Convex-compact sets and Banach discs

I. Monterde* and V. Montesinos†

Abstract

Every relatively convex-compact convex subset of a locally convex space is contained in a Banach disc. Moreover, an upper bound for the class of sets which are contained in a Banach disc is presented. If the topological dual E' of a locally convex space E is the $\sigma(E', E)$-closure of the union of countably many $\sigma(E', E)$-relatively countably compacts sets, then every weakly (relatively) convex-compact set is weakly (relatively) compact.

1 Introduction

The possibility to include a certain bounded subset A of a locally convex space $(E, \mathcal{T})$ in a *Banach disc* (i.e., a bounded absolutely convex set in $(E, \mathcal{T})$ such that $E_A := \bigcup_n nA$, endowed with the norm $\|\cdot\|_A$ given by the Minkowski gauge of A, is a Banach space) has a big impact on its structure (in particular, the set A becomes *strongly bounded*, i.e., bounded on bounded subsets of the space $(E', \sigma(E', E))$ —the topological dual E' of $(E, \mathcal{T})$ endowed with the topology $\sigma(E', E)$ of the pointwise convergence on all points in E), and is the basic fact in the proof of the important Banach-Mackey theorem (see, for example, [4, §20.11(3)]). It is then convenient to be able to check if this happens with a minimum of requirements. This is so for sequentially complete absolutely convex bounded subsets of a locally convex space ([4, §20.11(2)]) and for convex relatively countably compact subsets ([2, p. 17]).

Let $(E, \mathcal{T})$ be a locally convex space. An *adherent point of a filter* $(F_i)_I$ *in* E is an element in $\bigcap_I \overline{F_i}$. An *adherent point of a net* (x_i) in E is an adherent point (in the former sense) of the filter $(F_i := \{x_j : j \geq i\})$.

We collect in the following definition several of the most useful concepts when dealing with compactness in a general locally convex space.

Definition 1 *A subset A of a locally convex space $(E, \mathcal{T})$ is said to be*

*Universidad Politécnica de Valencia, nachomonterde@gmail.com

†Instituto de Matemática Pura y Aplicada. Universidad Politécnica de Valencia. vmontesinos@mat.upv.es. Research supported in part by Project MTM2005-08210 (Spain) and the Universidad Politécnica de Valencia.

Keywords: weakly compact sets, convex-compact sets, Banach discs.

- (relatively) countably compact ((R)NK) *if every sequence of points in A has an adherent point in A (in $\overline{A}$).*
- (relatively) sequentially compact ((R)SK) *if every sequence in A has a subsequence which converges to a point in A (in $\overline{A}$).*
- (relatively) compact ((R)K) *if every net in A has an adherent point in A (in $\overline{A}$)*
- (relatively) convex-compact ((R)CK) *if the following holds: suppose that $K_1 \supset K_2 \supset \dots$ is a decreasing sequence of closed convex subsets of E for which all the intersections $K_n \cap A$ are non-empty; then the sequence $(K_n \cap A)$ has an adherent point in A (in $\overline{A}$).*

Obviously, (R)K sets are (R)NK and (R)SK sets are (R)NK, too. It is easy to prove (see, for example, [4, §24.3(3)]) that every (R)NK set is (R)CK. The converse does not hold. A RCK set is always bounded ([4, §24.3(3)]). The closure of a RCK set does not need to be CK (see Example 9 below). The concept of (R)CK is due to Šmulian (see references in [1, Ch. III, §2]).
As we mentioned before, the following result holds:

Theorem 2 ([2], p.17) *Every convex RNK subset A of a locally convex space E is contained in a Banach disk $U \subset E$.*

In this paper, we extend this result to the class of RCK sets. We provide also an upper bound for classes of sets which are contained in a Banach disc together with some other results about CK sets; in particular, we prove that $\sigma(E,E')$-(R)CK implies $\sigma(E,E')$-(R)K when there is a sequence of $\sigma(E',E)$-RNK subsets of E' whose union is $\sigma(E',E)$-dense in E' (in particular, if $(E,\mathcal{T})$ has a coarser metrizable locally convex topology).

2 Banach discs

The following result is well known, so its proof will be omitted.

