Czechoslovak Mathematical Journal, 59 (134) (2009), 613–621

A NOTE ON WEAKLY LINDELÖF DETERMINED BANACH SPACES

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(Received December 12, 2007)

Abstract. We prove that weakly Lindelöf determined Banach spaces are characterized by the existence of a "full" projectional generator. Some other results pertaining to this class of Banach spaces are given.

Keywords: projectional generator, projectional resolution of the identity, weakly Lindelöf determined Banach space, Markushevich base, Corson compacta

MSC 2000: 46B26, 46B20

A Banach space is *weakly Lindelöf determined* (in short, WLD) if its dual unit ball, equipped with the weak*-topology, is Corson, i.e., it is a compact subspace of the topological space $\Sigma(\Gamma)$ consisting of all elements in $\mathbb{R}^\Gamma$ with only a countable number of non-zero coordinates. Here, $\mathbb{R}^\Gamma$ is endowed with the product (i.e., pointwise) topology $\mathscr{T}_p$. This note gives a characterization of WLD Banach spaces in terms of the existence of a certain projectional generator.

As it is usual, given a Banach space X, we denote by B_X its closed unit ball and by S_X its unit sphere. By slightly abusing the notation, given a subset W of a Banach space X we put $B_W := B_X \cap W$ and $S_W := S_X \cap W$. For other undefined notions and for notations we refer to [4].

Let X be a Banach space. A *Markushevich basis* (in short, an M-basis) in $X \times X^*$ is a biorthogonal system $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ such that $\{x_\gamma\}_{\gamma\in\Gamma}$ is *fundamental*, i.e. linearly dense in X and $\{x_\gamma^*\}_{\gamma\in\Gamma}$ is *total*, i.e. weak*-linearly dense in X^*.

The possibility of introducing a "system of coordinates" in a separable (or non-separable) Banach space effectively reduces some of the arguments associated to

Supported by a Grant CONACYT of the Mexican Government. Supported in part by Project MTM2005-08210, the Universidad Politécnica de Valencia and the Generalitat Valenciana (Spain).

certain constructions to analytic computations. In the case of a separable Banach space, a Schauder basis (if there exists one) or, more generally, a countable M-basis (which always exists) do the job. In the non-separable case, an M-basis is not always available, although the "natural" non-separable Banach spaces posses one. A related "coordinatewise" structure is a projectional resolution of the identity—i.e., a long sequence of norm-one projections somehow "splitting" the space. A device to produce in a natural way such a structure is a *projectional generator*, introduced by Valdivia (precedents can be traced back to John and Zizler, Vašak and Plichko) and Orihuela and Valdivia, see references in [2, Section 6.3].

Definition 1. A couple (N, Φ) is a *projectional generator* (in short, a PG) for a Banach space X if N is a 1-norming subset of X^* such that $\mathrm{span}_{\mathbb{Q}}(N)$, the linear span of N with rational coefficients, coincides with N, and $\Phi\colon N \to 2^X$ is a countably-valued mapping such that for all $W \subset N$ with $\mathrm{span}_{\mathbb{Q}}(W) = W$ we have

$$(\Phi(W))^{\perp} \cap \overline{B_W}^{w^*} = \{0\}.$$

A good account of this concept and the way a projectional resolution of the identity is constructed from it can be found, for example, in [2, Section 6.3]. Projectional generators appear in a natural way: for example, it is easy to prove that, in a WCG Banach space X (i.e., X has a linearly dense weakly compact subset K), the couple (X^*, Φ), where $\Phi(x^*)$ is an element of K where x^* attains its maximum, is a PG. We illustrate this fact in Proposition 16. This PG is a particular instance of the following class.

Definition 2. Given a Banach space X, we say that a PG (N, Φ) for X is *full* if $N = X^*$.

Definition 3. Let $W \subset X^*$ be a non-empty subset of X^*. A set $G \subset X$ in a Banach space X is said to *countably support* W if

$$\mathop{\mathrm{supp}}_{G}(x^*) := \{x \in G; \langle x, x^* \rangle \neq 0\}$$

is countable for every $x^* \in W$.

