned in 3 has been solved recently in the negative by A. Suárez-Granero in a forthcoming paper. Assuming CH, he provides an example of a space where the best constant in the quantitative Krein's Theorem is 2. Thus the positive result in our paper is in fact optimal.

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