Theorem 3 *Let A be a bounded subset of a complete locally convex space E. Then, the map*

$$T : \ell_1(A) \longrightarrow E$$

given by

$$(\alpha_a)_{a\in A} \xrightarrow{T} \sum_{a\in A} \alpha_a\, a$$

is well defined and continuous and $D := T(B_{\ell_1(A)})$ is a Banach disc.

The following result extends [4, §20.11(2)] and [2, p.17, Lemma] to the class of convex and CK subsets of an arbitrary locally convex space.

Theorem 4 *Every convex, RCK subset A of a locally convex space $(E,\mathcal{T})$ is contained in a Banach disc $D \subset E$.*

Proof Let $\tilde{E}$ be the completion of E and D the Banach disc in $\widetilde{E}$ constructed in Theorem 3. We shall prove that, in fact, $D \subset E$. This will conclude the proof. To that end, let us denote by $\widetilde{E}_D$ the Banach space generated by D in $\widetilde{E}$ and let $\|\cdot\|_D$ be its norm. Given $a \in D$, it can be written as $a = \sum_{i=1}^{\infty} \alpha_i\, a_i$ (the sum converges in $\|\cdot\|_D$ and, in particular, also in $\mathcal{T}$), where $a_i \in A$, $\alpha_i \neq 0$ for every i and $\sum_{i=1}^{\infty} |\alpha_i| \leq 1$. We can split this sum as

$$a = \underbrace{\sum_{i=1}^{\infty} \beta_i\, b_i}_{b} - \underbrace{\sum_{i=1}^{\infty} \gamma_i\, c_i}_{c},$$

where $\beta_i > 0$, $\gamma_i > 0$, $b_i \in A$ and $c_i \in A$.
Let $s_n = \sum_1^n \beta_i$, $s = \sum_1^{\infty} \beta_i$ and $x_n = (1/s_n) \sum_1^n \beta_i\, b_i$. Then $x_n \in A$ and (x_n) $\|\cdot\|_D$-tends to $(1/s)\, b \in \widetilde{E}$.
Let K_n be the sequence of closed convex sets in $(E, \mathcal{T})$ defined as

$$K_n = \overline{\mathrm{conv}} \left[\{x_i : i \in \mathbb{N}\} \bigcap \left(\frac{1}{s}\, b + \frac{1}{n}\, D \right) \right]$$

Thus, $K_1 \supset K_2 \supset \cdots$ and $K_n \cap A \neq \emptyset$ (observe that D contains the open unit ball in the norm $\|\cdot\|_D$). Therefore, there exists $x \in E$ such that $x \in \bigcap_1^{\infty} \overline{(K_n \cap A)}$. Let $U(0)$ be any closed neighborhood of 0 in $(E, \mathcal{T})$. By the fact that D is bounded, there exists $n \in \mathbb{N}$ such that $(1/n)\, D \subset U(0)$. Then (all closures taken in $(E, \mathcal{T})$),

$$x \in \overline{K_n \cap A} \subset K_n \subset \overline{\mathrm{conv}}\, [(1/s)\, b + (1/n)\, D] =$$

$$= \overline{[(1/s)\, b + (1/n)\, D]} \subset \overline{[(1/s)\, b + U(0)]} = (1/s)\, b + U(0).$$

Therefore, $x = (1/s)\, b$, so $b \in E$. Analogously, $c \in E$. This implies, finally, that $a \in E$. ∎

If A is absolutely convex, we can be a little bit more precise, since we have $D = A$ if A is CK. In case that A is just RCK, we can only say that $A \subseteq D \subseteq \overline{A}$.

Corollary 5 *Let A be an absolutely convex, (R)CK subset of a locally convex space. Then A is a Banach disc (A is contained in a Banach disc D such that $D \subseteq \overline{A}$).*

Since convex RCK sets are contained in a Banach disc, we can use, for example, [4, §20.11(3)] to conclude the following result.

