A note on biorthogonal systems in weakly compactly generated Banach spaces

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Abstract

Using separable projectional resolutions of the identity, we provide a different proof of a result of Argyros and Mercourakis on the behavior of fundamental biorthogonal systems in weakly compactly generated (in short, WCG) Banach spaces. This result is used to discuss the example given by Argyros of a non-WCG subspace of a WCG space of the form $C(K)$.

1 Introduction

Our goal is to focus on a remarkable tool in the theory of WCG Banach space (see Theorem 18 below). This slight extension of a result of Argyros and Mercourakis [2] allows to check whether a certain Banach space X having a Markushevich basis (in short, an M-basis) is WCG. It is based on a particular behavior of the functional coefficients of the M-basis as soon as the space is WCG. This property is actually shared by each of the existing M-bases on X. While Argyros and Mercourakis used in their proof a combinatorial approach due to Argyros (see, e.g., [4, Lemma 1.6.2]), our argument is functional-analytic. It is based on the construction of a separable projectional resolution of the identity.
We show then, in detail, how to get from Theorem 18 a result of Argyros and Farmaki [3] (Theorem 20), and hence that of Johnson (Corollary 21) about unconditional long Schauder bases in WCG Banach spaces, see [2]. We conclude our paper with another application: We use Theorem 18 to give an alternative proof to the fact that the subspace of the WCG space of the form $C(K)$, constructed by Argyros, and described in [4, Section 1.6], is not itself WCG.

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2 Preliminaries

The notation used here is standard. If X is a Banach space, B_X (S_X) is the closed unit ball (the unit sphere) and X^* denotes its topological dual. Given a non-empty subset S of X, $\text{span}S$ denotes its linear span, and $\overline{\text{span}}S$ its closed linear span. We denote by $\text{span}_{\mathbb{Q}}S$ the set of all linear combinations of elements from S, with rational coefficients. If S is a set, $\#S$ is the cardinal number of S. The symbol ω stands for the first infinite ordinal. For non-defined concepts we refer to [6].

Dealing with non-separable Banach spaces, it is customary to introduce a kind of "reference system" that allows to replace vectors by their "coordinates". We have in mind projectional resolutions of the identity (and their embryo, projectional generators), and/or Markushevich bases. We provide the definition of those concepts below.

Definition 1 *Let X be a Banach space. A* Markushevich basis *(in short, an* M-basis*) for X is a biorthogonal system $\{x_i; x_i^*\}_{i\in I}$ in $X \times X^*$ such that $\{x_i;\ i \in I\}$ is* fundamental *(i.e., linearly dense in X) and $\{x_i^*;\ i \in I\}$ is* total *(i.e., weak*-linearly dense in X^*).*

Definition 2 *We say that a set $\Gamma \subset X$* countably supports *X^* if $\#\{\gamma \in \Gamma;\ \langle \gamma, x^*\rangle \neq 0\} \leq \aleph_0$ for every $x^* \in X^*$. Analogously, we say that a set $\Delta \subset X^*$* countably supports *X if $\#\{\delta \in \Delta;\ \langle x, \delta\rangle \neq 0\} \leq \aleph_0$ for every $x \in X$.*

Remark 3 It is simple to prove that, if $\{x_i; x_i^*\}_{i\in I}$ is a fundamental system in $X \times X^*$, the set $\{x_i^*;\ i \in I\}$ countably supports X. It is worth to recall (see, e.g., [6, Theorem 12.50 and Proposition 12.51]) that a Banach space X is *weakly Lindelöf determined* (WLD, in short) (i.e., (B_{X^*}, w^*) is a Corson compactum) if, and only if, there exists an M-basis $\{x_i; x_i^*\}_{i\in I}$ such that $\{x_i;\ i \in I\}$ countably supports X^*.

The following concept is useful for constructing a projectional resolution of the identity in a Banach space (Definition 5). For a thorough exposition and for the history of these see, e.g., [4, Chapter 6].

Definition 4 *A* projectional generator *(in short, a* PG*) for a Banach space $(X, \|\cdot\|)$ is a countably-valued mapping Ψ from a 1-norming linear subset $N \subset X^*$ such that, for every $\emptyset \neq W \subset N$ with $\text{span}_{\mathbb{Q}}W \subset W$, we have $\Psi(W)^\perp \cap \overline{W}^{w^*} = \{0\}$. If the set N can be replaced by X^*, we speak about a* full projectional generator *on X.*

Definition 5 *A* projectional resolution of the identity *(in short, a* PRI*) on a (non-separable) Banach space $(X, \|\cdot\|)$ is a long sequence $(P_\alpha)_{\omega\leq\alpha\leq\mu}$ of linear projections on X, where μ is the first ordinal with cardinality* dens X*, such that*
(i) $P_\omega \equiv 0$, P_μ is the identity mapping on X.
(ii) For every $\alpha, \beta \in [\omega, \mu]$, $P_\alpha P_\beta = P_{\min\{\alpha,\beta\}}$.

(iii) For every $\alpha \in (\omega, \mu]$, $\|P_\alpha\| = 1$.
(iv) For every $\alpha \in [\omega, \mu]$, $\operatorname{dens} P_\alpha X \le \#\alpha$.
(v) For every $x \in X$, *the mapping* $\alpha \mapsto P_\alpha x$, $\alpha \in [\omega, \mu]$, *is continuous when* $[\omega, \mu]$ *carries the order topology and* X *the norm topology.*

It was proved in [7] that a Banach space X has a full projectional generator if and only if X is WLD. The existence of a projectional generator for a Banach space $(X, \|\cdot\|)$ implies the existence of a projectional resolution of the identity on it (see, e.g., [4, Chapter 6]).

