On a problem of Namioka on norm-attaining functionals

María D. Acosta* and Vicente Montesinos†

Abstract

We prove that every Banach space containing an isomorphic copy of ℓ_1 can be renormed in such a way that, in the new norm, the set of norm-attaining functionals has an empty norm-interior. As a consequence, we prove the rightness of a conjecture of Isaac Namioka in a wide class of Banach spaces containing, for example, the weakly compactly generated ones.

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In [AR1] and [AR2] some isometric conditions were provided (mainly smoothness) such that if a Banach space satisfies one of these assumptions and the set of norm attaining functionals has a non empty norm-interior, then the space has to be reflexive.

In 1999 Isaac Namioka posed the following problem:
Question. *Assume that X is a Banach space such that for every equivalent norm, the set of norm attaining functionals has a nonempty norm-interior. Does X have to be reflexive?*

In [AR1, Corollary 7] it was proved that a separable Banach space that is not weakly sequentially complete, admits an equivalent norm for which the set of norm attaining functionals has an empty norm-interior. This result gives a partial answer to the previous Question. However, there are even

*Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain. dacosta@ugr.es. Supported in part by D.G.E.S., Project BFM 2003-01681 (Spain).

†Instituto de Matemática Pura y Aplicada. Universidad Politécnica de Valencia. vmontesinos@mat.upv.es. Research supported in part by Project BMF2002-01423 (Spain) and the Universidad Politécnica de Valencia.

some classical spaces which are weakly sequentially complete, non reflexive, and it was not known whether or not they can be renormed to fulfill the mentioned property.
We will prove that any space containing an isomorphic copy of ℓ_1 can be renormed such that the set of norm attaining functionals has an empty norm-interior. As a consequence of this fact and the result in [AR1, Corollary 7] quoted above, we answer in the positive Namioka's question in the class of separable Banach spaces and, more generally, in the class of Banach spaces with the separable complementation property.

In the following, for a Banach space $(X, \|\cdot\|)$, $B_{(X,\|\cdot\|)}$ will be its closed unit ball, $S_{(X,\|\cdot\|)}$ its unit sphere and X^* the topological dual of X. $NA(X, \|\cdot\|)$ will denote the set of norm attaining functionals, i.e.,

$$NA(X, \|\cdot\|) := \{x^* \in X^* : \text{there exists } x \in S_{(X,\|\cdot\|)}, \langle x, x^* \rangle = \|x^*\|\},$$

a set norm-dense in X^*, according to Bishop-Phelps Theorem.

Let's begin by renorming ℓ_1 with a norm having the sought property.

Theorem 1 *There is a Banach space $(X, \|\cdot\|)$ isomorphic to $(\ell_1, \|\cdot\|_1)$ such that $NA(X, \|\cdot\|)$ has an empty norm-interior.*

Proof:
Let

$$W := \left\{ \sum_{n=1}^{\infty} \lambda_n \frac{1}{2^n} e_{\sigma(n)} : |\lambda_n| = 1, \quad \forall n, \quad \sigma : \mathbb{N} \longrightarrow \mathbb{N} \text{ injective} \right\} \subset S_{(\ell_1, \|\cdot\|_1)}, \tag{1}$$

where e_n denotes the n-th vector of the canonical basis of ℓ_1, and put

$$A := \overline{\text{conv}}^{w^*}(W),$$

where w^* denotes the topology $\sigma(\ell_1, c_0)$ of the pointwise convergence on the elements of c_0. Since ℓ_1 has the Radon-Nikodým property, then $A = \overline{\text{conv}}^{\|\cdot\|_1}(\text{Ext}(A))$, where $\text{Ext}(A)$ denotes the set of extreme points of A. In view of the reverse Krein-Milman Theorem, $\text{Ext}(A) \subset \overline{W}^{w^*}$. Observe that

$$\overline{W}^{w^*} = \left\{ \sum_{n \in M} \lambda_n \frac{1}{2^n} e_{\sigma(n)} : M \subset \mathbb{N}, |\lambda_n| = 1, \quad \forall n \in M, \quad \sigma : M \longrightarrow \mathbb{N} \text{ injective} \right\}.$$