Remark 4. A well known result says that for every WCG Banach space X there exists an M-basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ in $X \times X^*$ with the following property: given $x^* \in X^*$ and $\varepsilon > 0$ the set $\{\gamma \in \Gamma; |\langle x_\gamma, x^* \rangle| > \varepsilon\}$ is finite (see, for example, [8, Theorem 6.8]). In particular, $\{x_\gamma; \gamma \in \Gamma\} \cup \{0\}$ is weakly compact, and $\{x_\gamma; \gamma \in \Gamma\}$ countably supports X^*. A sketch of the proof of this fact is given on page 9 prior to the statement of Proposition 16, and it is based on that result.

The following lemma will be used in the proof of Theorem 7. Although its proof is simple, we include it here for the sake of completeness.

Lemma 5. *Let X be a Banach space with a full PG (X^*, Φ). Then, every complemented subspace Y of X has also a full PG.*

Proof. Let $P\colon X \to Y$ be a continuous linear projection. Put $\hat{\Phi}(y^*) := P(\Phi(P^*y^*))$ for all $y^* \in Y^*$. We shall prove that $(Y^*, \hat{\Phi})$ is a PG for Y. Let $W \subset Y^*$ be such that $\operatorname{span}_{\mathbb{Q}}(W) = W$. Let $y^* \in \overline{B_W}^{w^*} \cap [\hat{\Phi}(W)]^{\perp}$. It follows that

$$P^*y^* \in P^*(\overline{B_W}^{w^*}) \subset \overline{P^*(B_W)}^{w^*} \subset \overline{\|P\| \cdot B_{P^*(W)}}^{w^*}.$$

It is easy to prove that $P^*y^* \in [\Phi(P^*(W))]^{\perp}$. Indeed, for every $w^* \in W$ and every $x \in \Phi(P^*w^*)$, we have $Px \in \hat{\Phi}(w^*)$, and so

$$\langle x, P^*y^* \rangle = \langle x, y^* \circ P \rangle = \langle Px, y^* \rangle = 0.$$

It follows that $\|P\|^{-1}P^*y^* \in \overline{B_{P^*(W)}}^{w^*} \cap [\Phi(P^*(W))]^{\perp}$ $(= \{0\})$, hence $P^*y^* = 0$ and so $y^* = 0$. $\square$

Remark 6. Every Banach space X with a full PG has an M-basis. This can be proved by a standard transfinite induction argument. Indeed, the classical Markushevich theorem gives an M-basis in every separable Banach space. Now assume that, given an uncountable cardinal number $\aleph$, the result has been proved for every Banach space with density $< \aleph$. Let X be a Banach space with a full PG and such that $\operatorname{dens} X = \aleph$. Then X has a PRI $(P_\alpha)_{\omega_0 \leqslant \alpha \leqslant \mu}$, where μ is the first ordinal with cardinal $\operatorname{dens} X$. Every subspace $(P_{\alpha+1} - P_\alpha)X$, $\omega_0 \leqslant \alpha < \mu$, has a full PG thanks to Lemma 5, so it has an M-basis by the induction hypothesis. A standard argument (see, e.g., [2, Proposition 6.2.4]) concludes that X itself has an M-basis.

The following result characterizes Banach spaces having a full PG. It is known that a WLD Banach space has such a PG (see, for example, [2, Proposition 8.3.1]). That a WLD is characterized by the existence of an M-basis (or just a linearly dense subset) that countably supports X^* is also known (see, for example, [3] or [4, Theorem 12.50]).

Theorem 7. *Let X be a Banach space. Then the following are equivalent.*

(i) *X is WLD.*

(ii) *X has a full PG.*

(iii) *X has an M-basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ such that $\{x_\gamma; \gamma \in \Gamma\}$ countably supports X^*.*

(iv) *X has a linearly dense subset set G which countably supports X^*.*

In fact, if one of the above conditions hold, then every M-basis $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ in $X \times X^*$ has the property that $\{x_\gamma; \gamma \in \Gamma\}$ countably supports X^*.

