BIORTHOGONAL SYSTEMS IN WEAKLY LINDELÖF SPACES

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Abstract

We study countable splitting of Markushevich bases in weakly Lindelöf Banach spaces in connection with the geometry of these spaces.

1 Introduction

A *Markushevich basis* (in short, an *M-basis*) of a Banach space X is a biorthogonal system $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ in $X \times X^*$ such that $\overline{\operatorname{span}}\,\Gamma = X$ and $\{\gamma^*\}_{\gamma\in\Gamma}$ separates points of X.
We say, typically, that an M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is *weakly compact*, resp. *weakly Lindelöf*, if $\Gamma \cup \{0\}$ is a compact set, resp. Lindelöf space, in its relativized weak topology from X.
A Banach space X is *weakly compactly generated* (in short, *WCG*) if there is a weakly compact set $K \subset X$ such that $X = \overline{\operatorname{span}}\,K$.

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A Banach space X is a *Vašák* space (or *weakly countably determined space*) if there is a sequence $\{B_n\}$ of weak* compact sets in X^{**} such that given $x \in X$ and $u \in X^{**} \setminus X$, there is n_0 such that $x \in B_{n_0}$ and $u \notin B_{n_0}$.
A compact space K is an *Eberlein compact* if K is homeomorphic to a weakly compact set (endowed with its weak topology) in a Banach space.
A compact space K is a *Gul'ko compact* if $C(K)$ is a Vašák space.
A compact space K is a *Corson compact* if, for some Γ, K is homeomorphic to a set $S \subset [-1, 1]^\Gamma$ in the pointwise topology and such that $\{\gamma;\ x(\gamma) \neq 0\}$ is countable for all $x \in S$.
If a Banach space X admits a Markushevich basis, then X is *weakly Lindelöf* (i.e. X in its weak topology is a Lindelöf space) if and only if B_{X^*} in its weak star topology is a Corson compact [F∼, Thm12.48].
Let X be a Banach space and let μ be the first ordinal of cardinality dens(X), the *density of* X, the smallest cardinality of a dense subset of X. A *projectional resolution of the identity* (in short, a *PRI*) on X is a long sequence of norm one projections $(P_\alpha)_{\omega_0 \leq \alpha \leq \mu}$ in X such that $P_{\omega_0} = 0$, $P_\mu = \text{Identity}_X$, $P_\alpha P_\beta = P_{\min(\alpha,\beta)}$, $\text{dens}(P_\alpha X) \leq \#\alpha$, for all α and the map $\alpha \mapsto P_\alpha x$ is continuous from the order topology on ordinals into the norm topology of X, for every $x \in X$.

2 The results

The purpose of this paper is to study countable coverings of Markushevich bases in several subclasses of weakly Lindelöf Banach spaces. We will show that such covering enjoying some extra property actually characterizes these subclasses and that *every* Markushevich basis in a particular subclass shares this property. We will give proofs to these results that use Lindenstrauss' technique of projectional resolutions. Some results in this paper can alternatively be shown by using the results of Farmaki [Fa].
For more information in this area we refer to e.g. [DGZ], [F], [F∼] and [Z].

Definition 1 *(a) An M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ of a Banach space X is* $\sigma-$shrinking *if $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$ so that for every neighborhood U of the origin in $X^{**}[\|\cdot\|]$ and for every $\gamma \in \Gamma$ there is $n \in I\!N$ such that $\gamma \in \Gamma_n$ and $\Gamma_n' \subset U$, where Γ_n' is the set of all accumulation points of the set Γ_n in $X^{**}[\omega^*]$.*
(b) An M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ of a Banach space X is weakly $\sigma-$shrinking *if $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$ so that for every neighborhood U of the origin in $X^{**}[\omega^*]$ and for every $\gamma \in \Gamma$, there is $n \in I\!N$ such that $\gamma \in \Gamma_n$ and $\Gamma_n' \subset U$.*

The main results in this paper are the following three theorems.

Theorem 2 *Let X be a Banach space. Then, the following are equivalent.*
(i) X is a subspace of a WCG Banach space.
(ii) X admits a $\sigma-$shrinking M-basis.
(iii) $B_{X^*}[w^*]$ is an Eberlein compact.
Moreover, if this is the case, then every M-basis of X is $\sigma-$shrinking.

Theorem 3 *Let X be a Banach space. Then the following are equivalent.*
(i) X is a Vašák space.
(ii) X admits a weakly $\sigma-$shrinking M-basis.
(iii) $B_{X^*}[w^*]$ is a Gul'ko compact.
Moreover, if this is the case, then every M-basis in X is weakly $\sigma-$shrinking.

Theorem 4 *Let K be a compact space. Then the following (i) and (ii) are equivalent.*
(i) K is a Corson compact.
(ii) $C(K)$ admits a pointwise Lindelöf M-basis.
If K is a Corson compact, then every M-basis $\{\gamma,\gamma^*\}$ of $C(K)$ such that $\{f_\alpha\}\subset\overline{span}^{\|\cdot\|}K$ is pointwise Lindelöf.