Definition 6 *Let* X *be a Banach space, let* $(T_i)_{i\in I}$ *be a family of continuous linear operators from* X *into* X, *and let* Γ *be a subset of* X. *We say that* $(T_i)_{i\in I}$ is subordinated to Γ, *or that* Γ is subordinated to $(T_i)_{i\in I}$, *if* $T_i(\gamma) \in \{\gamma, 0\}$ *for every* $i \in I$ *and for every* $\gamma \in \Gamma$.

The following concept can be found in [4, Definition 6.2.6].

Definition 7 *Let* X *be a (nonseparable) Banach space* X, *and let* μ *be the first ordinal with* card $\mu = \operatorname{dens} X$. *A* separable projectional resolution of the identity *(*SPRI, *in short) on* X *is a "long sequence"* $(P_\alpha)_{\omega\le\alpha\le\mu}$ *of linear projections on* X *such that:*
(i) $P_\omega \equiv 0$, P_μ *is the identity mapping on* X.
(ii) For every $\alpha, \beta \in [\omega, \mu]$, $P_\alpha P_\beta = P_{\min\{\alpha,\beta\}}$.
(iii) For every $\alpha \in [\omega, \mu]$ *we have* $\|P_\alpha\| < \infty$.
(iv) For every $\alpha \in [\omega, \mu)$ *the subspace* $(P_{\alpha+1} - P_\alpha)X$ *is separable.*
(v) For every $x \in X$ *we have* $x \in \overline{\operatorname{span}}\{(P_{\alpha+1} - P_\alpha)x;\ \omega \le \alpha < \mu\}$.

From now on, associated to a given PRI $(P_\alpha)_{\omega\le\alpha\le\mu}$ on a Banach space $(X, \|\cdot\|)$ we shall consider the long sequence $(Q_\alpha)_{\omega\le\alpha<\mu}$ of projections, where $Q_\alpha := P_{\alpha+1} - P_\alpha$ for all $\alpha \in [\omega, \mu)$].
We shall use the following simple lemma.

Lemma 8 *Let* $(X, \|\cdot\|)$ *be a Banach space and* $(P_\alpha)_{\omega\le\alpha\le\mu}$ *a PRI (a SPRI) on* X. *Let* Γ *be a subset of* X *subordinated to* $(P_\alpha)_{\omega\le\alpha\le\mu}$, *and let* Δ *be a subset of* X^* *subordinated to* $(P^*_\alpha)_{\omega\le\alpha\le\mu}$. *Then* $\Gamma \subset \bigcup_{\omega\le\alpha<\mu} Q_\alpha X$ *and* $\Delta \subset \bigcup_{\omega\le\alpha<\mu} Q^*_\alpha X^*$.

Proof. Assume first that $(P_\alpha)_{\omega\le\alpha\le\mu}$ is a PRI. Fix any $\gamma \in \Gamma$. If $\gamma = 0$, we are done. So, suppose that $\gamma \ne 0$. Then, by (v) in Definition 5, there exists $\alpha \in [\omega, \mu)$ such that $P_\alpha\gamma \ne 0$. Let α_0 be the first ordinal α in $[\omega, \mu)$ such that $P_\alpha\gamma \ne 0$. Then $\omega < \alpha_0$ and, because of (v) in Definition 5, α_0 is not a limit ordinal. Let $\alpha_0 - 1$ be its predecessor. Since $P_{\alpha_0}\gamma = \gamma$ we get $\gamma \in Q_{\alpha_0-1}X$. The statement about Δ can be proved similarly. It is enough to observe that given $x^* \in X^*$, the net $(P^*_\alpha x^*)_{\omega\le\alpha<\mu}$ is weak*-convergent to x^*.
If $(P_\alpha)_{\omega\le\alpha\le\mu}$ is a SPRI and $\gamma \ne 0$ then, by (v) in Definition 7, there exists $\alpha \in [\omega, \mu)$ such that $Q_\alpha\gamma \ne 0$. This implies that $P_{\alpha+1}\gamma = \gamma$ and $P_\alpha\gamma = 0$, hence $\gamma \in Q_\alpha X$. The argument for Δ is similar; we shall need the fact that

$x^* \in \overline{\mathrm{span}}^{w^*}\{Q_\alpha^* x^*;\ \omega \le \alpha < \mu\}$ for all $x^* \in X^*$. This can be proved easily: if for some $x^* \in X^*$ this is not the case, we can find $x \in X$ such that $\langle x, x^* \rangle = 1$ and $\langle x, Q_\alpha^* x^* \rangle = 0$ for all $\omega \le \alpha < \mu$. Then $\langle Q_\alpha x, x^* \rangle = 0$ for all $\omega \le \alpha < \mu$, hence $x \notin \overline{\mathrm{span}}\{Q_\alpha x;\ \omega \le \alpha < \mu\}$, a contradiction with (v) in Definition 7. ∎

We shall need the following enhanced statement about the existence of a PRI and a SPRI.

Proposition 9 *Let $(X, \|\cdot\|)$ be a nonseparable Banach space admitting a full PG. Let $\Gamma \subset B_X$ be a set which countably supports X^* and let $\Delta \subset B_{X^*}$ be a set which countably supports X. Then there exists a PRI (a SPRI) $(P_\alpha)_{\omega \le \alpha \le \mu}$ on $(X, \|\cdot\|)$ such that Γ is subordinated to $(P_\alpha)_{\omega \le \alpha \le \mu}$ and Δ is subordinated to $(P_\alpha^*)_{\omega \le \alpha \le \mu}$.*

Proof. We shall prove first the statement on the existence of a PRI. Let $\Phi_0 : X^* \to 2^X$ be a projectional generator on X. Put

$$\Phi(x^*) = \Phi_0(x^*) \cup \{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle \ne 0\big\}, \quad x^* \in X^*.$$

Clearly, Φ is also a projectional generator. We shall use a standard back-and-forth argument, see, e.g., [4, Section 6.1]. For every $x \in X$ we find a countable set $\Psi_0(x) \subset X^*$ such that

$$\|x\| = \sup\big\{\langle x, x^* \rangle;\ x^* \in \Psi_0(x) \ \text{ and } \ \|x^*\| \le 1\big\}.$$

Put

$$\Psi(x) = \Psi_0(x) \cup \{\delta \in \Delta;\ \langle x, \delta \rangle \ne 0\}, \quad x \in X.$$

Thus we defined $\Psi : X \to 2^{X^*}$. We still have

$$\|x\| = \sup\big\{\langle x, x^* \rangle;\ x^* \in \Psi(x) \ \text{ and } \ \|x^*\| \le 1\big\}$$

for every $x \in X$.

