BOUNDEDNESS OF BIORTHOGONAL SYSTEMS IN BANACH SPACES

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Abstract. We prove that every Banach space that admits a Markushevich basis also admits a bounded Markushevich basis.

1. Introduction

A *Markushevich basis* (in short, an *M-basis*) for a Banach space X is a biorthogonal system $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ in $X \times X^*$ such that $\{x_\gamma : \gamma \in \Gamma\}$ is *fundamental*, i.e., linearly dense in X, and $\{f_\gamma : \gamma \in \Gamma\}$ is *total*, i.e., w^*-linearly dense in X^*. The *boundedness constant* of the system is $\sup\{\|x_\gamma\|.\|f_\gamma\| : \gamma \in \Gamma\}$ (eventually $+\infty$). If the boundedness constant of an M-basis is a finite number K, we speak of a *K-bounded M-basis.* The main results of this note is the construction of a $(2(1+\sqrt{2})+\varepsilon)$-bounded M-basis (for every $\varepsilon > 0$) in every nonseparable Banach space which admits an M-basis.

The boundedness problem for an M-basis (or more generally a biorthogonal system) has received attention in the work of many mathematicians. In the separable case, Davis and Johnson [DJ73] (building up on the work of Singer [S73]) constructed a $(1+\varepsilon)$-bounded fundamental system, an essentially optimal result for fundamental systems (see, e.g., [HMVZ, Corollary 1.26]). An important ingredient in their work was the use of Dvoretzky's theorem on almost Euclidean sections. Their ideas were developed further by Ovsepian and Pełczyński [OP75], who constructed a bounded M-basis in every separable Banach space. Ultimately, Pełczyński [Pe76] and Plichko [Pl77] independently, constructed a $(1+\varepsilon)$-bounded M-basis in every separable Banach space. The existence in every separable Banach space of a 1-bounded M-basis (i.e., an *Auerbach basis*) is still open.

In non-separable spaces, the existence of a bounded M-basis (provided the space has some M-basis) was claimed by Plichko [Pl82]. His method yields a boundedness constant roughly 10 (see, e.g., [HMVZ, Theorem 5.13]). However, the proof of this result in [Pl82] (and its reproduction in [HMVZ], Theorem 5.13) is flawed. The (subtle) troublesome point in the proof (see in [HMVZ] the claim on page 171, line 10 from below; we follow the notation there) is that $\operatorname{span}\{x_\alpha : \alpha \in J_{\gamma+2} \setminus J_{\gamma-1}\}$ is dense in $G_\gamma^\perp \cap X$. This claim (and thus the statement of Plichko's theorem) is true whenever the original M-basis is strong, but it is false in general (see [HMVZ], Proposition 1.35.). Let us recall that an M-basis $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ is called *strong* if, for every $x \in X$, $x \in \overline{\operatorname{span}}\{\langle x, f_\gamma\rangle x_\gamma : \gamma \in \Gamma\}$. The class of Banach spaces having a strong M-basis is quite large. For example, *every Banach space belonging to a*

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$\mathcal{P}$-class has a strong M-basis [HMVZ, Theorem 5.1]. We recall here that a class $\mathcal{C}$ of Banach spaces is a *$\mathcal{P}$-class* if, for every $X \in \mathcal{C}$, there exists a projectional resolution of the identity $(P_\alpha)_{\omega\le\alpha\le\mu}$ (where μ is the first ordinal with cardinal dens X) such that $(P_{\alpha+1} - P_\alpha)X \in \mathcal{C}$ for all $\alpha \in [\omega, \mu)$. The class of all weakly compactly generated (resp. weakly countably determined, resp. weakly Lindelöf determined) Banach spaces is a $\mathcal{P}$-class.
However, there exists a Banach space with an M-basis admitting no strong M-basis ([HMVZ], Prop. 5.5).