P r o o f. (i) $\Rightarrow$ (ii) is proved, for example, in [2, Proposition 8.3.1]. We provide here a (somehow) streamlined proof. (B_{X^*}, w^*) is Corson; hence, for some non-empty Γ, it is a subspace of $(\Sigma(\Gamma), \mathscr{T}_p)$. Given $\gamma \in \Gamma$, let $\pi_\gamma\colon \Sigma(\Gamma) \to \mathbb{R}$ be the γ-th coordinate mapping; its restriction to B_{X^*}, denoted again π_γ, is an element in $C((B_{X^*}, w^*))$. In this last space, the algebra generated by the elements in X and the constant functions is norm-dense, so there exists a countable set $X_\gamma \subset X$ such that π_γ is in the norm-closure of the algebra $\mathscr{A}(X_\gamma, I_{\mathrm{const}})$ generated by X_γ and the constant function I_{const} on (B_{X^*}, w^*). Define $\Phi\colon X^* \to 2^X$ as

$$(1)\qquad \Phi(x^*) = \begin{cases} \{0\}, & \text{if } x^* = 0,\\ \{0\}, & \text{if } x^* \notin B_{X^*},\\ \displaystyle\bigcup_{\pi_\gamma(x^*)\neq\pi_\gamma(0)} X_\gamma, & \text{if } x^* \in B_{X^*},\ x^* \neq 0. \end{cases}$$

We `Claim` that (X^*, Φ) is a PG. To prove the `Claim` take $W \subset X^*$ such that $\operatorname{span}_{\mathbb{Q}} W = W$. Let $x^* \in \Phi(W)^\perp \cap \overline{B_W}^{w^*}$. Assume $x^* \neq 0$. Then there exists $\gamma \in \Gamma$ such that $\pi_\gamma(x^*) \neq \pi_\gamma(0)$. As $x^* \in \overline{B_W}^{w^*}$, there exists $w^* \in B_W$ such that $\pi_\gamma(w^*) \neq \pi_\gamma(0)$. Then $X_\gamma \subset \Phi(w^*)$. Since $x^* \in \Phi(W)^\perp$ and $\Phi(w^*) \subset \Phi(W)$, we have $\langle X_\gamma, x^*\rangle = 0$. Now, every element of $\mathscr{A}(X_\gamma, 1)$, where 1 denotes the constant 1 function, is of the form $f := a_0 + \sum\limits_{i=1}^{k} a_i \prod\limits_{j=1}^{n_i} x_{i,j}$, where a_0, a_i are constant functions and $x_{i,j} \in X_\gamma$. It follows that $f(x^*) = a_0 = f(0)$. Then, since π_γ is in the norm-closure of $\mathscr{A}(X_\gamma, 1)$ we get $\pi_\gamma(x^*) = \pi_\gamma(0)$, a contradiction.

(ii) $\Rightarrow$ (iii) We proved in Remark 6 that every Banach space with a full PG has an M-basis. We shall prove now that every M-basis $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ in $X \times X^*$ satisfies that $\{x_\gamma; \gamma \in \Gamma\}$ countably supports X^*. Given $x^* \in X^*$, put $\operatorname{supp}(x^*) := \{\gamma \in \Gamma; \langle x_\gamma, x^*\rangle \neq 0\}$, and let $\#S$ be the cardinal number of a set S.

Let $S := \{x^* \in X^*; \#\operatorname{supp}(x^*) \leqslant \aleph_0\}$; it is a linear subspace of X^*. Since S contains all x^*_γ, it is weak*-dense. We shall prove that $S \cap B_{X^*}$ is weak*-closed. Then the Banach-Dieudonné Theorem will yield that S is w^*-closed, hence $S = X^*$ and so $\{x_\gamma; \gamma \in \Gamma\}$ countably supports X^*, as we wish to prove. Let then $x^*_0 \in \overline{S \cap B_{X^*}}^{w^*}$.