For the construction of projections $P_\alpha : X \to X$ we shall need the following `Claim`. *Let $\aleph <$ dens X be any infinite cardinal and consider two nonempty sets $A_0 \subset X$, $B_0 \subset X^*$, with $\#A_0 \le \aleph$, $\#B_0 \le \aleph$. Then there exists sets $A_0 \subset A \subset X$, $B_0 \subset B \subset X^*$ such that $\#A \le \aleph$, $\#B \le \aleph$, $\overline{A}$, $\overline{B}$ are linear and $\Phi(B) \subset A$, $\Psi(A) \subset B$.*

In order to prove this, put $A = \bigcup_{n=1}^\infty A_n$, $B = \bigcup_{n=1}^\infty B_n$, where the sets

$$A_n = \mathrm{span}_{\mathbb{Q}}\big(A_{n-1} \cup \Phi(B_{n-1})\big), \quad B_n = \mathrm{span}_{\mathbb{Q}}\big(B_{n-1} \cup \Psi(A_n)\big), \quad n = 1, 2, \ldots,$$

are defined inductively. Then it is easy to verify that the sets A, B have all the proclaimed properties.

Having the sets A, B constructed, we observe that $A^\perp \cap \overline{B \cap B_{X^*}}^{w^*} \subset \Phi(B)^\perp \cap \overline{B \cap B_{X^*}}^{w^*} = \{0\}$. Therefore [8, Lemmas 3.33, 3.34] yield a linear projection $P : X \to X$, with $PX = \overline{A}$, $P^{-1}(0) = B_\perp$, $P^* X^* = \overline{B}^{w^*}$, and such that $\|P\| = 1$.

Fix any $\gamma \in \Gamma$. We shall prove that $P\gamma \in \{\gamma, 0\}$. If $\gamma \in PX$, then, trivially, $P\gamma = \gamma$. Second, assume that $\gamma \notin PX$ $(= \overline{A})$. Then $\gamma \notin \Phi(B)$, which implies that $\langle \gamma, b \rangle = 0$ for every $b \in B$, that is, that $\gamma \in B_\perp$. But $B_\perp = P^{-1}(0)$. Hence $P\gamma = 0$.

Fix any $\delta \in \Delta$. If $\delta \in P^*X^*$, then, trivially, $P^*\delta = \delta$. Further, assume that $\delta \notin P^*X^*$ $(= \overline{B}^{w^*})$. Then, in particular, $\delta \notin \Psi(A)$. It follows that $\langle a, \delta \rangle = 0$ for every $a \in A$. Since $A \subset PX$ we have $\langle a, P^*\delta \rangle = \langle Pa, \delta \rangle = \langle a, \delta \rangle$ $(= 0)$ for every $a \in A$. Hence $P^*\delta \in A^\perp \cap P^*X^*$ $(= A^\perp \cap \overline{B}^{w^*} \subset \Phi(B)^\perp \cap \overline{B}^{w^*} = \{0\})$, and so $P^*\delta = 0$.

Now, once knowing how to construct one projection $P : X \to X$, the construction of the whole PRI is standard, see, e.g., [4, Section 6.1].

In order to prove the SPRI's version, we proceed by induction on the density character of X. If X is separable there is nothing to prove. Assume that the statement holds for all Banach spaces satisfying the requirements and with density character less than μ, for some uncountable ordinal μ. Let X be a Banach space with density character μ, and sets Γ and Δ as stated. Use the first part to obtain a PRI $(P_\alpha)_{\omega \le \alpha \le \mu}$ on X. Since, for $\omega \le \alpha < \mu$, $Q_\alpha X$ has density character $\le \alpha$ $(< \mu)$, and the sets $Q_\alpha \Gamma$ and $Q_\alpha^* \Delta$ satisfy the required properties as regards the Banach space $Q_\alpha X$, there exits, by the inductive hypothesis, a SPRI on $Q_\alpha X$ subordinated to $Q_\alpha \Gamma$ and such that the family of adjoint projections is subordinated to $Q_\alpha^* \Delta$. Now, a standard technique (see, e.g., [4, Proposition 6.2.7]) give a SPRI on X with the required properties. ∎

Remark 10 If the set Γ in Proposition 9 is linearly dense, the assumption on the existence of a full projectional generator can be dispensed because then the multivalued mapping $\Phi_0 : X^* \to 2^X$ given by $\Phi_0(x^*) = \{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle \neq \emptyset\}$ for $x^* \in X^*$ is already a full projectional generator.

3 Core

In order to motivate (the main) Theorem 18, let us present some easy facts about biorthogonal systems in Banach spaces. We did not see them described in the literature, and we believe that they provide the right insight in the aforesaid result. First, we isolate a property of sets that plays an important role in the study of the structure of WCG Banach spaces; it was used by Amir and Lindenstrauss in their seminal paper [1].

