Weakly compact generating and shrinking Markuševič bases

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Abstract

It is shown that most of the well known classes of nonseparable Banach spaces related to the weakly compact generating can be characterized by elementary properties of the closure of the coefficient space of Markuševič bases for such spaces. In some cases, such property is then shared by all Markuševič bases in the space.

Let X be a Banach space and let $\langle \cdot, \cdot \rangle$ denote the canonical duality pairing between X and its dual space X^*. A system $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$, where $x_\gamma \in X$, $x^*_\gamma \in X^*$, $\gamma \in \Gamma$, is called a *Markuševič basis* for X if $\langle x_\gamma, x^*_\gamma\rangle = 1$ for every $\gamma \in \Gamma$, $\langle x_\gamma, x^*_{\gamma'}\rangle = 0$ whenever $\gamma, \gamma' \in \Gamma$ and $\gamma \neq \gamma'$, the linear span $\mathrm{sp}\{x_\gamma;\ \gamma \in \Gamma\}$ is dense in X and $\mathrm{sp}\{x^*_\gamma;\ \gamma \in \Gamma\}$ is weak* dense in X^*.
A compact space K is a *Corson compact* if for some set Γ, K is homeomorphic to a subset T of a product $[-1, 1]^\Gamma$ taken in its pointwise topology, such that

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for every $t \in T$, $\{\gamma \in \Gamma;\ t(\gamma) \neq 0\}$ is at most countable. A compact space K is called an *Eberlein compact* respectively a *uniform Eberlein compact* if for some set Γ, K is homeomorphic to a compact set in $c_0(\Gamma)$ (respectively $\ell_2(\Gamma)$) endowed with the weak topology.
A Banach space X is called *weakly compactly generated* (*WCG*) if it contains a weakly compact subset whose linear span is dense in X.We will call a set $S \subset X$ total if its linear span is dense in X. A Banach space X is called a *Vašák space* (i.e., *weakly countably determined Banach space*) if there exists a countable family $\mathcal{K}$ of weak* compact subsets of X^{**} such that, given any $x \in X$ and $x^{**} \in X^{**} \setminus X$, there is $K \in \mathcal{K}$ such that $x \in K$ and $x^{**} \notin K$. A Banach space X is called *weakly Lindelöf determined* (*WLD*) if B_{X^*} in its weak star topology is a Corson compact which is the same as if there is a set $\Delta \subset X$, with $\mathrm{sp}\Delta$ dense in X, such that Δ *countably supports* X^*, that is, for every $x^* \in X^*$ the set $\{\delta \in \Delta;\ \langle \delta, x^* \rangle \neq 0\}$ is at most countable (cf. e.g. [Fa97]). It is well known that every WLD space admits a Markuševič basis, see, e.g., [F, Propositions 8.3.1, 6.1.7, 6.2.4], [F~01, Theorem 12.50] and the proof of Theorem 4 below. A Banach space is called an *Asplund space* if every of its separable subspaces has separable dual. We refer to [DGZ93], [Fa97] and [F~01] for more on these concepts.

A result in [Fa87] has the following known consequence, see, e.g., [F, page 112 and Theorem 8.3.3]:

Theorem 0 *A Banach space X is simultaneously WCG and Asplund if and only if X admits a Markuševič basis $\{x_\gamma; x^*_\gamma\}_{\gamma \in \Gamma}$ which is shrinking, that is,* $\mathrm{sp}\{x^*_\gamma;\ \gamma \in \Gamma\}$ *is norm dense in X^*.*

Note that not every Markuševič basis on a WCG Asplund space is shrinking. Indeed, in any nonreflexive separable Asplund space there exists a Schauder basis which is not shrinking, [Zi68].

The purpose of this note is to show that the property of biorthogonality, together with known techniques of projectional resolutions of the identity operator make it possible to dualize and extend some results in [FGMZ04], [FGHZ03] and [FMZ05] to the case of Markushevic bases in the spirit of Theorem 0. More precisely, we show that most of the classes of Banach spaces related to the weakly compact generating, like WCG spaces, subspaces of WCG spaces, Vašák spaces can be characterized by replacing the norm topology in Theorem 0 by the topology of uniform convergence on

an appropriate family of subsets of the Markuševič basis for X. Note that the classes of spaces involved are known not to coincide (cf. e.g [Fa97] and references therein).
The approach to these spaces that uses Markuševič bases provides a good insight into these spaces (compare the statement in Theorems 2 and 3 for instance).
This is useful in questions in the renormings by smooth norms, smooth approximations, weak Asplund spaces, Asplund generated spaces and in the area of topology of special compacta (Eberlein, Corson, Gul'ko, Talagrand compacta), cf.e.g.[FGMZ], [FMZ], [HMVZ]).