Remarks.
An equivalent definition of a $\sigma-$shrinking M-basis $\{\gamma,\gamma^*\}_{\gamma\in\Gamma}$ is the following: *for every $\epsilon>0$, $\Gamma=\cup_{n=1}^{\infty}\Gamma_n^{\epsilon}$ so that*

$$(\Gamma_n^{\epsilon})'\subset\epsilon B_{X^{**}}\quad \textit{for each } n\in I\!N.$$

Examples of Banach spaces that are Vašák spaces but not subspaces of WCG spaces and examples of non-Vašák spaces whose dual balls are Corson compacts are discussed , e.g., in [F].
Note that a Banach space X is an Asplund WCG space if and only if X admits a shrinking M-basis $\{\gamma,\gamma^*\}_{\gamma\in\Gamma}$, i.e. an M-basis $\{\gamma,\gamma^*\}_{\gamma\in\Gamma}$ such that $\overline{\text{span}}^{\|\cdot\|}\{\gamma^*;\ \gamma\in\Gamma\}=X^*$ ([F, Thm. 8.3.3]).
Note that, if $\Gamma\subset X$ is bounded, an M-basis $\{\gamma,\gamma^*\}_{\gamma\in\Gamma}$ in a Banach space X is weakly compact if and only if $X^*=\overline{\text{span}}^{\,\tau}\{\gamma^*;\ \gamma\in\Gamma\}$, where τ is the topology of the uniform convergence on the set Γ. Indeed, if the M-basis is weakly compact, we can use the Mackey-Arens theorem to show the statement. On the other hand, if the condition holds and $(\gamma_n)_{n=1}^{\infty}$ is a sequence

of distinct points in Γ, given $f \in X^*$ and $\epsilon > 0$ find $g \in \text{span}\{\gamma^*;\ \gamma \in \Gamma\}$ such that $\sup |\langle \Gamma, f - g\rangle| < \epsilon$. Now, there exists $n_0 \in I\!N$ such that, for every $n \geq n_0$, $\langle \gamma_n, g\rangle = 0$, due to the orthogonality of the system, so $\limsup_{n\to\infty} |\langle \gamma_n, f\rangle| \leq \epsilon$ and this implies that $\gamma_n \overset{\omega}{\to} 0$. Thus the M-basis is weakly compact by Eberlein's theorem. Note that *an M-basis* $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ *is* $\sigma-$weakly compact *(i.e.* $\Gamma \cup \{0\}$ *is a weakly* $\sigma-$*compact set in* X*) if and only if for every* $\gamma \in \Gamma$ *there exists* $\delta_\gamma > 0$ *such that* $\{\delta_\gamma \gamma\}_{\gamma\in\Gamma} \cup \{0\}$ *is weakly compact*. This then means that X is WCG and every WCG space admits a weakly compact M-basis (see e.g. [F∼, Thm. 11.12]).

It is not true that every M-basis in a WCG Banach space is necessarily $\sigma-$weakly compact. Indeed, assuming the Continuum Hypothesis, let X be a WCG space of density character $\aleph_1$ and Y be a non-WCG subspace of X ([R]). Then let $\{\gamma, \gamma^*\}_{\gamma\in\Gamma_1}$ be an M-basis of Y extended to an M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ of X ([JZ, Proposition 4]). By *extending* an M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma_1}$ to $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ we mean that new elements γ are added, γ^* are extended to X for $\gamma \in \Gamma_1$, and new elements γ^* are added. If $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ were $\sigma-$weakly compact so it would be $\{\gamma, \gamma^*\}_{\gamma\in\Gamma_1}$, which is a contradiction as Y is not WCG.

This fact is in contrast with the result of Johnson (see [R] or [D]) who showed that if X is a WCG Banach space and $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is an unconditional basis of X, then $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is necessarily $\sigma-$weakly compact.

By the Hahn-Banach Theorem, an M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is $\sigma-$shrinking if and only if for every $\epsilon > 0$, $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n^\epsilon$ so that $\#\{\gamma \in \Gamma_n^\epsilon;\ |\langle \gamma, x^*\rangle| \geq \epsilon\} < \aleph_0$, for all $x^* \in B_{X^*}$.

An M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is weakly $\sigma-$shrinking if and only if $\Gamma = \bigcup_{n\in I\!N} \Gamma_n$ so that for each $\varepsilon > 0$, for each $\gamma_0 \in \Gamma$ and for each $x^* \in B_{X^*}$ there is $n \in I\!N$ so that $\gamma_0 \in \Gamma_n$ and $\{\gamma \in \Gamma_n; |\langle \gamma, x^*\rangle| \geq \varepsilon\}$ is finite. Note that if $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is a weakly $\sigma-$shrinking M-basis of X, then $\{\gamma \in \Gamma; \langle \gamma, x^*\rangle \neq 0\}$ is countable for all $x^* \in X^*$. Indeed, observe that if $x^* \in X^*$ and $\varepsilon > 0$ are given, from the preceding remark we get that Γ is covered by such Γ_n that $\{\gamma \in \Gamma_n; |\langle \gamma, x^*\rangle| \geq \varepsilon\}$ is finite. Thus $\{\gamma \in \Gamma; |\langle \gamma, x^*\rangle| \geq \varepsilon\}$ is countable. It follows then from Theorem 3 that B_{X^*} in its weak* topology is a Corson compact if X is a Vašák space.