Definition 11 *We say that a subset Γ of a Banach space X has the* Amir-Lindestrauss property *(the* (AL)-property*, in short), if for every $x^* \in X^*$ and every $c > 0$, the set $\{\gamma \in \Gamma;\ |\langle \gamma, x^* \rangle| > c\}$ is finite.*

Proposition 12 *Let X be a Banach space and let $\Gamma \subset X$ be a set with the (AL)-property. Then Γ countably supports all of X^* and the set $\Gamma \cup \{0\}$ is weakly compact.*

Proof. Fix any $x^* \in X^*$. We have $\{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle \neq 0\} = \bigcup_{n=1}^{\infty} \{\gamma \in \Gamma;\ |\langle \gamma, x^* \rangle| > \frac{1}{n}\}$ where all the sets on the right side are, according to the (AL) property, finite. Hence, Γ countably supports x^*.
Let γ^{**} be any weak*-accumulation point of Γ, considered as a subset of X^{**}. If $\gamma^{**} \in X$, we are done. Otherwise find $x^* \in X^*$ and $c > 0$ such that $\langle \gamma^{**}, x^* \rangle > c > 0$. But then the set $\{\gamma \in \Gamma;\ \langle \gamma, x^* \rangle > c\}$ is infinite, a contradiction with the (AL)-property. ∎

Proposition 13 *Let $\{x_\lambda; f_\lambda\}_{\lambda \in \Lambda}$ be a total biorthogonal system in $X \times X^*$. Then the following assertions are equivalent:*

(i) The set $\{x_\lambda;\ \lambda \in \Lambda\}$ has the (AL) property.

(ii) The set $\{x_\lambda;\ \lambda \in \Lambda\} \cup \{0\}$ is weakly compact.

(iii) The set $\{x_\lambda;\ \lambda \in \Lambda\}$ is weakly relatively compact.

Proof. (i)⇒(ii) is included in Proposition 12. (ii)⇒(iii) is trivial. (iii)⇒(i). Assume that (iii) holds and (i) is false. Then there exist $x^* \in X^*$, $c > 0$ and an infinite one-to-one sequence $\lambda_1, \lambda_2, \ldots$ of elements of Λ such that $\langle x_{\lambda_i}, x^* \rangle > c$ for every $i = 1, 2, \ldots$ Let $x \in X$ be a weak-accumulation point of the sequence $(x_{\lambda_i})_{i=1}^{\infty}$. Then $\langle x, x^* \rangle \geq c$, and so $x \neq 0$. On the other hand, we can easily check that $\langle x, f_\lambda \rangle = 0$ for every $\lambda \in \Lambda$. And since the set $\{f_\lambda;\ \lambda \in \Lambda\}$ is total, we get that $x = 0$, a contradiction. ∎

The following simple proposition is a consequence of the orthogonality.

Proposition 14 *Let X be a Banach space and let $\{x_i; f_i\}_{i \in \mathbb{N}}$ be a biorthogonal system in $X \times X^*$. Assume that for some increasing sequence (n_p) in $\mathbb{N}$ the sequence $(\sum_{i=1}^{n_p} f_i)_{p=1}^{\infty}$ is bounded. Then all $x_1, x_2, \ldots$ lie in a hyperplane missing 0.*

Proof. Let $x^* \in X^*$ be a weak*-cluster point of the sequence $(\sum_{i=1}^{n_p} f_i)_{p=1}^{\infty}$. If $j \in \mathbb{N}$ and $p \in \mathbb{N}$ are such that $n_p \geq j$, then $\left\langle x_j, f_1 + \cdots + f_{n_p} \right\rangle = 1$. Hence $\langle x_j, x^* \rangle = 1$ for all $j \in \mathbb{N}$. ∎

Corollary 15 *Let X be a Banach space and let $\{x_i, f_i\}_{i \in \mathbb{N}}$ be a biorthogonal system in $X \times X^*$. Assume that $\{x_i;\ i \in \mathbb{N}\}$ has the (AL)-property. Then $\|\sum_{i=1}^{n} f_i\| \to \infty$ as $n \to \infty$.*

Proof. Assume that the conclusion does not hold. Find then an increasing sequence (n_p) in $\mathbb{N}$ such that $(\sum_{i=1}^{n_p} f_i)_{p=1}^{\infty}$ is a bounded sequence. It then follows from Proposition 14 that $\{x_i;\ i \in \mathbb{N}\}$ is in a hyperplane missing 0, and this violates the (AL)-property. ∎

We think that the origin of Theorem 18 below can be traced back to the following theorem, due to V. Pták.

Theorem 16 (Pták [10]) *Let X be a Banach space. Then the following assertions are equivalent:*

(i) X *is reflexive.*

(ii) *For every biorthogonal system* $\{x_n; x_n^*\}_{n=1}^\infty$ *in* $X \times X^*$ *such that* $\{x_n^*;\ n \in \mathbb{N}\}$ *is bounded, the sequence* $\left(\sum_{k=1}^n x_k\right)_{n=1}^\infty$ *is unbounded.*

(iii) *For every biorthogonal system* $\{x_n; x_n^*\}_{n=1}^\infty$ *in* $X \times X^*$ *such that* $\{x_n;\ n \in \mathbb{N}\}$ *is bounded, the sequence* $\left(\sum_{k=1}^n x_k^*\right)_{n=1}^\infty$ *is unbounded.*

We provide below a new proof of this result, based on a well-known characterization of reflexivity due to James that we quote here for the sake of completeness.

Theorem 17 (James [9]) *Let X be a Banach space. Then the following assertions are equivalent:*

(i) X *is not reflexive.*

(ii) *For every* $0 < \theta < 1$ *there are a sequence* (x_n) *in* S_X *and a sequence* (x_n^*) *in* S_{X^*} *such that*

$$\begin{cases} \langle x_n, x_m^* \rangle = \theta & \text{for all} \quad n \geq m, \\ \langle x_n, x_m^* \rangle = 0 & \text{for all} \quad n < m. \end{cases}$$