Our approach to the problem uses ideas from several of the above mentioned papers, including [Pl82]. The essential new ingredient is the use of the Δ-system lemma (see Lemma 2), which solves the difficulties in [Pl82]. We are also able to reduce the boundedness constant, by incorporating Dvoretzky's theorem together with the Walsh-matrices-mixing technique used in [OP75].
In the special case of WCG spaces, an adaptation of the proof in the separable case by Plichko leads to a constant $2 + \varepsilon$ (for every $\varepsilon > 0$) [Pl79], which is essentially optimal ([Pl86]).
This alternative approach uses the existence of many projections in the WCG space. In the end of our note we indicate how to obtain a (more or less formal) generalization of the $2 + \varepsilon$ result for wider classes of Banach spaces ($\mathcal{P}$-classes). We refer to [HMVZ] for more results and references related to boundedness of biorthogonal systems.

Our notation is standard. B_X is the closed unit ball of a Banach space X, S_X its unit sphere. Given a non-empty subset S of a Banach space, let span$S$ be the linear span of S, and $\mathrm{span}_{\mathbb{Q}}S$ the set of all linear combinations with rational coefficients of elements in S. The closed linear span of S is denoted $\overline{\mathrm{span}}S$. Given two subspaces F and G of a Banach space X, we put $F \hookrightarrow G$ if F is a subspace of G. We denote by $|S|$ the cardinality of a set S. The *density character* of X, dens X, is the smallest ordinal Ω such that X has a dense subset with cardinal $|\Omega|$. We identify, as usual, an ordinal number Ω with the segment $[0, \Omega)$, and a cardinal number with the initial ordinal having this cardinality. The ordinal number of $\mathbb{N}$ is denoted by ω and its cardinal number by $\aleph_0$. If $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ is an M-basis for X and $x \in X$, the *support* of x (with respect to the M-basis) is the set $\mathrm{supp}\,(x) := \{\gamma \in \Gamma : \langle x, f_\gamma\rangle \neq 0\}$. Analogously, if $f \in X^*$, $\mathrm{supp}\,(f) := \{\gamma \in \Gamma : \langle x_\gamma, f\rangle \neq 0\}$.

For convenience, we formulate the main tools used in the proof of our theorem.

Theorem 1 (Dvoretzky)**.** *Let $N \in \mathbb{N}$, $\varepsilon > 0$. Then there exists a natural number $K := K(N, \varepsilon)$, such that for every Banach space $(X, \|\cdot\|)$ of dimension at least K, there exists a linear space $Y \hookrightarrow X$ of dimension N, which is $(1 + \varepsilon)$-isomorphic to ℓ_2^N.*

A family $\{A_\lambda\}_{\lambda\in\Lambda}$ of sets is called a Δ-*system* (with *root* B, possibly empty) if $A_\lambda \cap A_\alpha = B$ for all distinct $\lambda, \alpha \in \Lambda$.

Lemma 2 (Δ-system lemma, see, e.g., [Ju80], Lemma 0.6)**.** *Let $\Lambda > \omega$ be a regular cardinal and $\{A_\lambda\}_{\lambda\in\Lambda}$ a family of finite subsets of Λ. Then there exists a subfamily $\Omega \subset \Lambda$ of cardinality Λ that is a Δ-system.*

By a more or less standard argument, we obtain the next mild strengthening of the previous result.

Corollary 3. *Let $\Lambda > \omega$ be a regular cardinal, X a Banach space with an M-basis $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$, $\{v_\lambda\}_{\lambda\in\Lambda}$ a long sequence of finitely supported vectors in X with supports $\{A_\lambda\}_{\lambda\in\Lambda}$ and only rational coefficients $\langle v_\lambda, f_\gamma\rangle$. Then there exists a subset*

$\Omega \subset \Lambda$ of cardinality Λ and a finite set $B \subset \Lambda$ such that $A_\lambda \cap A_\alpha = B$ for all $\lambda, \alpha \in \Omega$, and the coefficients of v_λ on B are independent of $\lambda \in \Omega$.

