Function theory on infinite dimensional spaces

11-14 December 2007

Facultad de Ciencias Matemáticas

Universidad Complutense de Madrid

X

CONFERENCE

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Proceedings of
Function Theory on Infinite Dimensional Spaces X
Pages 31–35

A note on σ-finite dual dentability indices

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ABSTRACT

This short note announces a forthcoming paper [9] on the subject of characterizing Banach spaces admitting uniformly Gâteaux smooth equivalent norms in terms of σ-finite dual dentability indices.

Key words: Dentability indices, uniformly Gâteaux smooth norms, weak compactness, uniform Eberlein compacts.

2000 Mathematics Subject Classification: Primary: 46B03, 46B20, 46B26.

1. Introduction

Banach spaces admitting uniformly Gâteaux smooth equivalent norms were characterized in [4] as those having a uniform Eberlein compact dual unit ball (equipped with the weak*-topology). In terms of Walsh-Paley martingales (a device used by Enflo, James and Pisier in renormings of superreflexive Banach spaces by equivalent uniformly Fréchet smooth norms), it was done by Troyanski [16]. A different technique to deal with the superreflexive case was used by Lancien [11]. Here we use Lancien approach in the uniformly Gâteaux smooth case.
Our notation is standard. Let M be a bounded subset of X. Given $f \in X^*$, we denote $|f|_M := \sup_{x\in M} |f(x)|$ and, for a bounded set $S \subset X^*$, we let $\mathrm{diam}_M(S) := \sup\{|f-g|_M; f,g \in S\}$, the M-diameter of S. Let $\varepsilon > 0$ be given. We say that the

The first author was supported by grants AVOZ 101 905 03 and IAA 100 190 610, and by the Universidad Politécnica de Valencia. The second author was supported in part by Project MTM2005-08210 and the Universidad Politécnica de Valencia. The third author was supported by grants AVOZ 101 905 03 and IAA 100 190 502.

dual norm $\|\cdot\|$ on X^* is (M,ε)-*LUR* if $\limsup_n |f_n - f|_M \leq \varepsilon$ whenever $f, f_n \in S_{X^*}$ are such that $\lim_n \|f_n + f\| = 2$. The dual norm $\|\cdot\|$ on X^* is called σ-*LUR* if for every $\varepsilon > 0$, there is a decomposition $B_X = \bigcup_{k=1}^{\infty} M_k^{\varepsilon}$ such that $\|\cdot\|$ is $(M_k^{\varepsilon}, \varepsilon)$-LUR for every $k \in \mathbb{N}$. We say that the dual norm $\|\cdot\|$ on X^* is M-*LUR* if it is (M,ε)-LUR for every $\varepsilon > 0$. The dual norm $\|\cdot\|$ on X^* is called *weak**-*LUR* if it is M-LUR for every finite subset M of X. We say that the norm $\|\cdot\|$ on X is M-*uniformly Gâteaux smooth* if $\lim_n |f_n - g_n|_M = 0$ whenever $f_n, g_n \in S_{X^*}$ are such that $\lim_n \|f_n + g_n\| = 2$. We say that the norm $\|\cdot\|$ on X is *strongly uniformly Gâteaux smooth* if it is M-uniformly Gâteaux smooth for some bounded linearly dense set M in X. Using the Šmulyan duality (see, e.g., [2, Section I.1]), we can also define that $\|\cdot\|$ on X is *uniformly Gâteaux smooth* [2, Definition II.6.5] if it is M-uniformly Gâteaux smooth for every finite subset M of X [2, Lemma II.6.6].

The notion of dual σ-LUR norms represents a sort of a common roof over uniformly Gâteaux smooth and Fréchet smooth norms (see Theorem 2 and Theorem 4 below). It is closely related to weak compactness (see [6] and [7]). In particular, the existence of such a norm in a weakly Lindelöf determined space implies that this space is necessarily a subspace of a weakly compactly generated space [6]. We recall that a Banach space X is *weakly Lindelöf determined* if (B_{X^*}, w^*) is a Corson compact space (for definitions see, e.g., [2, Chapter VI], [3], and [5, Chapter 12]). By a *weak**-*slice* of a set $D \subset X^*$ we understand the intersection of D with a weak*-open halfspace in X^*. Given a bounded set $M \subset X$, $\varepsilon > 0$, and $D \subset B_{X^*}$, we introduce the (M,ε)-*dentability derivative of* D by

$$D'_{(M,\varepsilon)} = \{f \in D;\ \operatorname{diam}_M(S) \geq \varepsilon \text{ for each weak*-slice } S \text{ of } D \text{ containing } f\}$$