(iii) *For every* $0 < \theta < 1$ *there is a sequence* (x_n) *in* S_X *such that*

$$\inf \left\{ \|u\|;\ u \in \operatorname{conv}\{x_1, x_2, \ldots\} \right\} \geq \theta$$

and

$$\operatorname{dist}\left(\operatorname{conv}\left\{x_1, \ldots x_n\right\}, \operatorname{conv}\left\{x_{n+1}, x_{n+2}, \ldots\right\}\right) \geq \theta \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof of Theorem 16. (i)⇒(ii). Assume that the space X is reflexive, and let $\{x_n; x_n^*\}_{n\in\mathbb{N}}$ be a biorthogonal system in $X \times X^*$ such that $\{x_n^*;\ n \in \mathbb{N}\}$ is bounded. Let Y denote the closed linear span of $\{x_n;\ n \in \mathbb{N}\}$; this is a reflexive space. Let $q : X^* \to X^*/Y^\perp$ the canonical quotient mapping. From the reflexivity it follows that $(X^*/Y^\perp)^* = Y$, and so the system $\{q(x_n^*); x_n\}_{n\in\mathbb{N}}$ is total and biorthogonal in $(X^*/Y^\perp) \times Y$. Since $\{q(x_n^*); n \in \mathbb{N}\}$ is a weakly relatively compact set, it has, by Proposition 13, the (AL)-property. It follows from Corollary 15 that $\|\sum_{k=1}^n x_k\| \to_n \infty$.

(i)⇒(iii). If X is reflexive, so is X^*. Given a biorthogonal system $\{x_n; x_n^*\}_{n\in\mathbb{N}}$ in $X \times X^*$, it can also be seen as a biorthogonal system $\{x_n^*; x_n\}_{n\in\mathbb{N}}$ in $X^* \times X^{**}$. If $\{x_n;\ n \in \mathbb{N}\}$ is bounded, it follows from the first part of this proof that $\|\sum_{k=1}^n x_k^*\| \to_n \infty$.

(ii)⇒(i). Assume that X is not reflexive. Theorem 17 says, in particular, that there exist two sequences (x_n) in S_X and (x_n^*) in S_{X^*} such that $\langle x_n, x_m^* \rangle = \frac{1}{2}$

if $n \geq m$, and $\langle x_n, x_m^* \rangle = 0$ if $n < m$. Let $d_1 := 2x_1,\ d_n := 2(x_n - x_{n-1}),\ n = 2, 3, \dots$ Then, it is clear that the family $\{d_n; x_n^*\}_{n\in\mathbb{N}}$ is a biorthogonal system in $X \times X^*$. Moreover, $\{x_n^*;\ n \in \mathbb{N}\}$ is bounded. Observe, too, that $\sum_{k=1}^{n} d_k = 2x_n$ for all $n \in \mathbb{N}$. We obtain thus a contradiction with (ii).

(iii)⇒(i). Starting from the assumption that X is not reflexive, we proceed as in the proof of (iii)⇒(i). Once we have the two sequences (x_n) and (x_n^*), put $d_n^* = 2(x_n^* - x_{n+1}^*)$ for $n \in \mathbb{N}$. The system $\{x_n; d_n^*\}_{n\in\mathbb{N}}$ is again a biorthogonal system and the set $\{x_n;\ n \in \mathbb{N}\}$ is bounded. Now $\sum_{k=1}^{n} d_n^* = 2(x_1^* - x_{n+1}^*)$ for all $n \in \mathbb{N}$. We obtain again a contradiction, this time with (iii). ∎

The next theorem (and its proof in particular) is the main objective of our paper.

Theorem 18 *Let X be a Banach space. Let $K \subset X$ be a non-empty weakly compact set, and let $\{x_\lambda; f_\lambda\}_{\lambda\in\Lambda}$ be a fundamental biorthogonal system in $X \times X^*$. Put $\Lambda^0 = \{\lambda \in \Lambda;\ \langle k, f_\lambda \rangle \neq 0$ for some $k \in K\}$. Then there exists a splitting $\Lambda^0 = \bigcup_{m=1}^{\infty} \Lambda_m^0$ such that $\lim_{n\to\infty} \left\| \sum_{i=1}^{n} f_{\lambda_i} \right\| = \infty$ for every fixed $m \in \mathbb{N}$ and for every one-to-one sequence $\lambda_1, \lambda_2, \dots$ in Λ_m^0.*

Proof. Let Y denote the closed linear span of the set K; this will be a WCG subspace of X. By a well-known result of Amir and Lindenstrauss [1], there is a linearly dense set $\Gamma \subset B_Y$ with the property (AL); see, for instance [5, Theorem 1]. For $\lambda \in \Lambda$ let g_λ be the restriction of f_λ to Y and put $\Delta^0 = \{g_\lambda;\ \lambda \in \Lambda^0\}$. Observe that $g_\lambda = 0$ whenever $\lambda \in \Lambda \setminus \Lambda^0$. The set Δ^0 countably supports Y by Remark 3. Also, the set Γ countably supports Y^* by Proposition 12. Let $(P_\alpha)_{\omega\leq\alpha\leq\mu}$ be a SPRI on Y found for these Γ and Δ^0 by Proposition 9. Fix $\alpha \in [\omega, \mu)$.