Put $W_1 = \operatorname{sp}_Q\{x^*_0\}$; this is a countable set. $\Phi(W_1)$ is also a countable set. Let us enumerate it as $\Phi(W_1) = \{x^1_1, x^1_2, \dots\}$. Find $x^*_1 \in S \cap B_{X^*}$ so that $|\langle x^1_1, x^*_1 - x^*_0\rangle| < 1$. Put $W_2 = \operatorname{sp}_Q\{x^*_0, x^*_1\}$. The set $\Phi(W_2)$ is again countable. Write then $\Phi(W_2) = \{x^2_1, x^2_2, \dots\}$. Find $x^*_2 \in S \cap B_{X^*}$ so that $|\langle x^i_j, x^*_2 - x^*_0\rangle| < \frac{1}{2}$ for each $i, j \in \{1, 2\}$. Put $W_3 = \operatorname{sp}_Q\{x^*_0, x^*_1, x^*_2\}$. The set $\Phi(W_3)$ is again countable. Write then $\Phi(W_3) = \{x^3_1, x^3_2, \dots\}$. Find $x^*_3 \in S \cap B_{X^*}$ so that $|\langle x^i_j, x^*_3 - x^*_0\rangle| < \frac{1}{3}$ for each

$i,j \in \{1,2,3\}$. Continuing in a obvious way, we get a sequence $(x_n^*)_{n=1}^\infty$ in $S \cap B_{X^*}$, "rationally" linear countable sets $W_1 \subset W_2 \subset \ldots \subset X^*$, and vectors x_j^i, $i,j \in \mathbb{N}$. Put $W := W_1 \cup W_2 \cup \ldots$; then $\mathrm{sp}_Q W = W$. Let $y^* \in B_{X^*}$ be a weak* cluster point of the sequence (x_n^*). Pick any $x \in \Phi(W)$. Then $x = x_j^i$ for suitable $i,j \in \mathbb{N}$. Then for $n \in \mathbb{N}$, with $n > \max\{i,j\}$, we have $\left|\langle x, x_n^* - x_0^*\rangle\right| = \left|\langle x_j^i, x_n^* - x_0^*\rangle\right| < 1/n$, and hence $\langle x, y^* - x_0^*\rangle = 0$. We thus showed that $y^* - x_0^* \in \Phi(W)^\perp$. On the other hand $\frac{1}{2}(y^* - x_0^*) \in \overline{B_W}^{w^*}$. Therefore $y^* - x_0^* = 0$. And, since y^*, as a weak* cluster point of the sequence (x_n^*), has at most a countable support, we can conclude that $x_0^* = y^* \in S \cap B_{X^*}$.

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i) is obvious; the mapping $x^* \mapsto (\langle \gamma, x^*\rangle)_{\gamma\in\Gamma}$ from X^* into $\mathbb{R}^\Gamma$ shows that (B_{X^*}, w^*) is a Corson compactum. $\square$

Remark 8. Lemma 5 and Remark 6 give that every *complemented* subspace of a Banach space with a full PG has an M-basis. This holds, too, for an *arbitrary* (closed) subspace Y of a Banach space X with a full PG. The proof uses again a transfinite induction argument on the density of Y. Assume first that Y is separable. Then the result follows from the classical Markushevich theorem. Let $\aleph$ be an uncountable cardinal number. Assume that the result holds for every Banach space of density $< \aleph$ which is a subspace of a Banach space with a full PG. Let Y be a subspace of density $\aleph$ of X. Since X has a PG, there exists a complemented subspace Z of X such that $Y \subset Z \subset X$ and $\operatorname{dens} Z = \operatorname{dens} Y$. Let μ be the first ordinal with cardinal $\operatorname{dens} Z$. Lemma 5 ensures that Z has also a full PG, so there exists a PRI $(P_\alpha)_{\omega_0 \leqslant \alpha \leqslant \mu}$ on Z such that every P_α fixes Y (see the proof of [2, Proposition 6.1.10]), and then the long sequence of their restrictions to Y provides a PRI on Y. By the induction hypothesis, every $(P_{\alpha+1} - P_\alpha)Y$, $\omega_0 \leqslant \alpha < \mu$, has an M-basis. Finally, a standard argument gives an M-basis on Y.