It is then easy to check that

$$\operatorname{Ext}(A) \subset \overline{W}^{w^*} \cap S_{\ell_1} = W \tag{2}$$

and so, $A = \overline{\operatorname{conv}}^{\|\cdot\|_1}(W)$.
We will prove that the space $X = \ell_1$ with the equivalent norm $\|\cdot\|$ whose closed unit ball is given by the set B, where

$$B := B_{\ell_1} + A,$$

satisfies the desired statement.
First of all, B is a w^*-compact subset of ℓ_1, which is also convex, circled and contains B_{ℓ_1}, hence B is the closed unit ball of an equivalent dual norm $\|\cdot\|$ in ℓ_1.
From this point on we use the identification $X^* \equiv \ell_\infty$. We will denote also by $\|\cdot\|$ the dual norm of $\|\cdot\|$ on X^*. Being $\|\cdot\|$ in X^* a supremum on the sum of two sets, we have, for $x^* = (x^*_n) \in X^*$,

$$\|x^*\| = \|x^*\|_\infty + \sup\left\{\sum_{n=1}^{\infty} \frac{1}{2^n}|x^*_{\sigma(n)}| : \sigma : \mathbb{N} \longrightarrow \mathbb{N} \text{ injective}\right\}.$$

Since $B = B_{\ell_1} + A$, a functional $x^* \in X^*$ attains $\|x^*\|$ if, and only if, $\sup|\langle B_{\ell_1}, x^*\rangle|$ and $\sup|\langle A, x^*\rangle|$ are attained. It is known that a functional x^* on $(\ell_1, \|\cdot\|_1)$ attains its norm, if, and only if, $\sup_n |x^*_n|$ is attained at some n. We will show that x^* *attains its supremum on* A *if, and only if, the set*

$$M := \{m \in \mathbb{N} : |x^*_m| \geq \limsup |x^*_n|\}$$

is infinite.
If M is infinite, it is very easy to check that there is an injective mapping $\tau : \mathbb{N} \longrightarrow \mathbb{N}$ such that $M = \tau(\mathbb{N})$ and $|x^*_{\tau(n)}| \geq |x^*_{\tau(n+1)}|$ for every n. Then, if we fix an injective mapping $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ we get that, for every $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{1}{2^k}|x^*_{\sigma(k)}| \leq \sum_{k=1}^{n} \frac{1}{2^k}|x^*_{\tau(k)}| \leq \sum_{n=1}^{\infty} \frac{1}{2^n}|x^*_{\tau(n)}|.$$

Clearly, the supremum of x^* on A is attained at

$$a_0 := \sum_{n=1}^{\infty} \frac{1}{2^n}\lambda_n e_{\tau(n)},$$

where $\{\lambda_n\}$ is a choice of normalized scalars satisfying that

$$\lambda_n x^*_{\tau(n)} = |x^*_{\tau(n)}|, \qquad \forall n \in \mathbb{N}.$$

We proved that $x^* \in NA(X, \|\cdot\|)$ if the set M is infinite.
Suppose now that M is finite, say $M := \{n_1, \ldots, n_k\}$ where $|x^*_{n_1}| \geq \ldots \geq |x^*_{n_k}|$. Assume that x^* attains the supremum at A.
Every closed convex subset of ℓ_1 has the Radon-Nikodým property, hence the Krein-Milman property. In particular, so it does the face $A(x^*)$ of A defined by x^*, a non-empty set by assumption. Every extreme point of $A(x^*)$ is an extreme point of A, hence an element in W by (2).
Let $s \in W$ be an element where x^* attains its supremum. As W is symmetric, we can assume that all coordinates involved are non-negative. We have necessarily weights $1/2, \ldots, 1/2^k$ at $n_1, \ldots, n_k$, respectively. As regard to the rest, put n for the coordinate of s with weight $1/2^{k+1}$ and find $p \in \mathbb{N} \setminus M$ such that $|x^*_p| > |x^*_n|$ and $r > k+1$, where $1/2^r$ is the weight at p (such a p exists in our situation). Keep all weights in place but at n and p, where they are interchanged, defining a new element of W. Then, clearly,

$$\frac{1}{2^{k+1}}|x^*_n| + \frac{1}{2^r}|x^*_p| < \frac{1}{2^{k+1}}|x^*_p| + \frac{1}{2^r}|x^*_n| \ .$$