Let $\alpha > 1$ be an ordinal number and assume that we already defined a dentability derivatve $D^{(\beta)}_{(M,\varepsilon)}$ for every ordinal $\beta < \alpha$. If $\alpha - 1$ exists, we define the α-th (M,ε)-*dentability derivative of* D as $D^{(\alpha)}_{(M,\varepsilon)} = (D^{(\alpha-1)}_{(M,\varepsilon)})'_{(M,\varepsilon)}$. Otherwise, we put $D^{(\alpha)}_{(M,\varepsilon)} = \bigcap_{\beta<\alpha} D^{(\beta)}_{(M,\varepsilon)}$. We observe a simple fact that, if D is convex and weak*-closed, then so is $D'_{(M,\varepsilon)}$.

Definition 1. Let $(X, \|\cdot\|)$ be a Banach space. Let a bounded set $M \subset X$ and $\varepsilon > 0$ be given. We say that M *has finite* (resp. *countable*) ε-*dual index* if $(B_{X^*})^{(\alpha)}_{(M,\varepsilon)} = \emptyset$ for some finite (resp. countable) ordinal number α. The first ordinal with this property, if it exists, is called *the* ε-*dual index of* M.

We say that a Banach space $(X, \|\cdot\|)$ *has* σ-*finite* (resp. σ-*countable*) *dual index* if, for every $\varepsilon > 0$, there is a decomposition $B_X = \bigcup_{k=1}^{\infty} M_k^{\varepsilon}$ such that each set M_k^{ε} has finite (resp. countable) ε-dual index.

2. The results

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space. Then the following assertions are equivalent.*

(i) *X admits an equivalent uniformly Gâteaux smooth norm.*

(ii) *X has σ-finite dual index.*

Theorem 3. *Let $(X, \|\cdot\|)$ be a Banach space. Then the following assertions are equivalent.*
(i) *X admits an equivalent strongly uniformly Gâteaux smooth norm.*
(ii) *There exists a bounded linearly dense set $M \subset X$ that has finite ε-dual index for every $\varepsilon > 0$.*

Theorem 4. *Assume that X has σ-countable dual index. Then X^* admits an equivalent dual σ-LUR, and hence weak*-LUR norm.*

Theorem 5. *Assume that a bounded set M in a Banach space X has countable ε-dual index for every $\varepsilon > 0$. Then X^* admits an equivalent dual M-LUR norm.*

Examples and remarks

As it is usual, $\langle P \rangle$ denotes that a Banach space has an equivalent norm with property P. If this concerns a dual space X^*, $\langle P \rangle^*$ denotes that the equivalent norm on X^* is, moreover, a dual norm. The following diagram (see Figure 1) summarizes some of the information given by the former results, and establish some connections among them. The thick-boxed examples justify that the broken-line implications do not hold (this is emphasized by a cross). The description of those examples is provided below. The meaning of the acronyms should be clear from the context. For example, SUG means *strongly uniformly Gâteaux*, and so on.

(i) A Banach space X is said to be *strongly generated by a Banach space Z* if there exists a bounded linear operator $T : Z \to X$ such that, for every weakly compact subset M of X and for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $M \subset nT(B_Z) + \varepsilon B_X$ (see [15]). Every Banach space strongly generated by a superreflexive Banach space admits an equivalent norm that is M-uniformly Gâteaux smooth for every weakly compact set $M \subset X$ (see, e.g., [8]); thus such a norm is then uniformly Gâteaux smooth. For a finite measure μ, the space $L_1(\mu)$ is strongly generated by the Hilbert space $L_2(\mu)$. Let X_0 be the Rosenthal subspace of $L_1(\mu)$, for a certain finite measure μ, that is not weakly compactly generated ([14]). By Theorem 2, X_0 has σ-finite dual index. The space X_0 is weakly Lindelöf determined as it is a subspace of the weakly compactly generated space $L_1(\mu)$ (see, e.g., [5, Chapters 11 and 12]). Assume that X_0 contained a bounded linearly dense set M that had countable ε-dual index for every $\varepsilon > 0$. By Theorem 5, ${X_0}^*$ would then admit an equivalent dual M-locally uniformly rotund norm. Thus X_0 would be weakly compactly generated ([6, Theorem 1]). Therefore, *X_0 is a space that has σ-finite dual index but for no $\varepsilon > 0$, X_0 contains a bounded linearly dense set having countable ε-dual index.*