Proof. Apply Lemma 2 to the family $\{A_\lambda\}_{\lambda\in\Lambda}$ to obtain a Δ-system $\{A_\lambda\}_{\lambda\in\Lambda'}$ with root B such that $|\Lambda'| = |\Lambda|$. The set $\mathbb{Q}^B$ is countable. We define a mapping $r : \Lambda' \to \mathbb{Q}^B$ by $r(v_\lambda)(b) = v_\lambda(b)$ for all $b \in B$ and $\lambda \in \Lambda'$. Assume that $|\{\lambda \in \Lambda' : r(\lambda) = \mathbf{q}\}| < |\Lambda|$ for all $\mathbf{q} \in \mathbb{Q}^B$. We have $\Lambda' = \bigcup_{\mathbf{q}\in\mathbb{Q}^B}\{\lambda \in \Lambda' : \ r(\lambda) = \mathbf{q}\}$. Since $\mathbb{Q}^B$ is countable and $|\Lambda'|$ $(> \omega)$ is regular we obtain a contradiction, hence there exists $\mathbf{q} \in \mathbb{Q}^B$ such that $|\Omega| = |\Lambda|$, where $\Omega := \{\lambda \in \Lambda' : \ r(\lambda) = \mathbf{q}\}$. For all $\lambda \in \Omega$ and $b \in B$ we get $v_\lambda(b) = \mathbf{q}(b)$. □

Recall (see, e.g., [J78]) that every non-limit cardinal is regular, and thus in particular every cardinal is a limit of a transfinite increasing sequence of regular cardinals. We rely on orthonormal matrices with special properties, described below.

Lemma 4. *Given $n \in \mathbb{N}$, there exists an orthonormal matrix $W := (a^n_{k,j})_{0\le k,j<2^n}$ with real coefficients, such that*

$$a^n_{k,0} = 2^{-\frac{n}{2}} \qquad \text{for } \ 0 \le k < 2^n, \tag{1}$$

$$\sum_{j=1}^{2^n-1} |a^n_{k,j}| < 1 + \sqrt{2} \qquad \text{for } \ 0 \le k < 2^n. \tag{2}$$

Such matrices were used by Ovsepian and Pełczyński in [OP75]. For a concrete example of Walsh matrices see, e.g., [HMVZ, Lemma 5.17] or [LT77, Lemma 1.f.5].

The following is the main result of this note.

Theorem 5. *Let X be a Banach space with an M-basis $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$, and let $\varepsilon > 0$. Then X admits an M-basis $\{x'_\gamma; f'_\gamma\}_{\gamma\in\Gamma}$ such that $\|x'_\gamma\|.\|f'_\gamma\| \le 2(1+\sqrt{2}) + \varepsilon$ for every $\gamma \in \Gamma$. Moreover,* $\operatorname{span}\{x_\gamma : \ \gamma \in \Gamma\} = \operatorname{span}\{x'_\gamma : \ \gamma \in \Gamma\}$ *and* $\operatorname{span}\{f_\gamma : \ \gamma \in \Gamma\} = \operatorname{span}\{f'_\gamma : \ \gamma \in \Gamma\}$.

Proof. For convenience, we may assume without loss of generality that Γ is an ordinal of cardinality $|\Gamma|$. We are going to find a system consisting of a splitting $\Gamma = \bigcup_{\lambda\in\Gamma} A_\lambda$, where all A_λ are countable and pairwise disjoint, together with biorthogonal systems $\{x'_\gamma; f'_\gamma\}_{\gamma\in A_\lambda}$, so that

A. $\operatorname{span}\{x'_\gamma : \gamma \in A_\lambda\} = \operatorname{span}\{x_\gamma : \gamma \in A_\lambda\}$

B. $\operatorname{span}\{f'_\gamma : \gamma \in A_\lambda\} = \operatorname{span}\{f_\gamma : \gamma \in A_\lambda\}$

C. $\|x'_\gamma\|\|f'_\gamma\| \le 2(1+\sqrt{2}) + \varepsilon$, for all $\gamma \in A_\lambda$, $\lambda \in \Gamma$

The existence of such a system clearly implies the statement of the theorem. We construct the A_λ's and the biorthogonal system associated to each of them by using induction in $\lambda \in \Gamma$.