We made an effort to systematically present the results of this paper together with some folklore results in the area in order to provide the reader with a compact information on this subject.

Theorem 1 *A Banach space X is WCG if and only if it admits a Markuševič basis $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ such that $\mathrm{sp}\{x^*_\gamma;\ \gamma \in \Gamma\}$ is dense in X^* in the topology of uniform convergence on the set $\{x_\gamma;\ \gamma\in\Gamma\}$.*

It follows that the Markuševič basis from Theorem 1 is weakly compact, that is, the set $\{x_\gamma;\ \gamma\in\Gamma\}\cup\{0\}$ is weakly compact. Indeed, any sequence of distinct elements in it converges to 0 in the topology of the pointwise convergence on $\{x^*_\gamma\}$ (by the orthogonality) and thus converges to 0 in the weak topology by the uniformity in the condition in Theorem 1. Note that it follows from [Ro74] and from known results on Markuševič bases that there exists a WCG space X with a Markuševič basis such that the set $\{x_\gamma;\ \gamma\in\Gamma\}\cup\{0\}$ cannot be written as the union of countably many weakly compact sets [AM] [FMZ05].
On the other hand, W. Johnson proved that any unconditional basis in WCG space can be so decomposed (see Proposition 1.3 in [Ro74]).
Note also that $\mathrm{sp}\{x^*_\gamma;\ \gamma\in\Gamma\}$ from the Markuševič basis in Theorem 1 may not be weak* sequentially dense in X^*, i.e not every element of X^* can be reached as the weak star limit of a sequence from $\mathrm{sp}\{x^*_\gamma;\ \gamma\in\Gamma\}$. Indeed, according to S. Banach, [Ba32, Theorem 1, Annexe] there exists a weak* dense subspace $Y\subset c_0^*$ which is not weak* sequentially dense in c_0^*. Then, [F~01, Theorem 6.41] yields a Markuševič basis $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ such that $\mathrm{sp}\{x^*_\gamma;\ \gamma\in\Gamma\}\subset Y$. See also Godun [Go78] who showed such a phenomenon in every separable quasireflexive Banach space.

Given a system of vectors $x_\gamma \in X,\ \gamma \in \Gamma$ and a set $\Delta \subset \Gamma$, we define a semimetric

$$\rho_\Delta(x_1^*, x_2^*) = \sup_{\gamma\in\Delta} |\langle x_\gamma, x_1^* - x_2^*\rangle|, \quad x_1^*, x_2^* \in X^*.$$

For $x^* \in X^*$ and a set $M \subset X^*$ we define

$$\rho_\Delta(x^*, M) = \inf\{\rho_\Delta(x^*, y^*);\ y^* \in M\}.$$

Theorem 2 *A Banach space X is a subspace of a WCG space if and only if X admits a Markuševič basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ with the following property: there are sets $\Gamma_n \subset \Gamma,\ n \in \mathbb{N}$, such that*

$$\forall\epsilon > 0\ \forall\gamma \in \Gamma\ \exists n \in \mathbb{N} \ \ \textit{so that}\ \gamma \in \Gamma_n \ \ \textit{and}$$

$$\forall\ x^* \in B_{X^*} \ \ \rho_{\Gamma_n}(x^*, \mathrm{sp}\{x_\gamma^*;\ \gamma \in \Gamma\}) < \epsilon.$$

In this case every Markuševič basis in X has this property.

It should be noted that the above sets Γ_n's usually overlap, see [FGMZ04, Theorem 2] for how to get a "non-overlapping" version of the above theorem.

Theorem 3 *A Banach space X is a Vašák space if and only if X admits a Markuševič basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ with the following property: there are sets $\Gamma_n \subset \Gamma,\ n \in \mathbb{N}$, such that*

$$\forall\epsilon > 0\ \forall\gamma \in \Gamma\ \forall x^* \in B_{X^*}\ \exists n \in \mathbb{N}\ \textit{so that}\ \gamma \in \Gamma_n\ \textit{and}$$

$$\rho_{\Gamma_n}(x^*, \mathrm{sp}\{x_\gamma^*;\ \gamma \in \Gamma\}) < \epsilon.$$

In this case every Markuševič basis in X has this property.

Theorem 4 *A Banach space X is WLD if and only if it has a Markuševič basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ with one of the equivalent properties listed in Proposition 1 below.*
If this is the case, all Markuševič bases in X share the same properties.