(ii) Let X be the Ciesielski-Pol space $C(K)$, where K is a scattered compact of finite height (see e.g., [2, Chapter VI]). Thus B_X has countable ε-dual index for every $\varepsilon > 0$ ([12]). However, X does not admit any equivalent uniformly Gâteaux smooth norm. Indeed, otherwise X would be a subspace of a weakly compactly generated Banach space ([4], see, e.g., [5, Theorem 12.18]). However, this is not the case as there is no bounded linear injection of X into any $c_0(\Gamma)$ ([2, Chapter VI]). Thus *the Ciesielski-Pol space does not have σ-finite dual index* by Theorem 2. This space is somehow an optimal example. Indeed, for every $\varepsilon > 0$ the ε-dual index of B_X is not only countable but it is also the smallest

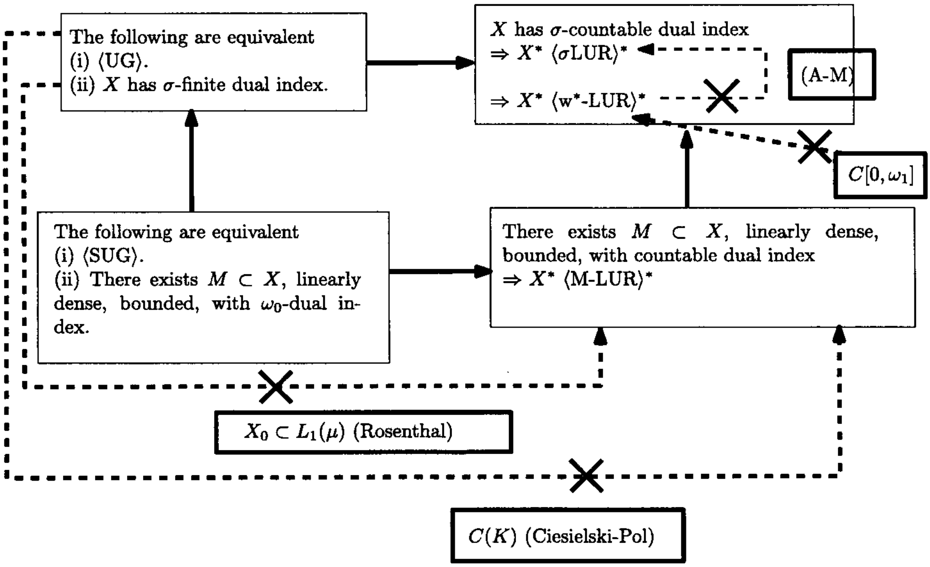

Figure 1: Some connections and counterexamples

possible for a space that does not have a σ-finite dual index. Namely, we have that

$$\sup\left\{\alpha;\ \alpha \text{ is the dual index of } B_X\ \right\} < \omega^2.$$

This follows from the separable determination of this index and from a computation made by P. Hájek and G. Lancien in [10].

(iii) The space X in [1, page 421] admits a dual weak*-LUR norm ([13]) but does not have σ-countable dual index. Indeed, otherwise, it would admit an equivalent dual σ-LUR norm by Theorem 4. Thus X would be a subspace of a weakly compactly generated space as X is weakly Lindelöf determined ([6]). However, as it is proved in [1], X is not a subspace of a weakly compactly generated space.

(iv) If M is the unit ball of the space $C[0,\omega_1]$, then for every $\varepsilon > 0$ there is an ordinal α such that $(B_{X^*})^{(\alpha)}_{(M,\varepsilon)} = \emptyset$. This is so as $C[0,\omega_1]$ is an Asplund space (see, e.g., [2, Theorem 12.29]), and hence its dual is weak* dentable. However, $C[0,\omega_1]$ does not have σ-countable dual index as otherwise $C[0,\omega_1]$ would admit an equivalent dual strictly convex norm by Theorem 3, which is not the case by a classical Talagrand's result (see, e.g., [2, page 313]).

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