We start by putting $A_1 := \{0\}$ (the first element in Γ), and letting $\{x'_0; f'_0\}$ be a (single-element) biorthogonal system in $\operatorname{span}\{x_0\} \times \operatorname{span}\{f_0\}$ with $\|x'_0\| = \|f'_0\| = 1$. Suppose we achieved this for all $\lambda < \beta \in \Gamma$. It remains to obtain the objects A_β and $\{x'_\gamma; f'_\gamma\}_{\gamma\in A_\beta}$. To this end we are going to construct an increasing sequence $\{A^j\}_{j=1}^\infty$, $A^j \subset A^{j+1}$, of finite subsets of Γ, so that $A_\beta = \bigcup_{j=1}^\infty A^j$. We are simultaneously going to build finite biorthogonal systems $\{x^j_\gamma; f^j_\gamma\}_{\gamma\in A^j}$, $j \in \mathbb{N}$, and a sequence of finite sets $\{C^j\}_{j=1}^\infty$ satisfying the following conditions for all $j \in \mathbb{N}$.

1. $\operatorname{span}\{x_\alpha : \alpha \in A^j\} = \operatorname{span}\{x^j_\alpha : \alpha \in A^j\}$.
2. $\operatorname{span}\{f_\alpha : \alpha \in A^j\} = \operatorname{span}\{f^j_\alpha : \alpha \in A^j\}$.
3. $C^j = \{\alpha \in A^j : \|x^j_\alpha\|\|f^j_\alpha\| \le 2(1+\sqrt{2}) + \varepsilon\}$.
4. $A^j \subset C^{j+1}$.
5. $x^{j+1}_\gamma = x^j_\gamma$ whenever $\gamma \in C^j$.
6. $f^{j+1}_\gamma = f^j_\gamma$ whenever $\gamma \in C^j$.

7. $\quad \mathrm{span}\{x^j_\alpha :\ \alpha \in A^j\} \subset \mathrm{span}\{x^{j+1}_\alpha : \alpha \in C^{j+1}\}$.
8. $\quad \mathrm{span}\{f^j_\alpha :\ \alpha \in A^j\} \subset \mathrm{span}\{f^{j+1}_\alpha : \alpha \in C^{j+1}\}$.

The existence of such systems now implies the inductive step in the proof of the main theorem.
Indeed, we put $A_\beta := \bigcup_{j=1}^\infty A^j$ $(= \bigcup_{j=1}^\infty C^j)$. If $\gamma \in C^j$ for some $j \in \mathbb{N}$ then, by 5., $x^j_\gamma = x^{j+l}_\gamma$ for all $l \in \mathbb{N}$, and so we can put $x'_\gamma := x^j_\gamma$. Similarly, by 6., $f^j_\gamma = f^{j+l}_\gamma$ for all $l \in \mathbb{N}$, and we put $f'_\gamma = f^j_\gamma$. The biorthogonality of $\{x'_\gamma; f'_\gamma\}_{\gamma\in A_\beta}$ follows from the fact that $\{x^j_\gamma; f^j_\gamma\}_{\gamma\in A^j}$ is biorthogonal for every $j \in \mathbb{N}$. Conditions A. and B. are checked easily: on one hand, if $\gamma \in A_\beta$, then $\gamma \in C^j$ for some $j \in \mathbb{N}$, so by 1.,

$$x'_\gamma = x^j_\gamma \in \mathrm{span}\{x_\alpha :\ \alpha \in A^j\} \subset \mathrm{span}\{x_\alpha :\ \alpha \in A_\beta\},$$

and, since $\gamma \in A^j$, by 7.,

$$\begin{aligned} x_\gamma \in \mathrm{span}\{x^j_\alpha :\ \alpha \in A^j\} &\subset \mathrm{span}\{x^{j+1}_\alpha :\ \alpha \in C^{j+1}\} \\ &= \mathrm{span}\{x'_\alpha :\ \alpha \in C^{j+1}\} \subset \mathrm{span}\{x'_\alpha :\ \alpha \in A_\beta\}. \end{aligned}$$