This result is less general than the one stated in Corollary 9. However, to prove it we did not need the full strength of Theorem 7.

It is known that a subspace of a WLD Banach space is itself WLD. This result is a consequence of the fact that the continuous image of a Corson compactum is again Corson compactum, a deep result of Gul'ko and, independently, Valdivia [12]. In Corollary 9 below we shall prove in a more simple way the hereditability by subspaces of the WLD property. This fact has been also proved in [3]; the proof here is even simpler.

A compact topological space K is *angelic* if for every non-empty subset A of K, every element in $\overline{A}$ is the limit of a sequence in A. We shall use the following fact. *Let $f\colon K \to T$ be a continuous onto mapping, K an angelic compact space, T a compact space. Then T is also angelic.* Indeed, let $\emptyset \neq B \subset T$. The family

$\mathscr{A} := \{A \subset K, A \text{ closed}; f(A) = \overline{B}\}$ is non-empty and has, by Zorn's Lemma, a minimal element, say A. Let $A_0 := \{a \in A; f(a) \in B\}$. Observe that $f(A_0) \subset B$ by the definition of A_0. If $b \in B$, then there exists $a \in A$ such that $b = f(a)$. Hence $a \in A_0$. Therefore $f(A_0) = B$. Now, $\overline{B} = \overline{f(A_0)} = f(\overline{A_0}) \subset f(A) = \overline{B}$, hence $f(\overline{A_0}) = \overline{B}$. Note that $\overline{A_0} \subset A$ and that A is minimal. It follows that $\overline{A_0} = A$. Given $b \in \overline{B}$ there exists $a \in A$ such that $f(a) = b$ and, by angelicity, there exists a sequence (a_n) in A_0 such that $a_n \to a$. It follows that (b_n) $(:= f(a_n))$ is a sequence in B which converges to b.

Corollary 9. *Let X be a WLD Banach space. Then, every subspace Y of X is again WLD.*

Proof. By Remark 8, Y has an M-basis $\{y_\gamma; y^*_\gamma\}_{\gamma\in\Gamma}$. By the preceding observation, (B_{Y^*}, w^*) is angelic and therefore, by the Banach-Dieudonné Theorem, $\{y_\gamma; \gamma \in \Gamma\}$ countably supports Y^*. It follows from Theorem 7 that Y is WLD. $\square$

Remark 10. Under Martin's Axiom (MA_{ω_1}), every Corson compactum has property (M), a result of Archangelskii, Šapirovskii and Kunen (see, for example, [6] and [8]). If this is the case, as a consequence of Corollary 9 we obtain the result mentioned above that the continuous image of a Corson compactum is again Corson.

Another simple consequence of Theorem 7 and Proposition 1 in [3] is the following corollary (see [12] and [14]).

Corollary 11. *Let X be a WLD Banach space and let Y be a closed subspace of X. Then, every M-basis (resp. norming M-basis, resp. uniformly minimal M-basis) on Y can be extended to an M-basis (resp. norming M-basis, resp. uniformly minimal M-basis) on X.*

Proof. Proceed by induction on the density character of X. If X is separable, the result follows from [7] (resp. [11], resp. [10]). Let $\aleph$ be an uncountable cardinal number. Assume that the corollary has been proved for all WLD Banach spaces of density $< \aleph$. Suppose that X is a WLD Banach space of density $\aleph$. Let $\{y_\gamma;\ y^*_\gamma\}_{\gamma\in\Gamma}$ an M-basis on Y. Let G be a total subset of X countably supporting X^*. Then $\{y_\gamma;\ \gamma \in \Gamma\} \cup G$ is a total subset of X and countably supports X^*. By [3, Proposition 1] there exists $(P_\alpha)_{\omega_0 \leqslant \alpha \leqslant \mu}$, a PRI on X subordinated to $\{y_\gamma;\ \gamma \in \Gamma\} \cup G$, i.e., $P_\alpha(x) \in \{x, 0\}$ for every $x \in \{y_\gamma;\ \gamma \in \Gamma\} \cup G$. Using the aforesaid PRI, and letting $Q_\alpha := P_{\alpha+1} - P_\alpha$, $\omega_0 \leqslant \alpha < \mu$, extend the M-basis $Q_\alpha\{y_\gamma;\ \gamma \in \Gamma\}$ of $Q_\alpha Y$ to an M-basis (resp. norming M-basis, resp. uniformly minimal M-basis) of $Q_\alpha X$, by the induction hypothesis. Now, using [2, Proposition 6.2.4], "glue together" those M-bases in one on X, which becomes an extension of $\{y_\gamma;\ y^*_\gamma\}_{\gamma\in\Gamma}$ with the required properties. $\square$