Proposition 1 *Given a Markuševič basis* $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ *in a Banach space* X*, then the following are equivalent.*
(i) *For every* $x^* \in X^*$ *the set* $\{\gamma \in \Gamma;\ \langle x_\gamma, x^*\rangle \neq 0\}$ *is at most countable.*
(ii) $\mathrm{sp}\{x^*_\gamma;\ \gamma \in \Gamma\}$ *is sequentially dense in* X^* *endowed with the topology of the pointwise convergence on the set* $\{x_\gamma; \gamma \in \Gamma\}$*.*
(iii) $\mathrm{sp}\{x^*_\gamma;\ \gamma \in \Gamma\}$ *is countably dense in* X^* *in the weak* topology, i.e. every element of* X^* *is in the closure in this topology of a countable set in* $\mathrm{sp}\{x^*_\gamma;\ \gamma \in \Gamma\}$.

It is known that a Banach space X admits an equivalent uniformly Gâteaux smooth norm if and only if its dual unit ball B_{X^*} in the weak* topology is a uniform Eberlein compact, see [FGZ01], [FGMZ04] for definitions and proofs. For $M \subset X^*$ and $\kappa \in \mathbb{N}$ we define

$$\mathrm{sp}_\kappa M = \Big\{ \sum_{i=1}^{\kappa} a_i x^*_i;\ a_i \in \mathbb{R},\ x^*_i \in M,\ i = 1, \dots, \kappa \Big\}.$$

Theorem 5 *The dual ball* (B_{X^*}, w^*) *of a Banach space* X *is a uniform Eberlein compact if and only if* X *admits a Markuševič basis* $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ *with the following property: there are numbers* $\kappa(n) \in \mathbb{N}$ *and sets* $\Gamma_n \subset \Gamma,\ n \in \mathbb{N}$*, such that*

$$\forall \epsilon > 0\ \forall \gamma \in \Gamma\ \exists n \in \mathbb{N} \ \text{ so that } \ \gamma \in \Gamma_n \ \text{ and}$$

$$\forall x^* \in B_{X^*} \ \ \rho_{\Gamma_n}(x^*, \mathrm{sp}_{\kappa(n)}\{x^*_\gamma;\ \gamma \in \Gamma_n\}) < \epsilon.$$

In this case every Markuševič basis in X *has this property.*

The next theorem characterizes a subclass (called in [FGHZ03] *strongly uniformly Gâteaux smooth* if $\mathrm{dens}\, X \le \omega_1$) of the uniformly Gâteaux smooth Banach spaces –spaces admitting an equivalent uniformly M-smooth norm, with $M \subset B_X$ a total set, see [FGMZ04, Theorem 8].

Theorem 6 *A Banach space* $(X, \|\cdot\|)$ *admits a total set* $\Gamma \subset B_X$ *such that*

$$\forall \varepsilon > 0\ \exists \kappa \in \mathbb{N}\ \forall x^* \in B_{X^*} \quad \#\{\gamma \in \Gamma;\ |\langle \gamma, x^*\rangle| \ge \varepsilon\} \le \kappa. \tag{1}$$

if and only if X *admits a Markuševič basis* $\{x_\lambda; x^*_\lambda\}_{\lambda\in\Lambda}$ *such that* $x_\lambda \in \mathrm{sp}\Gamma$ *for every* $\lambda \in \Lambda$*, and denoting* $\Delta = \{x_\lambda;\ \lambda \in \Lambda\}$*, we have*

$$\forall \varepsilon > 0\ \exists \kappa \in \mathbb{N}\ \forall x^* \in B_{X^*} \ \ \rho_\Delta(x^*, \mathrm{sp}_\kappa\{x^*_\lambda;\ \lambda \in \Lambda\}) < \epsilon.$$

Theorem 7 *Let $1 < p \leq 2$, let Γ be an uncountable set, and let e_γ, $\gamma \in \Gamma$, denote the canonical basis vectors in $\ell_p(\Gamma)$. A Banach space X admits a linear bounded mapping from $\ell_p(\Gamma)$ onto a dense subset of X if and only if X admits a Markuševič basis $\{x_\lambda; x^*_\lambda\}_{\lambda\in\Lambda}$, and a bounded linear operator $T : \ell_p(\Lambda) \to X$ such that $x_\lambda = Te_\lambda$ for every $\lambda \in \Lambda$.*

Note that the property exhibited in Theorem 7 characterizes, in case that dens $X \leq \omega_1$, the superreflexive generated spaces (see [FGHZ03]).

Proofs

The following simple statement will be of frequent use.