Corollary 6 *Every convex, RCK subset A of a locally convex space $(E, \mathcal{T})$ is strongly bounded, i.e., $\sup_{u \in B,\ x \in A} |u(x)| < \infty$, for each $\sigma(E', E)$-bounded set $B \in E'$.*

Further criteria for weak compactness use, for example, the interchangeable limit condition, as in [5] and [3]. Given a locally convex space $(E, \mathcal{T})$, we say that two sets, $A \subset E$ and $B \subset E'$, *interchange limits* (and we write $A \sim$

B) if $\lim_n \lim_m \langle x_n, x'_m \rangle = \lim_m \lim_n \langle x_n, x'_m \rangle$ whenever (x_n) (resp. (x'_m)) is a sequence in A (resp. in B) such that both iterated limits exists. Let $\mu(E, E')$ be the *Mackey topology on* E, i.e., the topology on E of the uniform convergence on the family of all absolutely convex and $\sigma(E', E)$ compact subsets of E'. A central result in [3] is that *a bounded subset of a* $\mu(E, E')$*-quasicomplete locally convex space* E *is* $\sigma(E, E')$*-RK if and only if it interchanges limits with every absolutely convex* $\sigma(E', E)$*-K subset of* E'. If A is RCK then $A \sim B$ for every absolutely convex and $\sigma(E', E)$-K subset of E'. This can be easily deduced from the following fact. Here, E'^* denotes the algebraic dual of the topological dual E' of E.

Lemma 7 *Every (R)CK set* A *in a locally convex space* $(E, \mathcal{T})$ *satisfies the following property: for every sequence* (f_n) *in* E' *and for every element* $z \in \overline{A}^{(E'^*, \sigma(E'^*, E'))}$*, there exists* $a \in A$ $\left(\in \overline{A}^{(E, \sigma(E, E'))}\right)$ *such that* $\langle z - a, f_n \rangle = 0$ *for all* $n \in \mathbb{N}$.

This can be proved just by considering the decreasing sequence of closed convex sets $K_n := \{x \in E : \sup\{|\langle z - x, f_i \rangle| : i = 1, 2, \ldots, n\} \leq 1/n\}$.
With the following example we bound the class of sets in $(E, \mathcal{T})$ which are included in a Banach disc.

Example 8 *There exists a locally convex space* $(E, \mathcal{T})$ *and a bounded subset of* E *interchanging limits with every absolutely convex* $\sigma(E', E)$*-K subset of* E' *and yet not included in a Banach disc.*

Proof. Let $(E, \mathcal{T}) := (\ell_1, \sigma(\ell_1, \varphi))$, where $\varphi \subset \ell_\infty$ is the linear space of all eventually zero sequences (so $\sigma(\ell_1, \varphi)$ is the topology on ℓ_1 of the pointwise convergence) and let $A := \prod_{n=1}^{\infty} [-n, n] \cap \ell_1$, a convex and bounded subset of E.
Observe that A is not $\beta(\ell_1, \varphi)$-bounded, where $\beta(\ell_1, \varphi)$ denotes the *strong* topology on ℓ_1 for the dual pair $\langle \ell_1, \varphi \rangle$, i.e., the topology of the uniform convergence on all the $\sigma(\varphi, \ell_1)$-bounded subsets of φ. In order to see this, notice that the set $M := [-1, 1]^{\mathbb{N}} \cap \varphi$ is $\sigma(\varphi, \ell_1)$-bounded and yet $\sup\{\langle n e_n, e_n \rangle : n \in \mathbb{N}\} = +\infty$, where e_n is the n-th vector of the canonical basis of ℓ_1.
We shall prove that $A \sim U$ for every absolutely convex and $\sigma(\varphi, \ell_1)$-compact subset of φ. The set U is $\beta(\varphi, \ell_1)$-bounded by the Banach-Mackey theorem (see, for example, [4, §20.11(3)]) The topology $\beta(\varphi, \ell_1)$ is compatible with the dual pair $\langle \mathbb{R}^{\mathbb{N}}, \varphi \rangle$ (this can be seen as follows: given $x := (x_n) \in \mathbb{R}^{\mathbb{N}}$, the sequence $(\sum_{k=1}^{n} x_k e_k)_n$ is in ℓ_1 and $\sigma(\mathbb{R}^{\mathbb{N}}, \varphi)$-converges to x, so x is in the $\sigma(\mathbb{R}^{\mathbb{N}}, \varphi)$-closure of a $\sigma(\ell_1, \varphi)$-bounded subset of ℓ_1). It follows then that U lies in a finite-dimensional subspace of φ, say $\operatorname{span}\{w_i : i = 1, 2, \ldots, k\}$. Assume now that for two sequences (a_m) in A and (u_n) in U the iterated limits