`Claim`: The set $\Delta_\alpha^0 := Q_\alpha^* Y^* \cap \Delta^0$ is countable (observe that, due to Lemma 8, $\Delta^0 = \bigcup_{\omega\leq\alpha<\mu} \Delta_\alpha^0$).
This can be seen as follows. Put $\Gamma_\alpha = \Gamma \cap Q_\alpha Y$ (notice again that, due to Lemma 8, $\Gamma = \bigcup_{\omega\leq\alpha<\mu} \Gamma_\alpha$). Thanks to the fact that Γ is subordinated to $(P_\alpha)_{\omega\leq\alpha\leq\mu}$, we have $Q_\alpha\Gamma = \Gamma_\alpha \cup \{0\}$, and then Γ_α is linearly dense in the (separable) subspace $Q_\alpha Y$. Find a countable dense subset Γ_α^0 of Γ_α. For $y \in Y$, we define supp $y = \{g_\lambda;\ \lambda \in \Lambda^0,\ \langle y, f_\lambda \rangle \neq 0\}$. If $\gamma \in \Gamma_\alpha$, then supp γ is a subset of Δ_α^0. Indeed, for $g_\lambda \in$ supp γ we have $\langle \gamma, Q_\alpha^* g_\lambda \rangle = \langle Q_\alpha \gamma, g_\lambda \rangle = \langle \gamma, g_\lambda \rangle \neq 0$. Hence $Q_\alpha^* g_\lambda = g_\lambda$, since Δ^0 is subordinated to $(Q_\alpha^*)_{\omega\leq\alpha\leq\mu}$. The set $\bigcup_{\gamma\in\Gamma_\alpha^0}$ supp γ ($\subset \Delta_\alpha^0$) is countable. Let us check that $\Delta_\alpha^0 = \bigcup_{\gamma\in\Gamma_\alpha^0}$ supp γ. Given $g_\lambda \in \Delta_\alpha^0$, we have $g_\lambda \neq 0$. So, there exists $\gamma \in \Gamma_\alpha^0$ such that $\langle \gamma, g_\lambda \rangle \neq 0$, and thus $g_\lambda \in$ supp γ. This proves the Claim.
Let $\{\delta_\alpha^1, \delta_\alpha^2, \dots\}$ be an enumeration of the set Δ_α^0. The set $Q_\alpha\Gamma$ is linearly dense in $Q_\alpha Y$. Then, for each $m \in \mathbb{N}$ we can find an element $\gamma_\alpha^m \in Q_\alpha\Gamma$ such that $\langle \gamma_\alpha^m, \delta_\alpha^m \rangle \neq 0$. Do all the above for every $\alpha \in [\omega, \mu)$.
Put

$$\Lambda_m^0 = \{\lambda \in \Lambda^0;\ f_\lambda = \delta_\alpha^m \ \text{ for some } \ \alpha \in [\omega, \mu)\}, \quad m \in \mathbb{N}.$$

Clearly, $\bigcup_{m=1}^{\infty} \Lambda_m^0 = \Lambda^0$. Further, for $m, l \in \mathbb{N}$ put

$$\Lambda_{m,l}^0 = \big\{\lambda \in \Lambda_m^0;\ f_\lambda = \delta_\alpha^m \text{ and } |\langle \gamma_\alpha^m, \delta_\alpha^m \rangle| > \tfrac{1}{l} \text{ for some } \alpha \in [\omega, \mu)\big\}.$$

Clearly, $\bigcup_{m,l=1}^{\infty} \Lambda_{m,l}^0 = \Lambda^0$.
Now fix any $m, l \in \mathbb{N}$ and consider a one-to-one sequence $\lambda_1, \lambda_2, \ldots$ in $\Lambda_{m,l}^0$. Find $\alpha_1, \alpha_2, \ldots$ in $[\omega, \mu)$ so that $f_{\lambda_i} = \delta_{\alpha_i}^m$, $i \in \mathbb{N}$. It remains to show that $\lim_{n\to\infty} \big\| \sum_{i=1}^n f_{\lambda_i} \big\| = \infty$. Assume not. Then there is an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that the sequence $\Big(\sum_{i=1}^{n_j} f_{\lambda_i} \Big)_{j=1}^{\infty}$ is bounded. Let $y^* \in Y^*$ be a weak*-cluster point of the latter sequence. For any fixed $k \in \mathbb{N}$, we then have

$$\begin{aligned} \big|\langle \gamma_{\alpha_k}^m, y^* \rangle\big| &\geq \liminf_{j\to\infty} \left| \left\langle \gamma_{\alpha_k}^m, \sum_{i=1}^{n_j} f_{\lambda_i} \right\rangle \right| \\ &= \liminf_{j\to\infty} \left| \left\langle \gamma_{\alpha_k}^m, \sum_{i=1}^{n_j} \delta_{\alpha_i}^m \right\rangle \right| = \big|\langle \gamma_{\alpha_k}^m, \delta_{\alpha_k}^m \rangle\big| > \frac{1}{l}. \end{aligned}$$

The last equality in the previous formula follows from the fact that the "long sequence" $(Q_\alpha Y)_{\omega \leq \alpha < \mu}$ is "orthogonal". But the sequence $\gamma_{\alpha_1}^m, \gamma_{\alpha_2}^m, \ldots$ is one-to-one. Thus we get a contradiction with the (AL)-property of the set Γ. ∎

Corollary 19 *Let X be a WCG Banach space, and let $\{x_\lambda; f_\lambda\}_{\lambda\in\Lambda}$ be a fundamental biorthogonal system in $X \times X^*$. Then there exists a splitting $\Lambda = \bigcup_{m=1}^{\infty} \Lambda_m$ such that $\lim_{n\to\infty} \big\| \sum_{i=1}^n f_{\lambda_i} \big\| = \infty$ for every fixed $m \in \mathbb{N}$ and for every one-to-one sequence $\lambda_1, \lambda_2, \ldots$ in Λ_m.*

Proof. Since X is WCG it contains a linearly dense and weakly compact subset K say, of X. The set Λ^0 defined in Theorem 18 for this K coincides with Λ. It is enough to apply Theorem 18. ∎

It should be noted that the result of Argyros and Mercourakis, [2, Theorem 2.2] was formulated for M-bases instead of just fundamental biorthogonal systems.

It was observed in [2], without going into details, that Corollary 19 has, as an important consequence, a result of Johnson (Corollary 21). Here we give a complete and different proof of this: From Theorem 18, we derive a result of Argyros and Farmaki (Theorem 20), which contains Corollary 21. Johnson's result is, somehow, a converse to Corollary 15 under unconditionality, and it has been used by Rosenthal [11] to prove that a WCG Banach space of the form $L_1(\mu)$, with a suitable finite measure μ, has a subspace that is not WCG.