Lemma 1 *Let $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ be a biorthogonal system in a Banach space X. Let $x^* \in X^*$, $\epsilon > 0$, and $\kappa \in \mathbb{N}$ be given. Then*
(i) $\#\{\gamma \in \Gamma;\ |\langle x_\gamma, x^*\rangle| \geq \epsilon\} \leq \kappa$ if and only if $\rho_\Gamma(x^*, \mathrm{sp}_\kappa\{x^*_\gamma;\ \gamma \in \Gamma\}) < \epsilon$.
(ii) $\#\{\gamma \in \Gamma;\ |\langle x_\gamma, x^*\rangle| \geq \epsilon\} < \omega$ if and only if $\rho_\Gamma(x^*, \mathrm{sp}\{x^*_\gamma;\ \gamma \in \Gamma\}) < \epsilon$.

Proof. (i) Necessity. Denote $F = \{\gamma \in \Gamma;\ |\langle x_\gamma, x^*\rangle| \geq \epsilon\}$. If $\gamma_0 \in \Gamma \setminus F$ then

$$\left|\left\langle x_{\gamma_0}, x^* - \sum_{\gamma\in F}\langle x_\gamma, x^*\rangle x^*_\gamma\right\rangle\right| = |\langle x_{\gamma_0}, x^*\rangle| < \epsilon.$$

If $\gamma_0 \in F$, then even

$$\left|\left\langle x_{\gamma_0}, x^* - \sum_{\gamma\in F}\langle x_\gamma, x^*\rangle x^*_\gamma\right\rangle\right| = 0.$$

It follows that

$$\rho_\Gamma(x^*, \mathrm{sp}_\kappa\{x^*_\gamma;\ \gamma \in \Gamma\}) \leq \rho_\Gamma(x^*, \mathrm{sp}\{x^*_\gamma;\ \gamma \in F\}) < \epsilon.$$

Conversely, find $F \subset \Gamma$, with $\#F \leq \kappa$, such that $\rho_\Gamma(x^*, \mathrm{sp}\{x^*_\gamma;\ \gamma \in F) < \epsilon$. Assume that there exists $\gamma_0 \in \Gamma \setminus F$ such that $|\langle x_{\gamma_0}, x^*\rangle| \geq \epsilon$. Then, for every $y^* \in \mathrm{sp}\{x^*_\gamma;\ \gamma \in F\}$ we get $\rho_\Gamma(x^*, y^*) \geq |\langle x_{\gamma_0}, x^* - y^*\rangle| = |\langle x_{\gamma_0}, x^*\rangle| \geq \epsilon$. Hence $\rho_\Gamma(x^*, \mathrm{sp}\{x^*_\gamma;\ \gamma \in F\}) \geq |\langle x_{\gamma_0}, x^*\rangle| \geq \epsilon$, a contradiction. Therefore $\{\gamma \in \Gamma;\ |\langle x_\gamma, x^*\rangle| \geq \epsilon\} \subset F$ and we are done.
(ii) follows immediately from (i). ∎

Let $(X, \|\cdot\|)$ be a non-separable Banach space. Let μ be the first ordinal whose cardinality is equal to the density $\operatorname{dens} X$ of X. A transfinite sequence of linear projections $(P_\alpha)_{\omega\leq\alpha\leq\mu}$ on X is called a *projectional resolution of identity* (*PRI*) if $P_\omega \equiv 0$, $P_\mu \equiv I_X$ (the identity on X) and for all $\alpha, \beta \leq \mu$ we have $\|P_\alpha\| = 1$, $\operatorname{dens} P_\alpha(X) \leq \#\alpha$ (the cardinality of α), $P_\alpha P_\beta = P_\beta P_\alpha = P_{\min(\alpha,\beta)}$, and for every $x \in X$ the mapping $\alpha \mapsto P_\alpha(x)$ from the ordinal segment $[\omega, \mu]$ in its standard topology into X is continuous. A *separable projectional resolution of the identity* (*separable PRI*) on X is a transfinite sequence $\{Q_\alpha : \omega \leq \alpha \leq \mu\}$ of linear projections on X such that $Q_\omega \equiv 0$, $Q_\mu \equiv I_X$, $(Q_{\alpha+1} - Q_\alpha)X$ is separable for $\omega \leq \alpha < \mu$, for all $\alpha, \beta \leq \mu$ we have $\|Q_\alpha\| < +\infty$, $Q_\alpha Q_\beta = Q_\beta Q_\alpha = Q_{\min(\alpha,\beta)}$, and for every $x \in X$, $x \in \overline{\operatorname{sp}}\{(Q_{\alpha+1} - Q_\alpha)(x); \alpha < \mu\}$. If Γ is a subset of X, a PRI (a separable PRI) $(P_\alpha)_{\omega\leq\alpha\leq\mu}$ on X is said to be *subordinated to* Γ if $P_\alpha(\gamma) \in \{0, \gamma\}$ for all $\gamma \in \Gamma$ and all $\alpha \in [\omega, \mu]$.