If B_{X^*} in its weak* topology is a Corson compact, then X admits an M-basis and every M-basis in X is weakly Lindelöf. On the other hand, if X admits a weakly Lindelöf M-basis, then B_{X^*} endowed with the weak* topology is a Corson compact [O], [VWZ], see e.g. [F∼, Ch 12].

Recall that an M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is *norming* if there exists $\lambda > 0$ such that for every $x \in S_X$,

$$\sup\langle x, \overline{\mathrm{span}}\,\{\gamma^*;\ \gamma \in \Gamma\} \cap B_{X^*}\rangle \geq \lambda.$$

A norming M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ of a Banach space X is $\sigma-$shrinking if and only if given $\epsilon > 0$, $\Gamma = \bigcup_{n\in I\!N} \Gamma_n^\epsilon$ so that for each $n \in I\!N$, $(\Gamma_n^\epsilon)' \subset X + \epsilon B_{X^{**}}$.
In order to prove one implication, let $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ be a norming M-basis (for some $\lambda > 0$) and $\Gamma_0 \subset \Gamma$ a set such that $\Gamma_0' \subset X + \epsilon B_{X^{**}}$. We shall prove that $\Gamma_0' \subset \epsilon(1 + 1/\lambda)B_{X^*}$.
This can be shown as follows:
Let $x^{**} \in \Gamma_0'$. Then $x^{**} = x + u^{**}$, where $x \in X$ and $u^{**} \in \epsilon B_{X^{**}}$. Choose $x^* \in \mathrm{span}\{\gamma^*;\ \gamma \in \Gamma\} \cap B_{X^*}$. Then

$$0 = \langle x^{**}, x^*\rangle = \langle x, x^*\rangle + \langle u^{**}, x^*\rangle,$$

so $|\langle x, x^*\rangle| < \epsilon$. As the basis is norming, we get $\|x\| < \epsilon/\lambda$, so $\|x^{**}\| < \epsilon(1 + 1/\lambda)$. The reverse implication is obvious.
This can be compared with [FMZ3], where a similar covering was required for B_X in order to characterize that X is a subspace of a WCG Banach space.

3 Proofs

We will now prove the main results in this paper.

Definition 5 *Given an M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ of a Banach space X and a PRI $(P_\alpha)_{\omega_0\leq\alpha\leq\mu}$ on X, we say that they are* subordinated (to each other) *whenever $P_\alpha(\gamma) = \gamma$ or 0 for every $\omega_0 \leq \alpha \leq \mu$ and $\gamma \in \Gamma$.*

The following result, a consequence of [JZ, Lemma 6], will be frequently used.

Lemma 6 *Let Z be a WCG Banach space generated by a weakly compact absolutely convex set K and X be a subspace of Z. Then any M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma_1}$ of X can be extended to an M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ of Z and a PRI $(P_\alpha)_{\omega_0\leq\alpha\leq\mu}$ can be constructed on Z such that it is subordinated to $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ and $P_\alpha(K) \subset K$ for all $\omega_0 \leq \alpha \leq \mu$ and all $\gamma \in \Gamma$. In particular, $P_\alpha X \subset X$ for all $\omega_0 \leq \alpha \leq \mu$.*

Remark. Part of the preceding statement (the construction of a subordinated PRI) can be proved in a more general context using the concept of a projectional generator (see, e.g., [F, Def. 6.1.6]). A Banach space Z is called *weakly Lindelöf determined* if $B_{Z^*}[\omega^*]$ is Corson. As it is well known, Z has, in this case, an M-basis, and every M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ of Z satisfies that

$$\Phi(z^*) := \{\gamma \in \Gamma : \ \langle \gamma, z^* \rangle \neq 0\}$$

is countable for every $z^* \in Z^*$ (see, e.g., [F∼, Prop. 12.51]). It is obvious that the couple (Z^*, Φ) is a projectional generator, so Z has a PRI $(P_\alpha)_{\omega_0\leq\alpha\leq\mu}$ (see, e.g., [F, Prop. 6.1.7]). This fact depends on the construction of two long sequences $(A_\alpha)_{\omega_0\leq\alpha\leq\mu}$ and $(B_\alpha)_{\omega_0\leq\alpha\leq\mu}$ of subsets $A_\alpha \subset Z$, $B_\alpha \subset Z^*$ where $\Phi(B_\alpha) \subset A_\alpha$ for all α. Then $P_\alpha(Z) = \overline{A_\alpha}$ and $P_\alpha^{-1}(0) = (B_\alpha)_\perp$ for all α (see, [F, Prop. 6.1.4] and the proof of [F, Prop. 6.1.7]). Assume $\gamma \notin P_\alpha(Z)$ for some $\gamma \in \Gamma$ and $\omega_0 \leq \alpha \leq \mu$. Then $\gamma \notin \Phi(B_\alpha)$, so $\langle \gamma, z^* \rangle = 0$ for all $z^* \in B_\alpha$. We have then $\gamma \in (B_\alpha)_\perp = P_\alpha^{-1}(0)$. It follows that, for any $\gamma \in \Gamma$ and $\omega_0 \leq \alpha \leq \mu$, $P_\alpha(\gamma) = \gamma$ or 0, and so $(P_\alpha)_{\omega_0\leq\alpha\leq\mu}$ is subordinated to $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$.