Theorem 20 [3, Lemma B] *Let X be a Banach space admitting an unconditional basis $\{x_\lambda; f_\lambda\}_{\lambda\in\Lambda}$. Let $K \subset X$ be a non-empty weakly compact set. Let $\Lambda^0 := \{\lambda \in \Lambda;\ \langle k, f_\lambda \rangle \neq 0$ for some $k \in K\}$. Then there exists a splitting $\Lambda^0 = \bigcup_{m=1}^{\infty} \Lambda_m^0$ such that for every $m \in \mathbb{N}$, the set $\{x_\lambda;\ \lambda \in \Lambda_m^0\} \cup \{0\}$ is weakly compact.*

Proof. Let Λ_m^0, $m \in \mathbb{N}$, be the sets found in Theorem 18 for our basis and for our set K. Fix one $m \in \mathbb{N}$. We shall show that the only weak*-cluster point of the set $\{x_\lambda;\ \lambda \in \Lambda_m^0\}$ in X^{**}, is 0. Assume this is not so. Find then $c > 0$, $\xi \in S_{X^*}$ and a one-to-one sequence $\lambda_1, \lambda_2, \ldots \in \Lambda_m^0$ so that $\big|\big\langle x_{\lambda_i}, \xi\big\rangle\big| > c$ for every $i \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, every $a_1, \ldots, a_n \in \mathbb{R}$, and for suitable $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ we have

$$\begin{aligned}\Big\|\sum_{i=1}^n a_i x_{\lambda_i}\Big\| &\geq C\Big\|\sum_{i=1}^n \varepsilon_i a_i x_{\lambda_i}\Big\| \geq C\Big\langle \sum_{i=1}^n \varepsilon_i a_i x_{\lambda_i}, \xi\Big\rangle \\ &= C\sum_{i=1}^n |a_i|.\big|\big\langle x_{\lambda_i}, \xi\big\rangle\big| \geq C\cdot c\sum_{i=1}^n |a_i|;\end{aligned}$$

here C denotes the "unconditional basis constant" of our basis. (Therefore, X contains an isomorphic copy of ℓ_1.) We have then, for every $n \in \mathbb{N}$,

$$\begin{aligned}\Big\|\sum_{i=1}^n f_{\lambda_i}\Big\| &= \sup\Big\{\Big\langle x, \sum_{i=1}^n f_{\lambda_i}\Big\rangle;\ x \in \operatorname{span}\big\{x_\lambda;\ \lambda \in \Lambda_m^0\big\},\ \|x\| \leq 1\Big\} \\ &= \sup\Big\{\Big\langle \sum_{i=1}^n a_i x_{\lambda_i}, \sum_{i=1}^n f_{\lambda_i}\Big\rangle;\ p \in \mathbb{N},\ a_1, \ldots, a_{n+p} \in \mathbb{R}, \\ &\qquad \Big\|\sum_{i=1}^{n+p} a_i x_{\lambda_i}\Big\| \leq 1\Big\} \\ &\leq \sup\Big\{\Big\langle \sum_{i=1}^n a_i x_{\lambda_i}, \sum_{i=1}^n f_{\lambda_i}\Big\rangle;\ a_1, \ldots, a_n \in \mathbb{R},\ d\Big\|\sum_{i=1}^n a_i x_{\lambda_i}\Big\| \leq 1\Big\} \\ &= \frac{1}{d}\sup\Bigg\{\Big\langle \sum_{i=1}^n a_i x_{\lambda_i}, \sum_{i=1}^n f_{\lambda_i}\Big\rangle;\ a_1, \ldots, a_n \in \mathbb{R},\ \Big\|\sum_{i=1}^n a_i x_{\lambda_i}\Big\| \leq 1\Big\} \\ &= \frac{1}{d}\sup\Bigg\{\sum_{i=1}^n a_i;\ a_1, \ldots, a_n \in \mathbb{R},\ \Big\|\sum_{i=1}^n a_i x_{\lambda_i}\Big\| \leq 1\Big\} \\ &\leq \frac{1}{d}\sup\Big\{\sum_{i=1}^n a_i;\ a_1, \ldots, a_n \in \mathbb{R},\ C\cdot c\sum_{i=1}^n |a_i| \leq 1\Big\} = \frac{1}{d\cdot C\cdot c} < \infty,\end{aligned}$$

where $d > 0$ is a constant such that $d\big\|\sum_{i=1}^j a_i x_{\lambda_i}\big\| \leq \big\|\sum_{i=1}^{j+p} a_i x_{\lambda_i}\big\|$ whenever $j, p \in \mathbb{N}$ and $a_1, \ldots, a_{j+p} \in \mathbb{R}$. But this contradicts the conclusion of Theorem 18. ∎

Corollary 21 (Johnson, see [11, Proposition 1.3]) *Let X be a WCG space admitting an unconditional basis $\{x_\lambda; f_\lambda\}_{\lambda\in\Lambda}$. Then there exists a splitting $\Lambda = \bigcup_{m=1}^\infty \Lambda_m$ such that for every $m \in \mathbb{N}$, the set $\{x_\lambda;\ \Lambda_m\} \cup \{0\}$ is weakly compact.*

Remark 22 Theorem 18 cannot be extended to subspaces of WCG spaces. Indeed, if so, then Corollary 21 would also be extendable. However, Theorem 21 does not work for unconditional basic sequences. Indeed, Argyros and Mercourakis proved, in [2], that there is a WCG space X with an unconditional basis containing a non-WCG subspace Y with an unconditional basis. The unconditional basis of Y (an unconditional basic sequence in X) cannot be σ-relatively weakly compact (i.e., a countable union of relatively weakly compacta), since then Y would be WCG.

We end our note by another application of Theorem 18. Argyros constructed an Eberlein compact space K such that the corresponding (WCG) space $C(K)$ of continuous functions on it contains a non-WCG subspace, see [4, Section 1.6]. Here we show that his combinatorial argument can be replaced by a simple reasoning profiting from Corollary 19 (which in turn was proved by the technology of SPRI).