Proposition 2 *Let X be a Banach space with a total subset Γ which countably supports X^*. Then X has a separable PRI subordinated to Γ.*

Proof. If X is separable there is nothing to prove. Assume now that the lemma holds for every Banach space with density character less than a certain uncountable cardinal $\aleph$. Let X be a Banach space with density character $\aleph$ and with a total subset Γ which countably supports X. By [FGMZ04, Proposition 1], X has a PRI $(P_\alpha)_{\omega\leq\alpha\leq\mu}$ subordinated to Γ. Now, for $\omega \leq \alpha < \mu$, the set $(P_{\alpha+1}-P_\alpha)\Gamma$ ($\subset \Gamma\cup\{0\}$) is total in $(P_{\alpha+1}-P_\alpha)X$ and countably supports the dual $((P_{\alpha+1}-P_\alpha)X)^*$. Moreover, $\operatorname{dens}(P_{\alpha+1}-P_\alpha)X$ is less than $\aleph$. Then, by the induction hypothesis, $(P_{\alpha+1} - P_\alpha)X$ has a separable PRI subordinated to $(P_{\alpha+1} - P_\alpha)\Gamma$.
Now, it is enough to use [Fa97, Proposition 6.2.7]. ∎

Proof of Theorem 1. Necessity. A well known result of Amir and Lindenstrauss [AL68] yields a weakly compact Markuševič basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ in X, i.e. an Markuševič basis $\{x_\gamma, x_\gamma^*\}$ such that $\{x_\gamma\} \cup 0$ is a weakly compact set in X. This means that for every $\epsilon > 0$ and every $x^* \in X^*$ the set $\{\gamma \in \Gamma;\ |\langle x_\gamma, x_\gamma^*\rangle| \geq \epsilon\}$ is finite. Now it is enough to apply Lemma 1. The sufficiency is obvious since then the set $\{x_\gamma : \gamma \in \Gamma\} \cup \{0\}$ must be weakly compact. Alternatively, one can use the Mackey-Arens theorem in this context. ∎

Proof of Theorems 2 resp. 5. The sufficiency follows from [FGMZ04, Thm. 2 resp. 6] and our Lemma 1. As regards the necessity, let $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$

be any Markuševič basis in X with all x_γ's in B_X. Write γ instead of x_γ. Thus we have that $\Gamma \subset B_X$. For this Γ find sets Γ_n^ϵ, $n \in \mathbb{N}$, $\epsilon > 0$, as in (ii) of [FGMZ04, Theorem 2 resp. 6]. Then the (countable) family of sets $\Gamma_n^{1/i}$, $n, i \in \mathbb{N}$, satisfies, according to Lemma 1, the condition of Theorem 2, resp. Theorem 5. ∎

Proof of Theorem 3. Combine [FGMZ04, Theorem 3] with Lemma 1 as it was done in the the previous proof. ∎

Proof of Theorem 4. The condition (i) in Proposition 1 implies that X is WLD. Conversely, assume that X is WLD, that is, there exists a total set $\Delta \subset B_X$ which countably supports X^*. By Proposition 2, we find in X a separable PRI $(P_\alpha)_{\omega \leq \mu}$ subordinated to the set Δ. Fix an arbitrary $\alpha \in [\omega, \mu)$. We note that $\Delta_\alpha := (P_{\alpha+1} - P_\alpha)\Delta \subset \Delta \cup \{0\}$ and that this set is total in the (separable) subspace $(P_{\alpha+1} - P_\alpha)X$. By the classical Markuševič Theorem (see, for example, [F~01, Theorem 6.41]), in $(P_{\alpha+1} - P_\alpha)X$, there exists a Markuševič basis $\{x_{\alpha,n}; x^*_{\alpha,n}\}_{n\in\mathbb{N}}$ such that $x_{\alpha,n} \in \mathrm{sp}\Delta$ for every $n \in \mathbb{N}$. Define $Q_\alpha : X \to (P_{\alpha+1} - P_\alpha)X$ by $Q_\alpha x = (P_{\alpha+1} - P_\alpha)x,\ x \in X$. Then, replacing PRI by separable PRI in the proof of [F, Proposition 6.2.4], we can conclude that the system $\{x_{\alpha,n}, Q^*_\alpha x^*_{\alpha,n}\}_{n\in\mathbb{N},\ \omega\leq\alpha<\mu}$ forms a Markuševič basis in X. It remains to check the cardinality condition in (i) of Proposition 1. Consider any $x^* \in X^*$ and any $n \in \mathbb{N}$. If $\alpha \in [\omega, \mu)$ satisfies $\langle x_{\alpha,n}, x^*\rangle \neq 0$, then $\langle \delta, x^*\rangle \neq 0$ for some $\delta \in \Delta_\alpha$ $(\subset \Delta \cup \{0\})$. Thus