Remark 12. Corollary 9 is a simple consequence of Corollary 11 and the fact that an M-basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ of a Banach space with a weak*-angelic dual unit ball has the property that $\{x_\gamma;\ \gamma\in\Gamma\}$ countably supports X^*.

We say that a Banach space X is *DENS* if the density of X is equal to the density of (X^*, w^*). Another simple consequence of Theorem 7 is the following well-known fact.

Corollary 13. *Every WLD Banach space is DENS.*

Proof. Let $\{x_\gamma;\ x_\gamma^*\}_{\gamma\in\Gamma}$ be an M-basis in X (which always exists, and $\{x_\gamma;\ \gamma\in\Gamma\}$ countably supports X^*, see Theorem 7), and let D be a weak*-dense subset of X^* with $\#D = w^*\text{-}\operatorname{dens} X^*$. Then $S := \{x_\gamma;\ \gamma\in\Gamma, \langle x_\gamma, d^*\rangle \neq 0 \text{ for some } d^*\in D\}$ is fundamental in X (assume not: there exists $0\neq x^*\in X^*$ such that $\langle s, x^*\rangle = 0$ for all $s\in S$. We can find $\gamma\in\Gamma$ such that $\langle x_\gamma, x^*\rangle \neq 0$. Find $d^*\in D$ such that $\langle x_\gamma, d^*\rangle\neq 0$. Then $x_\gamma\in S$ and $\langle x_\gamma, x^*\rangle = 0$, a contradiction), and $\#S = \#D$, so we have $\operatorname{dens} X \leqslant w^*\text{-}\operatorname{dens} X^*$ $(\leqslant \operatorname{dens} X)$ and so X is DENS. $\square$

The next theorem completes the information given in [9] and depends essentially upon the following Valdivia's result in [13]: (a) *Let X be an Asplund space. Then X has a biorthogonal system $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ such that $\overline{\operatorname{span}}^{w^*}\{x_\gamma^*;\ \gamma\in\Gamma\} = X^*$ and, moreover, $\{x_\gamma;\ x_\gamma^*\restriction_E\}_{\gamma\in\Gamma}$ is a shrinking M-basis in E, where $E := \overline{\operatorname{span}}\{x_\gamma;\ \gamma\in\Gamma\}$.*

We say that a Banach space is $\langle F\rangle$ if it has an equivalent Fréchet differentiable norm. An M-basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ in $X\times X^*$, where X is a Banach space, is *shrinking* if $\overline{\operatorname{span}}\{x_\gamma^*;\ \gamma\in\Gamma\} = X^*$. We say that a Banach space X is *Asplund* if every separable subspace of X has a separable dual.

Theorem 14. *Let E be a Banach space. Then, the following are equivalent.*
(i) *There is a subspace $X\subset E$ with a shrinking M-basis.*
(ii) *There is a subspace $Y\subset E$ which is Asplund and WCG.*
(iii) *There is a subspace $Z\subset E$ which is Asplund and DENS.*
(iv) *There is a subspace $U\subset E$ which is Asplund.*
(v) *There is a subspace $V\subset E$ which is $\langle F\rangle$ and WCG.*
(vi) *There is a subspace $W\subset E$ which is $\langle F\rangle$ and DENS.*
(vii) *There is a subspace $H\subset E$ which is $\langle F\rangle$.*

Proof. (i) $\Rightarrow$ (ii) Every Banach space with a shrinking M-basis is Asplund and WCG (and conversely, see, for example, [2, p. 112 and Theorem 8.3.3]).