Definition 7 *We will say that a PRI $(P_\alpha)_{\omega_0\leq\alpha\leq\mu}$ on a Banach space X is* $\sigma-$shrinking *if there is a countable collection $\{B_n\}_{n=1}^\infty$ of subsets of B_X such that for every $x_0 \in B_X$ and for every $\epsilon > 0$, there is $n_0 \in I\!N$ such that $x_0 \in B_{n_0}$ and $\limsup_{\alpha\uparrow\beta} \ \sup |\langle B_{n_0}, (P_\alpha^* - P_\beta^*)f\rangle| \leq \epsilon$, for all $f \in B_{X^*}$ and all limit ordinals $\beta \in (\omega_0, \mu]$.*

Proposition 8 *Let X be a Banach space with an M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ and a subordinated PRI $(P_\alpha)_{\omega_0\leq\alpha\leq\mu}$. Then $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is $\sigma-$shrinking if and only if $(P_\alpha)_{\omega_0\leq\alpha\leq\mu}$ is $\sigma-$shrinking.*

Proof: Assume first that $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is $\sigma-$shrinking. We may and do assume that $\Gamma \subset B_X$. Let $(\Gamma_n)_{n=1}^\infty$ be the covering of Γ given by the definition of $\sigma-$shrinking. Given $\varepsilon > 0$ let $n \in I\!N$ be such that $\Gamma_n' \subset \varepsilon B_{X^{**}}$. Suppose that, for some limit ordinal $\omega_0 < \beta \leq \mu$ and some $x^* \in B_{X^*}$,

$$\limsup_{\alpha\uparrow\beta} \ \sup |\langle \Gamma_n, (P_\beta^* - P_\alpha^*)(x^*)\rangle| > \varepsilon.$$

Then we can find an increasing net $(\alpha_i)_{i\in I}$ in $[\omega_0, \beta)$ such that $\alpha_i \to \beta$ and elements $\gamma_i \in \Gamma_n$ such that

$$|\langle \gamma_i, (P_\beta^* - P_{\alpha_i}^*)x^*\rangle| = |\langle (P_\beta - P_{\alpha_i})\gamma_i, x^*\rangle| > \varepsilon, \text{ for all } i \in I.$$

If $P_\beta\gamma_i = 0$ then $P_\alpha\gamma_i = 0$ for all $\alpha \le \beta$, so $P_\beta\gamma_i = \gamma_i$ and $P_{\alpha_i}\gamma_i = 0$ for all $i \in I$. It follows that $|\langle\gamma_i, x^*\rangle| > \varepsilon$ for all $i \in I$. Let γ^{**} be an accumulation point of $\{\gamma_i :\ i \in I\}$. Then $|\langle\gamma^{**}, x^*\rangle| \ge \varepsilon$, a contradiction. It follows that

$$\limsup_{\alpha\uparrow\beta}\ \sup|\langle\Gamma_n, (P^*_\beta - P^*_\alpha)(x^*)\rangle| \le \varepsilon \ \text{ for all } x^* \in B_{X^*}.$$

Now, a simple argument involving sets of the form

$$[a_1\Gamma_1 + a_2\Gamma_2 + \ldots + a_m\Gamma_m + \varepsilon B_X] \cap B_X,$$

where $\sum_{j=1}^m |a_j| \le K$, $\varepsilon > 0$, $m \in I\!N$, $K > 0$, proves that $(P_\alpha)_{\omega_0\le\alpha\le\mu}$ is $\sigma-$shrinking.
Assume now that $(P_\alpha)_{\omega_0\le\alpha\le\mu}$ is a $\sigma-$shrinking long sequence of projections on X which satisfy all properties of a PRI but not necessarily the requirement that μ be the first ordinal of cardinality dens$X$ (let's call it, for now on, a PRI$'$ on X), and let $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ be a subordinated M-basis. We shall prove that it is $\sigma-$shrinking. This will be done by transfinite induction on the density of X. If X is separable, then every M-basis on X is countable and the result is obvious. Assume that the result has been proved for every Banach space of density less than $\aleph$, a certain uncountable cardinal, having a $\sigma-$shrinking PRI$'$. Let X be a Banach space of density $\aleph$ with a $\sigma-$shrinking PRI$'$ and let $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ be a subordinated M-basis on X. We may and do assume that $\Gamma \subset B_X$.
Given $\gamma \in \Gamma$, let $b(\gamma)$ be the first ordinal in $(\omega_0, \mu]$ such that $P_{b(\gamma)}(\gamma) = \gamma$. Then $b(\gamma)$ has a predecessor $a(\gamma)$; it follows that, for all $\gamma \in \Gamma$, $\gamma \in [P_{a(\gamma)+1} - P_{a(\gamma)}](X)$. Define a well-order in each of the sets $\{\gamma \in \Gamma;\ a(\gamma) = \alpha\}$, $\alpha \in [\omega_0, \mu]$. This induces a lexicographic well-order $\prec$ in Γ and the mapping $a : \Gamma \to [\omega_0, \mu]$ is obviously increasing. Given $\epsilon > 0$ we can write $B_X = \bigcup_{n\in I\!N} B^\epsilon_n$ and