$$\lim_n \lim_m \langle a_m, u_n \rangle, \quad \lim_m \lim_n \langle a_m, u_n \rangle$$

exists. Put $u_n := \sum_{i=1}^{k} \lambda_i^n w_i$, $n \in \mathbb{N}$, where λ_i^n are real numbers. Let $u_0 := \sum_{i=1}^{k} \lambda_i^0 w_i$ be a $\sigma(\varphi, \ell_1)$-adherent point of the sequence (u_n) and $a_0 \in \mathbb{R}^{\mathbb{N}}$ a

$\sigma(\mathbb{R}^{\mathbb{N}}, \varphi)$-adherent point of the sequence (a_n). It follows that

$$\lim_n \lim_m \langle a_m, u_n \rangle = \lim_n \langle a_0, u_n \rangle, \quad \lim_m \lim_n \langle a_m, u_n \rangle = \lim_m \langle a_m, u_0 \rangle = \langle a_0, u_0 \rangle.$$

The element u_0 is also $\sigma(\varphi, \varphi)$-adherent to the sequence (u_n), so, in particular, λ_i^0 is adherent to the sequence $(\lambda_i^n)_n$ for $i = 1, 2, \ldots, k$. It follows that

$$\langle a_0, u_n \rangle = \sum_{i=1}^{k} \lambda_i^n \langle a_0, w_i \rangle \xrightarrow{n} \sum_{i=1}^{k} \lambda_i^0 \langle a_0, w_i \rangle = \langle a_0, u_0 \rangle$$

and this proves the assertion. Again by the Banach-Mackey theorem, A is not contained in a Banach disc as it is not $\beta(\ell_1, \varphi)$-bounded. ∎

3 Sometimes convex-compactness implies compactness

In ([2, p.9]), an example of an absolutely convex sequentially compact subset A in a locally convex space $(E, \mathcal{T})$ such that $\overline{A}$ is not countably compact is given. We can prove that, in fact, $\overline{A}$ is not even convex-compact. This provides an example of a relatively convex-compact set whose closure is not convex-compact.

Example 9 *There exists a locally convex space with an absolutely convex, sequentially compact (and then countably compact and so convex-compact) subset whose closure is not convex-compact.*

To present the example, take a X_n be a disjoint sequence of uncountable sets and define $X := \bigcup_{n=1}^{\infty} X_n$. For $f : X \to \mathbb{R}$, the support of f is defined as $\operatorname{supp} f := \{x \in X \mid f(x) \neq 0\}$. Let the vector space

$$E := \left\{ f : X \to \mathbb{R} \,|\, \exists\, n \in \mathbb{N} : \operatorname{supp} f \cap \bigcup_{m=n}^{\infty} X_m \text{ is countable} \right\}$$

be endowed with the restriction of the topology $\mathcal{T}_p$ in $\mathbb{R}^X$ of pointwise convergence on X, denoted again $\mathcal{T}_p$. Clearly, $(E, \mathcal{T}_p)$ turns out to be a locally convex space. By using a diagonal procedure, it is easy to see that the set

$$A := \{ f \in E \,|\, \operatorname{supp} f \text{ is countable}, \|\mathrm{f}\|_\infty \leq 1 \}$$

is sequentially compact. However, the closure

$$\overline{A}^{(E, \mathcal{T}_p)} \quad (= \{ f \in E \,|\, \|f\|_\infty \leq 1 \})$$

is not convex-compact. To see this, let f_n be the characteristic function of $\bigcup_{i=1}^{n} X_i$, $n \in \mathbb{N}$. The sequence (f_n) is in $\overline{A}^{(E, \mathcal{T}_p)}$ and $\mathcal{T}_p$-converges to $f \in \mathbb{R}^X$, the characteristic function of X, which is not in E. Consider now the sets

$$K_n = \overline{\operatorname{conv}} \{ f_i \}_n^{\infty}, \; n \in \mathbb{N}.$$