This contradicts the sup-attaining at s.
We just proved that in case that M is finite, x^* does not attain its supremum on A and so x^* does not attain its norm on $(X, \|\cdot\|)$.
Clearly the subset of elements of $X^* = \ell_\infty$ given by

$$\Big\{ z = (z_n) \in \ell_\infty : \{m \in \mathbb{N} : |z_m| \geq \limsup |z_n|\} \ \text{ is } \ \text{infinite} \Big\},$$

i.e., $NA(X\|\cdot\|)$, has an empty norm-interior. ∎

The following result extend the previous one to Banach spaces containing an isomorphic copy of ℓ_1.

Theorem 2 *Let $(X, \|\cdot\|)$ be a Banach space with an isomorphic copy of ℓ_1. Then X has an equivalent norm p such that the set of norm attaining functionals $NA(X, p) \subset X^*$ has an empty norm-interior.*

Proof. Consider ℓ_1 as a subspace (algebraically) of X, so $\|\cdot\|$ induces on ℓ_1 a norm (again denoted by $\|\cdot\|$) which is equivalent to $\|\cdot\|_1$, the canonical norm of ℓ_1. Then $(\ell_1, \|\cdot\|)^* = (\ell_\infty, \|\cdot\|)$, where $\|\cdot\|$ denotes also the dual norm both on X^* and on ℓ_∞ (in the last case, a norm equivalent to $\|\cdot\|_\infty$).

Denote by $q : X^* \longrightarrow \ell_\infty$ the quotient mapping. Now, $c_0 \subset \ell_\infty$ is a norming subspace for $(\ell_1, \|\cdot\|)$ (not necessarily 1-norming), hence, by [FHH, Exercise V.5.22], $N := q^{-1}(c_0)$ is a norming subspace of X^* for $(X, \|\cdot\|)$, in particular w^*-dense in X^*. We can now define on X an equivalent norm $|\cdot|$ in such a way that N is 1-norming for $(X, |\cdot|)$; precisely, $B(X, |\cdot|) := \overline{B(X, \|\cdot\|)}^{\sigma(X,N)}$. The topology $\sigma(X, N)$ on X of the pointwise convergence on N obviously induces on ℓ_1 the topology $\sigma(\ell_1, c_0)$. Let $|||\cdot|||$ be an equivalent norm on ℓ_1 such that the set $NA(\ell_1, |||\cdot|||)$ has an empty interior. Such a norm exists by the previous Proposition and it is a dual norm. We may and do assume that $B(\ell_1, |||\cdot|||) \supset B(X, |\cdot|) \cap \ell_1$. Observe that $B(\ell_1, |||\cdot|||)$ is $\sigma(\ell_1, c_0)$-compact (and so $\sigma(X, N)$-compact), and that $B(X, |\cdot|)$ is $\sigma(X, N)$-closed. It is trivial then that

$$W := B(\ell_1, |||\cdot|||) + B(X, |\cdot|)$$

is a bounded absolutely convex and $\sigma(X, N)$-closed subset of X containing the closed unit ball $B(X, |\cdot|)$, and so it is the closed unit ball of an equivalent norm p on X. Now, let $x^* \in NA(X, p)$. It is clear that its restriction $q(x^*)$ to ℓ_1 belongs to $NA(\ell_1, |||\cdot|||)$. Assume for a moment that $NA(X, p)$ had a non-empty interior. The restriction mapping $q : (X^*, p) \to (\ell_\infty, |||\cdot|||)$ is continuous and onto, then an open mapping, taking open sets onto open sets. We should have then that $NA(\ell_1, |||\cdot|||)$ has a non-empty interior, a contradiction. ∎

We can prove now the validity of Namioka's conjecture mentioned in the introduction in the case of separable Banach spaces.