$$\limsup_{\alpha\uparrow\beta}\ \sup|\langle B^\epsilon_n, (P^*_\beta - P^*_\alpha)x^*\rangle| \le \epsilon,$$

for all limit ordinal $\beta \in (\omega_0, \mu]$ and $x^* \in B_{X^*}$.
Define $\Gamma^\epsilon_n := \Gamma \cap B^\epsilon_n$, $n \in I\!N$. It follows that $\Gamma = \bigcup_{n\in I\!N} \Gamma^\epsilon_n$. Let $x^{**} \in (\Gamma^\epsilon_n)'$. Let $\mathcal{W}$ be the family of neighborhoods of x^{**} in $X^{**}[\omega^*]$ partially ordered by inclusion. Given $W \in \mathcal{W}$ let $g(W)$ be the first element (in the order $\prec$) in $\Gamma^\epsilon_n \cap W$. The net $\{g(W);\ W \in \mathcal{W}\}$ is w^*-convergent to x^{**} and the mapping $g : \mathcal{W} \to \Gamma^\epsilon_n$ is increasing. It follows that the mapping $a \circ g : \mathcal{W} \to [\omega_0, \mu]$ is

also increasing. Let $\beta := \lim_{W\in\mathcal{W}}[a\circ g(W)+1]$. If β is not a limit ordinal, then consider the Banach space $P_\beta(X)$ (whose density is less than $\aleph$), the long sequence $(P_\alpha)_{\omega_0\le\alpha\le\beta}$ of projections on it (a $\sigma-$shrinking PRI$'$ on $P_\beta(X)$ for the sets $B_n^\varepsilon \cap P_\beta(X)$) and carry on the construction in this setting to get, by the induction hypothesis, $\|x^{**}\| \le \epsilon$. If β is a limit ordinal, given $x^* \in B_{X^*}$ we get

$$\langle g(W), x^*\rangle = \langle (P_\beta - P_{a\circ g(W)})g(W), x^*\rangle = \langle g(W), (P^*_\beta - P^*_{a\circ g(W)})x^*\rangle,$$

and

$$\langle g(W), x^*\rangle \to \langle x^{**}, x^*\rangle.$$

As $g(W) \in B_n^\epsilon$, we get $|\langle x^{**}, x^*\rangle| \le \epsilon$ for all $x^* \in B_{X^*}$, so $\|x^{**}\| \le \epsilon$. ∎

We will use the following statement.

Lemma 9 *Let X be a Banach space, W be an absolutely convex and weakly compact subset of X and $(P_\alpha)_{\omega_0\le\alpha\le\mu}$ be a PRI on X such that $P_\alpha(W) \subset W$, for all α. Then, given $x^* \in X^*$ and a limit ordinal $\beta \in (\omega_0, \mu]$, $P^*_\alpha x^* \to P^*_\beta x^*$ uniformly on W when $\alpha \uparrow \beta$.*

Proof. Obviously, $P^*_\alpha x^* \xrightarrow{\omega^*} P^*_\beta x^*$ when $\alpha \uparrow \beta$, so

$$P^*_\beta x^* \in \overline{\bigcup_{\alpha<\beta} P^*_\alpha X^*}^{\omega^*} = \overline{\bigcup_{\alpha<\beta} P^*_\alpha X^*}^{\mu(X^*,X)},$$

where $\mu(X^*, X)$ denotes the Mackey topology on X^*, i.e., the topology of the uniform convergence on the family of absolutely convex and weakly compact subsets of X (see, e.g., [F$\sim$, Thm. 4.33]).
Given $\epsilon > 0$, find $y^* \in X^*$ and $\alpha_0 < \beta$ such that $\sup |\langle W, P^*_\beta x^* - P^*_{\alpha_0} y^*\rangle| < \epsilon$. Let $\alpha_0 \le \alpha < \beta$. Then $\sup |\langle P_\alpha(W), P^*_\beta x^* - P^*_{\alpha_0} y^*\rangle| < \epsilon$, as $P_\alpha(W) \subset W$. This implies $\sup |\langle W, P^*_\alpha x^* - P^*_{\alpha_0} y^*\rangle| < \epsilon$. Then

$$\sup |\langle W, P^*_\beta x^* - P^*_\alpha x^*\rangle| \le$$
$$\le \sup |\langle W, P^*_\beta x^* - P^*_{\alpha_0} y^*\rangle| + \sup |\langle W, P^*_\alpha x^* - P^*_{\alpha_0} y^*\rangle| < 2\epsilon.$$

∎

Lemma 10 *Let X be a WCG Banach space. Let $W \subset X$ be an absolutely convex and weakly compact set spanning X (i.e., $\overline{\text{span}}(W) = X$). Let $(P_\alpha)_{\omega_0 \leq \alpha \leq \mu}$ be a PRI on X such that $P_\alpha(W) \subset W$, for all α. Then $(P_\alpha)_{\omega_0 \leq \alpha \leq \mu}$ is $\sigma-$shrinking. If X is a subspace of a WCG Banach space, then X has a $\sigma-$shrinking PRI.*

Remark. By the well-known theorem of Amir and Lindenstrauss (see [AL] or, e.g., [F$\sim$, Thm. 11.6]) a WCG Banach space X generated by W as above has a PRI $(P_\alpha)_{\omega_0 \leq \alpha \leq \mu}$ such that $P_\alpha(W) \subset W$, for all α.