$$\#\{(\alpha, n) \in [\omega, \mu) \times \mathbb{N};\ \langle x_{\alpha,n}, x^*\rangle \neq 0\} \leq \#\{\delta \in \Delta;\ \langle x^*, \delta\rangle \neq 0\} \cdot \omega = \omega,$$

and (i) in Proposition 1 is verified.
Finally, assume that X is WLD and let $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ be *any* Markuševič basis in X. Put $Y = \{x^* \in X^*;\ \#\{\gamma \in \Gamma; \langle x_\gamma, x^*\rangle \neq 0\} \leq \omega\}$; this is a linear set. Take any ξ in the weak* closure of the intersection $Y \cap B_{X^*}$. Since (B_{X^*}, w^*) is a Corson compact, there exists a sequence in $Y \cap B_{X^*}$ which weak* converges to ξ ([F~01, Exercise 12.35]). Therefore $\xi \in Y \cap B_{X^*}$. We have thus proved that the latter set is weak* closed. Then Banach-Dieudonné Theorem ensures that Y is weak* closed. Moreover it contains $\{x^*_\gamma; \gamma \in \Gamma\}$, so $Y = X^*$ and (i) is verified. ∎

Proof of Proposition 1. (i)⇒(ii). Fix any $x^* \in X^*$. Enumerate the set $\{\gamma \in \Gamma;\ \langle x_\gamma, x^*\rangle \neq 0\}$ as $\{\gamma_1^o, \gamma_2^o, \ldots\}$. Find $x_1^* \in \mathrm{sp}\{x^*_\gamma;\ \gamma \in \Gamma\}$ so that $|\langle x_{\gamma_1^\circ}, x^* - x_1^*\rangle| < 1$. Enumerate $\{\gamma \in \Gamma;\ \langle x_\gamma, x_1^*\rangle \neq 0\}$ by $\{\gamma_1^1, \gamma_2^1, \ldots\}$. Find $x_2^* \in \mathrm{sp}\{x_\gamma^*;\ \gamma \in \Gamma\}$ so that $|\langle x_{\gamma_1^\circ}, x^* - x_2^*\rangle| < \frac{1}{2}$, $|\langle x_{\gamma_2^\circ}, x^* - x_2^*\rangle| < \frac{1}{2}$, $|\langle x_{\gamma_1^1}, x^* - x_2^*\rangle| < \frac{1}{2}$, and $|\langle x_{\gamma_2^1}, x^* - x_2^*\rangle| < \frac{1}{2}$. Assume that for some $i \in \mathbb{N}$ we found x_j^* with "support" on Γ given by $\{\gamma_1^j, \gamma_2^j \ldots\}$, $j = 1, 2, \ldots, i$. Find then $x_{i+1}^* \in \mathrm{sp}\{x_\gamma^*;\ \gamma \in \Gamma\}$ so that $|\langle x_{\gamma_l^j}, x^* - x_{i+1}^*\rangle| < \frac{1}{i+1}$ for all $j = 0, 1, \ldots, i$ and $l = 1, 2, \ldots, i$. Then we can easily see that $\langle x_\gamma, x^* - x_i^*\rangle \to 0$ as $i \to \infty$ for every $\gamma \in \Gamma$, and (ii) is proved.

(ii)⇒(i). Take any $x^* \in X^*$. Let $x_i^* \in \mathrm{sp}\{x_\gamma^*; \gamma \in \Gamma\}$, $i \in \mathbb{N}$, be such that $\langle x_\gamma, x^* - x_i^*\rangle \to 0$ as $i \to \infty$ for every $\gamma \in \Gamma$. Now if $\gamma \in \Gamma$ satisfies $\langle x_\gamma, x^*\rangle \neq 0$, then necessarily $\langle x_\gamma, x_i^*\rangle \neq 0$ for some $i \in \mathbb{N}$. Hence

$$\{\gamma \in \Gamma;\ \langle x_\gamma, x^*\rangle \neq 0\} \subset \bigcup_{i=1}^{\infty} \{\gamma \in \Gamma;\ \langle x_\gamma, x_i^*\rangle \neq 0\}$$

and the set on the right hand side is countable.