We obtain similar results for f'_γ and f_γ. Note that conditions 3. and 4. imply C..
It remains to check that $\bigcup_{\lambda\in\Gamma} A_\lambda = \Gamma$. This follows from the fact (see below) that in the construction of $\{A^j\}_{j\in\mathbb{N}}$ we start by taking $A^1 := \{\gamma_0\}$, where γ_0 is the first element in $\Gamma \setminus \bigcup_{\lambda<\beta} A_\lambda$, so $A_\beta \neq \emptyset$ while $\bigcup_{\lambda<\beta} A_\lambda \neq \Gamma$.
To start, put $A^1 = \{\gamma_0\}$, where γ_0 is the first element in $\Gamma \setminus \bigcup_{\lambda<\beta} A_\lambda$, $x^1_{\gamma_0} := x_{\gamma_0}$, and $f^1_{\gamma_0} := f_{\gamma_0}$. Put $C^1 := \{\gamma_0\}$ if $\|x_{\gamma_0}\|\|f_{\gamma_0}\| \le 2(1+\sqrt{2}) + \varepsilon$, $C^1 = \emptyset$ otherwise.
Let us describe the inductive step from j to $j+1$. Suppose that A^p, x^p_γ, f^p_γ for $p \le j$ have been constructed, such that 1.-8. are satisfied whenever the indices exist. Put $L = \{\lambda_1, \dots, \lambda_k\} = A^j \setminus C^j$, and find $C > 0$ such that $\sup\{\|x_\lambda\|, \|f_\lambda\| :\ \lambda \in L\} < C$. Put $N = 2^n - 1$, with $n \in \mathbb{N}$ large enough to have $2^{-n/2}C < \varepsilon$. Use Theorem 1 to find $K := K(N, \varepsilon)$. We are going to build a family $\{S_\lambda :\ \lambda \in L\}$ of finite pairwise disjoint subsets of Γ, disjoint also from $\bigcup_{\lambda<\beta} A_\lambda \cup \bigcup_{i\le j} A^i$, together with finite biorthogonal systems $\{y_\gamma; g_\gamma\}_{\gamma\in S_\lambda}$, $\lambda \in L$, such that, for all $\lambda \in L$,

a. $\quad S_\lambda = S^1_\lambda \cup S^2_\lambda$, $|S^1_\lambda| = N$, $S^1_\lambda \cap S^2_\lambda = \emptyset$.
b. $\quad \mathrm{span}\{x_\gamma : \gamma \in S_\lambda\} = \mathrm{span}\{y_\gamma : \gamma \in S_\lambda\}$.
c. $\quad \mathrm{span}\{f_\gamma : \gamma \in S_\lambda\} = \mathrm{span}\{g_\gamma : \gamma \in S_\lambda\}$.
d. $\quad \{g_\gamma : \gamma \in S^1_\lambda\}$ is $(1+\varepsilon)$-equivalent to the unit basis of ℓ^N_2.
e. $\quad \|y_\gamma\| \le 2 + 2\varepsilon$, for $\gamma \in \bigcup_{\lambda\in L} S_\lambda$.