Proof of Lemma 10. Given $\epsilon > 0$, let $B_n^\epsilon := (nW + \epsilon B_X) \cap B_X$, $n \in I\!N$. Given $x \in B_X$ we can find $y \in \text{span}(W)$ such that $\|x - y\| < \epsilon$. Now, $y \in nW$ for some $n \in I\!N$, so $x \in B_n^\epsilon$. By Lemma 9 we get

$$\sup |\langle nW, P_\beta^* x^* - P_\alpha^* x^* \rangle| \to 0, \text{ when } \alpha \uparrow \beta, \text{ for all } \omega_0 < \beta \leq \mu.$$

Then there exists $\alpha_0 < \beta$ such that

$$\sup |\langle nW, P_\beta^* x^* - P_\alpha^* x^* \rangle| < \epsilon, \text{ for all } \alpha \text{ such that } \alpha_0 \leq \alpha < \beta,$$

so

$$\sup |\langle B_n^\epsilon, P_\beta^* x^* - P_\alpha^* x^* \rangle| < 2\epsilon, \text{ for all } \alpha \text{ such that } \alpha_0 \leq \alpha < \beta,$$

and this proves the first part.
In order to prove the second part, observe first that if a Banach space X of density $\aleph$ is a subspace of a WCG Banach space, then it is also a subspace of a WCG Banach space Z of density $\aleph$ (indeed, let D be a dense subset of X such that $\#D = \aleph$ and let K be a weakly compact set generating Z. Given $x \in D$, find a countable set $N_x \subset K$ such that $x \in \overline{\text{span}} N_x$. Let $W := \bigcup_{x \in D} N_x$, a weakly relatively compact subset of Z. Then $Z_1 := \overline{\text{span}}(W)$ is a WCG Banach space of density $\aleph$ and containing X; it is enough to take now Z_1 as Z). Let μ be the first ordinal of cardinality $\aleph$. By Lemma 6 we can find a PRI $(P_\alpha)_{\omega_0 \leq \alpha \leq \mu}$ on Z such that $P_\alpha(K) \subset K$ and $P_\alpha(X) \subset X$ for all α, so $\sigma-$shrinking by the first part of the proof. It follows that $(P_\alpha|_X)_{\omega_0 \leq \alpha \leq \mu}$ is a $\sigma-$shrinking PRI on X. ∎

Corollary 11 *Let X be a WCG Banach space. Then every M-basis on X is $\sigma-$shrinking.*

Proof. It is enough to put together Lemma 6 (or just the remark following its proof), Proposition 8 and Lemma 10. ∎

We will now give an elementary proof to the following lemma. An alternative proof to it can be obtained by using the results in [Fa].

Lemma 12 *Assume that X admits a σ-shrinking M-basis. Then B_{X^*} in its weak* topology is an Eberlein compact.*

Proof. Let $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ be a $\sigma-$shrinking M-basis of X. We will construct a homeomorphism of B_{X^*} in its weak* topology onto a subset of $c_0(\Delta)$ in its weak topology for some Δ.
Given $n \in \mathbb{N}$, let $\{\Gamma_n^{1/m}\}_{n=1}^{\infty}$ be the sets that cover Γ for $U = (1/m)B_{X^{**}}$ (see Definition 1). For $i \in \mathbb{N}$, let the real valued function τ_i be defined on the reals by $\tau_i(t) = t + (1/i)$ for $t \leq -(1/i)$, $\tau_i(t) = 0$ for $t \in [-(1/i), (1/i)]$ and $\tau_i(t) = t - (1/i)$ if $t \geq (1/i)$.
The set Δ will be an infinite matrix whose first row is a display of Γ_1^1, followed by a disjoint display of Γ_2^1, then Γ_3^1, etc. The second row is the display of $\Gamma_1^{1/2}$ followed by a disjoint display of $\Gamma_2^{1/2}$, etc.
If $f \in B_{X^*}$ and $\gamma \in \Delta$ is in the i^{th} row, in the display $\Gamma_k^{1/i}$, we put $\Phi f(\gamma) = 2^{-(i+k)}\tau_i(f(\gamma))$. Then it is easy to see that Φ maps B_{X^*} into $c_0\,(\Delta)$. Indeed, due to the "weights" 2^{-i}, it suffices to note that on each row, the values are in c_0. This holds due to the properties of $\Gamma_n^{1/m}$ and due to the weights 2^{-k}. The map Φ is weak* to pointwise continuous and thus weak* to weak continuous. The one-to-one property follows from the observation that if t_1 and t_2 are two different real numbers then for sufficiently large i, $\tau_i(t_1) \neq \tau_i(t_2)$. Hence B_{X^*} in its weak* topology is homeomorphic to a weakly compact set in $c_0(\Delta)$.
∎

Proof of Theorem 2. (i)⇒ (ii) Let X be a subspace of the WCG Banach space Z. Then X admits a M-basis (see e.g. [JZ1]). Take any M-basis in X. This basis can be extended to an M-basis of Z (see Lemma 6). By Lemma 9, this extended M-basis is $\sigma-$shrinking, so the original M-basis on X is $\sigma-$shrinking, too.