They form a decreasing sequence of closed convex sets in E such that $K_n \cap \overline{A}^{(E,\mathcal{T}_p)} \neq \emptyset$. If $g \in K_n$ then $g(x) = 1$ for all $x \in \bigcup_{k=1}^{n} X_k$. Then the sequence $K_n \cap \overline{A}^{(E,\mathcal{T}_p)}$ has no adherent point in E. ∎

In Fréchet spaces or in locally convex spaces E with $\sigma(E',E)$-separable dual E', several concepts of weak compactness coincide (theorems of Eberlein and Eberlein-Šmulian, see for example [4, §24]). A criterium for weak compactness in the spirit of the Eberlein-Šmulian theorem is given in [2, 3.10]:

Theorem 10 *A locally convex space E which admits $\sigma(E',E)$-relatively countably compact sets $M_n \subset E'$, $n \in \mathbb{N}$, such that*

$$E' = \overline{\bigcup_{n=1}^{\infty} M_n}^{\sigma(E',E)}$$

is $\sigma(E,E')$-angelic *(i.e., every $\sigma(E,E')$-relatively countably compact subset of E is $\sigma(E,E')$-relatively compact, and its $\sigma(E,E')$-closure coincides with its $\sigma(E,E')$-sequential closure). In particular, the following classes of subsets coincide:*
(i) $\sigma(E,E')$-RNK, $\sigma(E,E')$-RSK, $\sigma(E,E')$-RK,
(ii) $\sigma(E,E')$-NK, $\sigma(E,E')$-SK, $\sigma(E,E')$-K.

We shall prove that there is a similar Eberlein-Šmulian theorem for the class of $\sigma(E,E')$-(R)CK sets. In fact, it can be stated for a more general class of sets (see the following definition) including the $\sigma(E,E')$-CK ones.

Definition 11 *A subset A of a locally convex space $(E,\mathcal{T})$ is said to be $\sigma(E,E')$-*(relatively) numerably compact *(briefly, $\sigma(E,E')$-(R)ΞK) if it is bounded and, given a sequence (a_n) in A and a $\sigma(E'^*,E')$-adherent point $a'^* \in E'^*$ of (a_n), then, for any sequence (x'_n) in E', there exists a point $a \in A \cap \overline{\operatorname{span}}\{a_n;\ n \in \mathbb{N}\}$ ($a \in \overline{\operatorname{span}}\{a_n;\ n \in \mathbb{N}\}$) such that $\langle a'^* - a, x'_n \rangle = 0$ for all $n \in \mathbb{N}$.*

It is easy to see that $\sigma(E,E')$-(R)CK sets are $\sigma(E,E')$-(R)ΞK. Indeed, they are bounded; the second condition can be proved just by considering the decreasing sequence of closed convex sets $K_n := \{x \in \overline{\operatorname{span}}\{a_n;\ n \in \mathbb{N}\};\ \sup\{|\langle a'^* - x, x'_i \rangle|;\ i = 1,2,\ldots,n\} \leq 1/n\}$.

Theorem 12 *Let $(E,\mathcal{T})$ be a locally convex space such that in E' there is a sequence (M_n) of $\sigma(E',E)$-RNK subsets such that $\bigcup_{n\in\mathbb{N}} M_n$ is $\sigma(E',E)$-dense in E' (in particular, this is the case if $E(\mathcal{T})$ has a locally convex topology coarser than $\mathcal{T}$ and metrizable, or more particulary, if E' is $\sigma(E',E)$-separable). Then every $\sigma(E,E')$-(R)ΞK set is $\sigma(E,E')$-(R)K.*

Proof Let us assume first that $(E',\sigma(E',E))$ is separable. Let A be a $\sigma(E,E')$-(R)ΞK subset of E and $a'^* \in \overline{A}^{\sigma(E'^*,E')}$ a $\sigma(E'^*,E')$-adherent point of a sequence (a_n) in A. By definition, given a countable subset $N \subset E'$, there exists $a_N \in A \cap \overline{\operatorname{span}}\{a_n;\ n \in \mathbb{N}\}$ ($a_N \in \overline{\operatorname{span}}\{a_n;\ n \in \mathbb{N}\}$), such that $a_N|_N = a'^*|_N$.