Example 23 We first recall the Argyros' construction of the compact space K. Given an element $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we put $\sigma|_n = (\sigma(1), \sigma(2), \ldots, \sigma(n))$. According to Talagrand, for $n \in \mathbb{N}$, let $\mathcal{A}_n$ be the family of all $A \subset \mathbb{N}^{\mathbb{N}}$ with the property: whenever σ, τ are distinct elements of A, then $\sigma|_n = \tau|_n$, and $\sigma(n+1) \neq \tau(n+1)$. It is clear that each $A \in \mathcal{A}_n$ is at most countable. Define then

$$K = \left\{ \tfrac{1}{n}\chi_A;\ A \in \mathcal{A}_n,\ n \in \mathbb{N} \right\}.$$

Thus K is a subset of the compact space $[0,1]^{\mathbb{N}^{\mathbb{N}}}$ endowed with the product topology. It is easy to check that K is closed. Therefore, K is a compact space. Moreover, K can be continuously injected into $c_0(\mathbb{N}^{\mathbb{N}})$ endowed with the weak topology. Therefore K is an Eberlein compact space, see [4, Section 1.6] for details.
Next we shall construct a candidate for a non-WCG subspace of the Banach space $C(K)$. For $\lambda \in \mathbb{N}^{\mathbb{N}}$ we define the evaluation function

$$\pi_\lambda(k) = k(\lambda), \quad k \in K.$$

Clearly, $\pi_\lambda \in C(K)$. Define then the subspace Y of $C(K)$ as the closed linear hull of $\left\{\pi_\lambda;\ \lambda \in \mathbb{N}^{\mathbb{N}}\right\}$.
We `claim` that Y is not WCG. For $\lambda \in \mathbb{N}^{\mathbb{N}}$ we put

$$f_\lambda(y) = y(\chi_{\{\lambda\}}), \quad y \in Y;$$

clearly $f_\lambda \in Y^*$. Then $\left\{\pi_\lambda; f_\lambda\right\}_{\lambda \in \mathbb{N}^{\mathbb{N}}}$ is a fundamental biorthogonal system in $Y \times Y^*$. Assume that Y is WCG. Let $\mathbb{N}^{\mathbb{N}} = \bigcup_{m=1}^{\infty} \Lambda_m$ be a partition found in Corollary 19 for our system. [4, Lemma 1.6.1] (due to Talagrand) yields $m, n \in \mathbb{N}$ and an infinite set $A = \{\lambda_1, \lambda_2, \ldots\} \subset \Lambda_m$ such that $A \in \mathcal{A}_n$.
Now, consider any $l \in \mathbb{N}$. We realize that $\frac{1}{n}\chi_{\{\lambda_1,\ldots,\lambda_l\}}$ belongs to the compact space K. Hence, for any $\lambda \in \mathbb{N}^{\mathbb{N}}$ we have

$$\left\langle \pi_\lambda, \sum_{i=1}^{l} f_{\lambda_i} \right\rangle = \sum_{i=1}^{l} f_{\lambda_i}(\pi_\lambda) = \sum_{i=1}^{l} \pi_\lambda\left(\chi_{\{\lambda_i\}}\right) = \sum_{i=1}^{l} \chi_{\{\lambda_i\}}(\lambda)$$

$$= \quad n \cdot \tfrac{1}{n}\chi_{\{\lambda_1,\dots,\lambda_l\}}(\lambda) = n \cdot \pi_\lambda\big(\tfrac{1}{n}\chi_{\{\lambda_1,\dots,\lambda_l\}}\big),$$

Therefore,

$$\Big\langle y, \sum_{i=1}^{l} f_{\lambda_i} \Big\rangle = n \cdot y\big(\tfrac{1}{n}\chi_{\{\lambda_1,\dots,\lambda_l\}}\big) \le n\|y\|$$

for every y from the linear span of $\{\pi_\lambda;\ \lambda \in \mathbb{N}^{\mathbb{N}}\}$, and so $\big\| \sum_{i=1}^{l} f_{\lambda_i} \big\| \le n$ for every $l \in \mathbb{N}$. But this contradicts the conclusion of Corollary 19. ∎

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References

[1] D. Amir, J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. **88** (1968), 35–44.

[2] S. Argyros, S. Mercourakis, *Examples concerning heredity problems of WCG*, Proc. Amer. Math. Soc. **133** (2005), 773–785.

[3] S. Argyros, V. Farmaki, *On the structure of weakly compact subsets of Hilbert spaces and applications to the geometry of Banach spaces,* Trans. Amer. Math. Soc. **289** (1985), 409–427.

[4] M. Fabian, "Gateaux Differentiability of Convex Functions and Topology —Weak Asplund Spaces", John Wiley & Sons, New York, 1997.

[5] M. Fabian, G. Godefroy, V. Montesinos, V. Zizler, *Inner characterizations of weakly compactly generated Banach spaces and their relatives,* J. Math. Anal. and Appl., **297** (2004), 419–455.

[6] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, and V. Zizler, *Functional Analysis and Infinite Dimensional Geometry*, Canad. Math. Soc. Books in Mathematics **8**, Springer-Verlag, New York, 2001.

[7] A. González and V. Montesinos, *A note on weakly Lindelöf determined Banach spaces*, to appear in Czechoslovak Math. J.

[8] P. Hájek, V. Montesinos, J. Vanderwerff, V. Zizler, *Biorthogonal Systems in Banach spaces*, Canad. Math. Soc. Books in Mathematics, Springer-Verlag, CMS, November, 2007.

[9] R.C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. **52** (1950), 518–527.

[10] V. Pták, *Biorthogonal systems and reflexivity of Banach spaces*, Czechoslovak Math. J. **9** (84) (1959), 319–326.

[11] H.P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces,* Compositio Math. **28** (1974), 83–111.

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