(ii)⇒(iii) Assuming (ii), we apply Lemma 12 to see that B_{X^*} in its weak* topology is homeomorphic to a weakly compact set in $c_0\,(\Delta)$ considered in its weak topology for some Δ. This proves (iii).

The implication (iii)$\Rightarrow$(i) is well known (see e.g. [F$\sim$, Thm. 12.12]).

Proof of Theorem 3.
First note that (i) and (iii) are equivalent [F, Thm 7.2.5]. (i)$\Rightarrow$(ii). We will use the approach of Sokolov [S]. A Vašák space X admits a separable PRI $(P_\alpha)_{\omega_0\le\alpha\le\mu}$, i. e., a long sequence of continuous projections not necessarily of norm one such that μ is the first ordinal with cardinality dens$X$, $P_{\omega_0}=0$, P_μ is the identity operator, $P_\alpha P_\beta = P_{\min\{\alpha,\beta\}}$ for all $\omega_0\le\alpha,\beta\le\mu$, $(P_{\alpha+1}-P_\alpha)X$ is separable for all $\omega_0\le\alpha<\mu$ and $x\in\overline{\text{span}}\{(P_{\alpha+1}-P_\alpha)(x);\omega_0\le\alpha<\mu\}$ for all $x\in X$. Then X has an M-basis $\{\gamma,\gamma^*\}_{\gamma\in\Gamma}$ subordinated to $(P_\alpha)_{\omega_0\le\alpha\le\mu}$ (see the proof of [F, Prop. 6.2.4]). We may an do assume $\Gamma\subset B_X$. Let $\{\gamma_n^\alpha\}_{n\in I\!N}=\{\gamma\in\Gamma;\ (P_{\alpha+1}-P_\alpha)\gamma=\gamma\}$, $\omega_0\le\alpha<\mu$.
Let $B_m\subset B_{X^{**}}$, $m\in\mathbb{N}$, be the weak* closed sets witnessing that X is Vašák, i.e., for every $x\in B_X$ there is $N\subset\mathbb{N}$ so that $x\in\bigcap_{m\in N}B_m\subset X$; we may an do assume that for every $m,n\in I\!N$ there exists $k\in I\!N$ such that $B_m\cap B_n=B_k$. Put then $\Gamma=\bigcup_{m,n=1}^\infty\Gamma_{m,n}$, where

$$\Gamma_{m,n}=\{\gamma_n^\alpha;\ \omega_0\le\alpha<\mu\}\cap B_m,\quad m,n\in\mathbb{N}.$$

Now, fix any $x^*\in X^*$, $\gamma_0\in\Gamma$, and $\epsilon>0$.
Let $N:=\{m_1,m_2,\dots\}\subset I\!N$ such that

$$\gamma_0\in\bigcap_{k=1}^\infty B_{m_k}\subset X.$$

We **claim** that $\#\{\gamma\in\Gamma_{m,n}:\langle\gamma,x^*\rangle>\varepsilon\}<\aleph_0$ for some $m\in I\!N$. Assume not. Let α_1 be the first ordinal $\alpha\in[\omega_0,\mu)$ such that $\gamma=\gamma_n^\alpha\in B_{m_1}$ and $\langle\gamma,x^*\rangle>\varepsilon$. Let $\alpha_2>\alpha_1$ be the first ordinal $\alpha\in[\omega_0,\mu)\setminus\{\alpha_1\}$ such that $\gamma=\gamma_n^\alpha\in B_{m_1}\cap B_{m_2}$ and $\langle\gamma,x^*\rangle>\varepsilon$. Continue in this way. We get a sequence $\alpha_1<\alpha_2<\dots$ converging to some $\alpha\le\mu$; then $(\gamma_n^{\alpha_k})_{k\in I\!N}$ weak-clusters to some point $x\in\bigcap_{k=1}^\infty B_{m_k}\subset X$. It follows that $(Q_\beta\gamma_n^{\alpha_k})_{k\in I\!N}$ weak-clusters to $Q_\beta x$ for every $\omega_0\le\beta<\mu$, so $Q_\beta x=0$ for every $\omega_0\le\beta<\mu$. Then $x=0$, a contradiction with $\langle x,x^*\rangle\ge\varepsilon$, and this proves the claim.