(i)⇒(iii). Let Y denote the set of all $x^* \in X^*$ which lie in the weak* closure of a countable subset of $\mathrm{sp}\{x_\gamma^*;\ \gamma \in \Gamma\}$. We want to show that $Y = X^*$. Clearly, Y is linear. Let ξ be any element of the weak* closure of B_Y. (i) guarantees that (B_{X^*}, w^*) is a Corson compact, hence ξ can be reached as the weak* limit of a sequence $(x_i^*)_{i=1}^\infty$ in B_Y. Now, for every $i \in \mathbb{N}$ we can find a suitable at most countable set $C_i \subset \mathrm{sp}\{x_\gamma^*;\ \gamma \in \Gamma\}$ so that x_i^* lies in the weak* closure of C_i. Then ξ lies in the (at most countable) set $\bigcup_{i=1}^\infty C_i$, and so $\xi \in Y$. Now, the Banach-Dieudonné Theorem guarantees that Y is weak* closed. But Y contains $\{x_\gamma^*; \gamma \in \Gamma\}$. Therefore $Y = X^*$.

(iii)⇒(i). Fix any $x^* \in X^*$. Find an at most countable set $C \subset \mathrm{sp}\{x_\gamma^*;\ \gamma \in \Gamma\}$ so that x^* belongs to the weak* closure of C. Then

$$\{\gamma \in \Gamma;\ \langle x_\gamma, x^*\rangle \neq 0\} \subset \bigcup_{y^* \in C} \{\gamma \in \Gamma;\ \langle x_\gamma, y^*\rangle \neq 0\},$$

and the latter set is countable. ∎

Proof of Theorem 6. The sufficiency is trivial.
The necessity. Assume first that X is separable. By [F˜01, Theorem 6.41], there exists a Markuševič basis $\{x_n; x_n^*\}_{n\in\mathbb{N}}$ in X such that $x_n \in \mathrm{sp}\Gamma$ and $\|x_n\| < \frac{1}{n}$ for every $n \in \mathbb{N}$. Then for every $\epsilon > 0$ and for every $x^* \in B_{X^*}$ we have $\#\{n \in \mathbb{N},\ |\langle x_n, x^*\rangle| \geq \frac{1}{\epsilon}\} < \frac{1}{\epsilon}$ and Lemma 1 finishes the proof.

Second, assume that X is non-separable. Let $\Gamma \subset B_X$ be a total set satisfying (1). This set countably supports X^*. Thus, by Proposition 2, there exists a separable PRI $(P_\alpha)_{\omega \le \alpha < \mu}$ in X subordinated to Γ. Fix any $\alpha \in [\omega, \mu)$ and denote $\Gamma_\alpha = (P_{\alpha+1} - P_\alpha)\Gamma$ $(\subset \Gamma \cup \{0\})$. Fix any $\alpha \in [\omega, \mu)$. Let $\{x_{\alpha,n}; x^*_{\alpha,n}\}_{n\in\mathbb{N}}$ be a Markuševič basis in the (separable) subspace $(P_{\alpha+1} - P_\alpha)X$ such that $x_{\alpha,n} \in \operatorname{conv}(\Gamma_\alpha \cup -\Gamma_\alpha)$ and $\|x_{\alpha,n}\| < \frac{1}{n}$ for every $n \in \mathbb{N}$; this can be done owing to [F˜01, Theorem 6.41] and by an eventual rescaling. Define $Q_\alpha : X \to (P_{\alpha+1} - P_\alpha)X$ by $Q_\alpha x = (P_{\alpha+1} - P_\alpha)x,\ x \in X$. Then, replacing PRI by separable PRI in the proof of [F, Proposition 6.2.4], we can conclude that the system $\{x_{\alpha,n}, Q^*_\alpha x^*_{\alpha,n}\}_{n\in\mathbb{N},\omega\le\alpha<\mu}$ forms a Markuševič basis in X. Now fix any $x^* \in B^*_X$ and consider any $\omega \le \alpha < \mu$ and any $n \in \mathbb{N}$ such that $|\langle x_{\alpha,n}, x^*\rangle| \ge \epsilon$. Then $n < \frac{1}{\epsilon}$ and there is $\gamma \in \Gamma_\alpha$ so that $|\langle \gamma, x^*\rangle| \ge \epsilon$. Now, if $\kappa \in \mathbb{N}$ was found for our ϵ and our x^* by (1), we can estimate that

$$\#\{(\alpha, n) \in [\omega, \mu) \times \mathbb{N};\ |\langle x_{\alpha,n}, x^*\rangle| \ge \epsilon\} < \frac{1}{\epsilon} \cdot \kappa.$$