Theorem 3 *If a separable Banach space X is not reflexive then it has an equivalent norm $|||\cdot|||$ such that $NA(X, |||\cdot|||)$ has an empty norm-interior.*

Proof:
Assume that X is not reflexive. If X is not weakly sequentially complete then, by [AR1, Corollary 7], there exists such a norm $|||\cdot|||$. If, on the contrary, X is weakly sequentially complete, we have two possibilities: 1) Every bounded sequence (x_n) in X has a weakly Cauchy subsequence. This obviously implies that X is reflexive, a contradiction. 2) There exists a bounded sequence (x_n) in X with no weakly Cauchy subsequence. According to Rosenthal's dichotomy [Ro], there exists a subsequence equivalent to the canonical ℓ_1-basis, so X contains an isomorphic copy of ℓ_1 and, by Theorem 2, X has such a norm $|||\cdot|||$. ∎

The spaces which can be renormed such that the set of norm attaining functionals has a nonempty norm-interior have the following stability property:

Proposition 4 *Assume that* $(X, \|\cdot\|)$ *is a Banach space and* $Y \subset X$ *is a complemented subspace of* $(X, \|\cdot\|)$ *such that* Y *admits an equivalent norm* $|||\cdot|||$ *satisfying that* $NA(Y, |||\cdot|||)$ *has an empty norm-interior. Then* $|||\cdot|||$ *can be extended to a norm* $|||\cdot|||$ *in* X *with the same property.*

Proof:
Let us consider the decomposition $X = Y \oplus M$. Define a norm on X by

$$|||y + m||| := \max\{|||y|||, \|m\|\} \qquad (y \in Y, m \in M).$$

This norm induces $|||\cdot|||$ on Y. Of course, $X^* = Y^* \oplus M^*$ and the dual norm in given by

$$|||y^* + m^*||| = |||y^*||| + \|m^*\| \qquad (y^* \in Y^*, m^* \in M^*),$$

where we used the same symbol to denote a norm and the corresponding dual norm. A functional $x^* = y^* + m^*$ attains its norm if, and only if, both y^* and m^* attain their corresponding norm, that is

$$NA(X, |||\cdot|||) = NA(Y, |||\cdot|||) + NA(M, |||\cdot|||).$$

Since any ball in X^* contains a product of balls of Y^* and M^* and, by assumption, $NA(Y, |||\cdot|||)$ has an empty norm-interior, then the subset $NA(X, |||\cdot|||)$ has also an empty norm-interior. ■

Recall that a Banach space X is *weakly Lindelöf determined* (in short, WLD), if (B_{X^*}, w^*) is a *Corson compact*, i.e., a compact subspace of a product of lines such that every element has only a countable number of non-zero coordinates. Every *weakly compactly generated* Banach space (i.e., a Banach space with a weakly compact linearly dense subset) is WLD. It is well known that a WLD Banach space X has the *separable complementation property*, i.e., the fact that for every separable subspace $Y \subset X$ there exists a separable space Z such that $Y \subset Z \subset X$ and Z is complemented in X (for these concepts and properties see, for example, [Fa, Chap. 7] and [FHH, Chap. 12]).
The following result extends the class of spaces where Namioka's conjecture holds true:

Theorem 5 *If a non-reflexive Banach space* X *has the separable complementation property, in particular, if* X *is WLD, then it has an equivalent norm* $|||\cdot|||$ *such that* $NA(X, |||\cdot|||)$ *has an empty norm-interior.*

Proof:
If X is not reflexive, it contains, by the Eberlein-Šmulyan Theorem, a non-reflexive and separable closed subspace Y. There exists then a complemented and separable subspace Z containing Y, so in particular Z is not reflexive either. It follows from Theorem 3 that Z can be renormed with a norm $|||\cdot|||$ such that $NA(Z, |||\cdot|||)$ has an empty norm-interior. We apply now Proposition 4 to define a norm $|||\cdot|||$ on X with the same property. ∎

Remark. There are Banach spaces X with the separable complementation property and without a Projectional Resolution of the Identity; in particular, they are not WLD, see [DGZ, Definition VI.1.1 and Example VI.8.6].

Question. Is Namioka's conjecture true in the case of a general (non-separable) Banach space?

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