(ii)$\Rightarrow$(i). Assume that X contains a weakly $\sigma-$ shrinking M-basis $\{\gamma,\gamma^*\}_{\gamma\in\Gamma}$, and $\Gamma=\bigcup_{n=1}^\infty\Gamma_n$ from the definition. For $i\in\mathbb{N}$, let $\tau_i(t)$ be a function on the real line such that $\tau_i=0$ on $[-\frac{1}{i},+\frac{1}{i}]$ and $\tau_i(t)=t-\frac{1}{i}$ on $[\frac{1}{i},\infty)$ and $\tau_i(t)=t+\frac{1}{i}$ on $(-\infty,\frac{-1}{i}]$. Let Δ be the infinite matrix whose first row consists of countably many disjoint copies of Γ_1, call them Γ_1^1, Γ_1^2 etc. the second row consists of countably many disjoint copies of Γ_2, call them Γ_2^1, Γ_2^2 etc. Define the map φ from B_{X^*} into $\ell_\infty(\Delta)$ by $\varphi(x^*)(\gamma_n^i) = \tau_i(\langle \gamma_n^i, x^* \rangle)$ where γ_n^i is an element of Γ_n^1. Then it can be checked that φ is one-to-one continuous map from the weak* topology of B_{X^*} into the pointwise topology of $\ell_\infty(\Delta)$. Then X is a Vašák space by [F, Thm. 7.2.5 (vi)].

(iii)⇒(i) is well known (see, e.g., [F, Thm. 7.1.9]). ∎

Before proving Theorem 4, we state and prove the following simple fact.

Lemma 13 *Let K be a compact space and $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ be an M-basis of $C(K)$. Then the following are equivalent.*
(i) $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is pointwise Lindelöf.
(ii) $\{\gamma \in \Gamma;\ \gamma(k) \neq 0\}$ is countable for all $k \in K$.
In this case, K is a Corson compact.

Proof. (i)⇒(ii) Assume that $\Gamma \cup \{0\}$ is pointwise Lindelöf. Let $k \in K$ and $p \in I\!N$. Let $\mathcal{U}$ be the open cover of $\Gamma \cup \{0\}$ formed by $U = \{f \in C(K); |f(k)| < \frac{1}{p}\}$ and by the sets $U_\gamma = \{f \in C(K); \langle f, \gamma^* \rangle > \frac{1}{2}\}$, $\gamma \in \Gamma$. Let $\mathcal{V}$ be a countable subcover of $\mathcal{U}$. As $0 \notin U_\gamma$ for all γ, the subcover $\mathcal{V}$ has to be formed by U and by some U_{γ_i}, $i = 1, 2, \ldots$ As $\gamma \notin U_{\gamma'}$ for $\gamma \neq \gamma'$, all but countably many γ's are in U, i.e. for all but countably many γ's, $|\gamma(k)| < \frac{1}{p}$. This holds for all p. Hence $\gamma(k) = 0$ for all but countably many γ's.
In particular, this shows that K is a Corson compact.

(ii)⇒(i) Let $\mathcal{U}$ be an open cover of $\Gamma \cup \{0\}$ in the pointwise topology. Then there is $U \in \mathcal{U}$ such that $0 \in U$. Assume that (ii) is satisfied. Then $\gamma \in U$ for all but countably many γ. Thus $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is pointwise Lindelöf. ∎

Proof of Theorem 4. (i)⇒(ii) If K is Corson, then $C(K)$ is Lindelöf in the pointwise topology (Corson, see e.g. [F∼]). Moreover, there is an M-basis $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ so that $\{\gamma^*;\ \gamma \in \Gamma\} \subset \overline{\text{span}}^{\|\cdot\|} K$ (see e.g. [DGZ, Thm. VI.7.6] and [F, Prop. 6.2.4]). We shall prove that such an M-basis is pointwise Lindelöf. We may and do assume that $\Gamma \subset B_{C(K)}$. $\Gamma \cup \{0\}$ is obviously closed in the topology of the pointwise convergence on the set $\{\gamma^*;\ \gamma \in \Gamma\}$, so it is also closed in the topology of the pointwise convergence on $\overline{\text{span}}^{\|\cdot\|}(K)$. Observe now that every pointwise limit in $C(K)$ of a net of elements in $\Gamma \cup \{0\}$ is also in $B_{C(K)}$. It follows easily that $\Gamma \cup \{0\}$ is also pontwise closed, hence it is pointwise Lindelöf.

(ii)⇒(i) If $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is pointwise Lindelöf, then by Lemma 13, K is a Corson compact.
As an application of the methods studied in this paper we present a new proof of the following well known result [BRW] (see also [Gu], [MR] and [NT]). For a different new proof of this result see [FMZ3].

Theorem 14 *Let K be an Eberlein compact. Let φ be a continuous map of K onto $\varphi(K)$. Then $\varphi(K)$ is an Eberlein compact.*

Proof. The space $C(\varphi(K))$ is a subspace of the WCG space $C(K)$. Let $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ be an M-basis of $C(\varphi(K))$ with $\|\gamma\| \leq 1$ for all γ. By Theorem 2, $\{\gamma, \gamma^*\}_{\gamma\in\Gamma}$ is $\sigma-$shrinking and thus by Lemma 12, $B_{C(\varphi(K))^*}$ is an Eberlein compact. Hence such is its closed subset $\varphi(K)$. ∎

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