(ii) $\Rightarrow$ (iii) Every WCG space is WLD. Apply now Corollary 13.

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i) follows from Valdivia's result (a).

(i) $\Rightarrow$ (v) See, for example, [4, Theorem 11.23].

(v) $\Rightarrow$ (vi) follows again from Corollary 13.

(vi) $\Rightarrow$ (vii) is trivial.

(vii) $\Rightarrow$ (iv) Every $\langle F\rangle$ space is Asplund (see, for example, [4, Cor. 10.9]). $\square$

Remark 15. As it is well-known (see, for example, [2, Theorem 8.3.3], in the framework of Asplund spaces all concepts WCG, subspace of WCG, WCD and WLD coincide. However, this is not the case with the concept DENS: the Banach space $C[0,\omega_1]$ is DENS (see, for example, [15]), Asplund ($[0,\omega_1]$ is scattered) but not WLD ($[0,\omega_1]$ is not Corson).

If K is the Kunen compactum (see, for example, [8]) then $C(K)$ is Asplund, not WLD (K is not Corson) and no non-separable subspace of $C(K)$ has an M-basis (see, for example, [15]).

The following well-known result is less general than Theorem 7. However, it gives in a very natural way a full projectional generator (X^*,Φ) in a WCG Banach space X, reencountering Amir-Lindenstrauss classical result in [1] (see also [4, Theorem 11.6]). The fact that, moreover, the range of the mapping Φ is contained in a weakly compact set generating X gives the well-known result that X contains a weakly compact M-basis, i.e., an M-basis $\{x_\gamma,x^*_\gamma\}_{\gamma\in\Gamma}$ such that $\{x_\gamma;\ \gamma\in\Gamma\}\cup\{0\}$ is a weakly compact set.

Proposition 16. *Let X be a WCG Banach space generated by an absolutely convex and weakly compact set K. Then, X has a full (single-valued) projectional generator (X^*,Φ) such that $\Phi(x^*)\in K$ for all $x^*\in X^*$*

Proof. Given $x^*\in X^*$, let $\Phi(x^*)$ be an element in K such that $\langle\Phi(x^*),x^*\rangle = \sup|\langle K,x^*\rangle|$. We `Claim` that (X^*,Φ) is a PG. In order to prove the `Claim` let $W\subset X^*$ be such that $\operatorname{span}_{\mathbb{Q}}(W)=W$. By the Mackey Arens theorem, let $x^*\in \Phi(W)^\perp\cap\overline{B_W}^{w^*}=\Phi(W)^\perp\cap\overline{B_W}^{\mu(X^*,X)}$, where $\mu(X^*,X)$ is the Mackey topology on X^* of the dual pair $\langle X,X^*\rangle$, i.e., the topology of the uniform convergence on the family of all absolutely convex weakly compact subsets of X. Note that $\overline{B_W}^{\mu(X^*,X)}\subset \overline{B_W}^{\mathscr{T}_K}$, where $\mathscr{T}_K$ is the topology on X^* of the uniform convergence on K. Let (x_n^*) be a sequence in B_W such that $x_n^*\xrightarrow{\mathscr{T}_K}x^*$. Fix $\varepsilon>0$ and find $n_0\in\mathbb{N}$ such that $\sup|\langle K,x^*-x_n^*\rangle|<\varepsilon$ for all $n\geqslant n_0$. Then, in particular, $|\langle\Phi(x_n^*),x_n^*\rangle| = |\langle\Phi(x_n^*),x^*-x_n^*\rangle|<\varepsilon$ for all $n\geqslant n_0$. This implies that $\sup|\langle K,x_n^*\rangle|<\varepsilon$ for all $n\geqslant n_0$, so $\sup|\langle K,x^*\rangle|\leqslant\varepsilon$. As $\varepsilon>0$ is arbitrary, we get $x^*|_K\equiv 0$, and so $x^*=0$, This proves the `Claim` and the result. $\square$

Acknowledgements. We thank an anonymous referee for his/hers careful reading of the manuscript and his/hers valuable comments that helped to improve the final form of this paper.

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