Finally, Lemma 1 completes the proof. ∎

Proof of Theorem 7. The sufficiency part is trivial. Let us prove the necessity. To achieve this, assume for simplicity that $\ell_p(\Gamma)$ is a dense subset of X and that $\|f\| \le \|f\|_{\ell_p}$ for every $f \in \ell_p(\Gamma)$. Fix any $x^* \in X^*$. Then the restriction $x^*{}_{|\ell_p(\Gamma)}$ lies in $\ell_p(\Gamma)^*$ $(\equiv \ell_q(\Gamma))$ where $q = \frac{p}{p-1}$. Thus the set $\{\gamma \in \Gamma;\ \langle e_\gamma, x^*\rangle \ne 0\}$ is at most countable which means that the set $\{e_\gamma;\ \gamma \in \Gamma\}$ countably supports all elements of X^*. Then we can apply Proposition 2 and get a separable PRI $(P_\alpha)_{\omega\le\alpha\le\mu}$ on X subordinated to the set $\widetilde{\Gamma} := \{e_\gamma;\ \gamma \in \Gamma\}$. Fix any $\alpha \in [\omega, \mu)$. Put $\widetilde{\Gamma}_\alpha = (P_{\alpha+1} - P_\alpha)\widetilde{\Gamma}$. Note that $\widetilde{\Gamma}_\alpha \subset \widetilde{\Gamma} \cup \{0\}$ and that $\widetilde{\Gamma}_\alpha$ is linearly dense in the (separable) subspace $(P_{\alpha+1} - P_\alpha)X$. By [F˜01, Theorem 6.41], we find a Markuševič basis $\{x_{\alpha,n}; x^*_{\alpha,n}\}_{n\in\mathbb{N}}$ in the subspace $(P_{\alpha+1} - P_\alpha)X$ such that $x^\alpha_n \in \operatorname{sp}\widetilde{\Gamma}_\alpha$ $(\subset \ell_p(\Gamma))$ and $\|x^\alpha_n\|_{\ell_p} = 1$ for every $n \in \mathbb{N}$. Define $Q_\alpha : X \to (P_{\alpha+1} - P_\alpha)X$ by $Q_\alpha x = (P_{\alpha+1} - P_\alpha)x,\ x \in X$. Performing this for every $\omega \le \alpha < \mu$, we get the system $\{\frac{1}{n} x_{\alpha,n}; n Q^*_\alpha x^*_{\alpha,n}\}_{n\in\mathbb{N},\ \omega\le\alpha<\mu}$, which will be a Markuševič basis in X, see, e.g, the proof of [Fa97, Proposition 6.2.4]. For every element $(a_{\alpha,m};\ \omega \le \alpha < \mu,\ m \in \mathbb{N})$ of $\ell_p([\omega, \mu) \times \mathbb{N})$, with finite support, we define

$$T(a_{\alpha,m}) = \sum_{m=1}^{\infty} \sum_{\omega\le\alpha<\mu} a_{\alpha,m} \tfrac{1}{m} x_{\alpha,m}.$$

This is a linear mapping from a dense subset of $\ell_p([\omega,\mu)\times \mathbb{N})$ into X. Now, using Hölder inequality and a disjoint support argument in the last of the following inequalities, we can estimate

$$\|T(a_{\alpha,m})\| \le \sum_{m=1}^{\infty} \frac{1}{m}\Big\| \sum_{\omega\le\alpha<\mu} a_{\alpha,m}x_{\alpha,m}\Big\|$$

$$\le \Big(\sum_{m=1}^{\infty}\frac{1}{m^q}\Big)^{\frac{1}{q}}\Big(\sum_{m=1}^{\infty}\Big\|\sum_{\omega\le\alpha<\mu} a_{\alpha,m}x_{\alpha,m}\Big\|^p\Big)^{\frac{1}{p}}$$

$$\le C\Big(\sum_{m=1}^{\infty}\Big\|\sum_{\omega\le\alpha<\mu} a_{\alpha,m}x_{\alpha,m}\Big\|_{\ell_p}^p\Big)^{\frac{1}{p}} \le C\Big(\sum_{m=1}^{\infty}\sum_{\omega\le\alpha<\mu}|a_{\alpha,m}|^p\Big)^{\frac{1}{p}} = C\|(a_{\alpha,m})\|_{\ell_p};$$

here we put $\Big(\sum_{m=1}^{\infty}\frac{1}{m^q}\Big)^{\frac{1}{q}} = C$. Therefore the mapping T can be extended to the whole space $\ell_p([\omega,\mu)\times\mathbb{N})$. Now, every canonical basic vector from this space is mapped by T to $\frac{1}{m}x_{\alpha,m}$ with a suitable $m\in\mathbb{N}$ and $\omega\le\alpha<\mu$. Therefore the range of T is dense in X and the proof is finished. ∎

Open problem. Characterize Banach spaces X such that every subspace of X is WCG.

Remark. It is likely that in this problem additional axioms of set theory may play a role (see, for example, the use of Martin's axiom in [Av05] and [MeSt]).

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