Let D be a countable and $\sigma(E',E)$-dense subset of E' and let $x' \in E'$ be an arbitrary point. Let us consider the points $a_{D\cup x'}$ and a_D in E. They coincide on D, so $a_{D\cup x'} = a_D$. Moreover, $\langle a'^*, x'\rangle = \langle a_{D\cup x'}, x'\rangle$ $(= \langle a_D, x'\rangle)$. Therefore $a'^*|_{E'} = a_D|_{E'}$, and so $a'^* \in E$ and A is $\sigma(E,E')$-(R)K since it is bounded.
Assume now that $(E,\mathcal{T})$ satisfies the requirement and let (a_n) be any sequence in A. Let us consider the separable locally convex space $F = \overline{\text{span}}\{a_n;\ n \in \mathbb{N}\}$. Its dual is $F' = q(E') = E'/F^\perp$, where $q : E' \to E'/F^\perp$ is the canonical mapping. It is easy to see that $q(M_n)$ is $\sigma(F',F)$-RNK and that $\bigcup_{n\in\mathbb{N}} q(M_n)$ is dense in $(F', \sigma(F',F))$. Furthermore, the dual of $(F',\sigma(F',F))$ is F, which is separable. Therefore we can apply Theorem 10 to conclude that $q(M_n)$ is $\sigma(F',F)$-RK, and so metrizable in $(F',\sigma(F',F))$. Thus, $q(M_n)$ is separable, and so it is $(F',\sigma(F',F))$, too.
We claim now that $A \cap F$ is $\sigma(E,E')$-(R)ΞK. Indeed, let f'^* be a $\sigma(F'^*,F')$-adherent point of a given sequence (x_n) in F, and let (f'_n) be a sequence in F'. The element $f'^* \circ q$ $(\in E'^*)$ is a $\sigma(E'^*,E')$-adherent point to (x_n) in E, and there exists a sequence (e'_n) in E' such that $q(e'_n) = f'_n$ for all $n \in \mathbb{N}$. By the assumption, we can find $a \in A \cap \overline{\text{span}}\{x_n;\ n \in \mathbb{N}\} \subset A \cap F$ $(a \in \overline{\text{span}}\{x_n;\ n \in \mathbb{N}\}$ $(\subset F))$ such that $\langle e'^* - a, e'_n\rangle = 0$, i.e., $\langle f'^* - a, f'_n\rangle = 0$, for all $n \in \mathbb{N}$. This proves the claim.
We can then apply the first part of the proof to the set $A\cap F$ to obtain that the set $\{a_n : n \in \mathbb{N}\}$ is $\sigma(F,F')$-RNK (with an adherent point in A (in $\overline{A}^{\sigma(F,F')}$)). This implies that A is $\sigma(E,E')$-(R)NK. By Theorem 10, A is $\sigma(E,E')$-(R)K. ∎

We can extend now Theorem 10 to include the class of $\sigma(E,E')$-(R)ΞK sets (and so, in particular, the $\sigma(E,E')$-(R)CK sets).

Theorem 13 *Let $(E,\mathcal{T})$ be a locally convex space which admits $\sigma(E',E)$-relatively countably compact sets $M_n \subset E'$, $n \in \mathbb{N}$, such that*

$$E' = \overline{\bigcup_{n=1}^{\infty} M_n}^{\sigma(E',E)}.$$

Then, the following classes of sets coincide:
(i) $\sigma(E,E')$-K, $\sigma(E,E')$-SK, $\sigma(E,E')$-NK, $\sigma(E,E')$-CK, $\sigma(E,E')$-ΞK.
(ii) $\sigma(E,E')$-RK, $\sigma(E,E')$-RSK, $\sigma(E,E')$-RNK, $\sigma(E,E')$-RCK, $\sigma(E,E')$-RΞK.

Acknowledgments. We thank an anonymous referee for his/her remarks that helped to improve the final version of this paper.

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