Finding the above system is the main step of our construction. We have $|\beta| < \Gamma$ and so there exists a regular cardinal R, $\beta < R \le \Gamma$. Denote $\{B_{\lambda,\alpha}\}_{\lambda\in L,\alpha<R}$ a system of pairwise disjoint subsets of $\Gamma \setminus \bigcup_{\delta<\beta} A_\delta$, each of them of cardinality K. By Theorem 1, we have that every $\mathrm{span}\{f_\gamma :\ \gamma \in B_{\lambda,\alpha}\}$ contains a $(1+\varepsilon)$-isometric copy $G_{\lambda,\alpha}$ of ℓ^N_2. Since the pair of finite dimensional spaces

$$(\mathrm{span}\{f_\gamma :\ \gamma \in B_{\lambda,\alpha}\}, \mathrm{span}\{x_\gamma :\ \gamma \in B_{\lambda,\alpha}\})$$

is a dual pair, it follows by standard linear algebra that there exist a splitting $B_{\lambda,\alpha} = D_{\lambda,\alpha} \cup E_{\lambda,\alpha}$, $D_{\lambda,\alpha} := \{\gamma^{\lambda,\alpha}_1, \dots, \gamma^{\lambda,\alpha}_N\}$, $D_{\lambda,\alpha} \cap E_{\lambda,\alpha} = \emptyset$, and a finite biorthogonal system $\{h_\gamma; z_\gamma\}_{\gamma\in B_{\lambda,\alpha}}$, with properties

$$\mathrm{span}\{h_\gamma :\ \gamma \in B_{\lambda,\alpha}\} = \mathrm{span}\{f_\gamma :\ \gamma \in B_{\lambda,\alpha}\},$$

$$\mathrm{span}\{z_\gamma :\ \gamma \in B_{\lambda,\alpha}\} = \mathrm{span}\{x_\gamma :\ \gamma \in B_{\lambda,\alpha}\},$$

$$\{h_\gamma\}_{\gamma\in D_{\lambda,\alpha}} \text{ is } (1+\varepsilon)\text{-equivalent to the unit basis of } \ell^N_2.$$

Let $G_{\lambda,\alpha} := \mathrm{span}\{h_\gamma :\ \gamma \in D_{\lambda,\alpha}\}$.
Fix $\lambda \in L$, $\alpha < R$, and $\gamma \in D_{\lambda,\alpha}$. Put $X_\gamma := z_\gamma \restriction_{G_{\lambda,\alpha}}$.
Clearly, $1 \le \|X_\gamma\| \le 1 + \varepsilon$. Denote X_γ again the Hahn-Banach norm-preserving extension of X_γ from $G_{\lambda,\alpha} \hookrightarrow X^*$ to the whole X^*, so $X_\gamma \in X^{**}$. Since obviously $\overline{\mathrm{span}}_{\mathbb{Q}}\{x_\zeta\}_{\zeta\in\Gamma} = X$, a standard application of Helly's theorem (see, e.g.,

[F$^{\sim}$, Exercise 3.36]) provides an element $\tilde{x}_\gamma \in \operatorname{span}_{\mathbb{Q}}\{x_\zeta : \zeta \in \Gamma\}$ such that $\|\tilde{x}_\gamma\| < \|X_\gamma\| + \varepsilon$ $(< 1 + 2\varepsilon)$ and $\tilde{x}_\gamma \restriction_{G_{\lambda,\alpha}} = X_\gamma$. Denote by $F_m^{\lambda,\alpha} \subset \Gamma$ the finite support sets of $\tilde{x}_{\gamma_m^{\lambda,\alpha}}$, $m \in \{1, \dots, N\}$. Apply Corollary 3 to the given M-basis $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ and to each system $\{\tilde{x}_{\gamma_m^{\lambda,\alpha}}\}_{\alpha<R}$, $m \in \{1, \dots, N\}$, $\lambda \in L$, to obtain a (single) subset $R' \subset R$ of cardinality $|R|$, such that the following conditions hold. There exists finite sets $\Delta_{\lambda,m} \subset \Gamma$, such that for all $\alpha < \xi \in R', m \in \{1, \dots, N\}$, $\lambda \in L$,

1. $F_m^{\lambda,\alpha} \cap F_m^{\lambda,\xi} = \Delta_{\lambda,m}$ (and $\operatorname{supp}(\tilde{x}_{\gamma_m^{\lambda,\alpha}} - \tilde{x}_{\gamma_m^{\lambda,\xi}}) \cap \Delta_{\lambda,m} = \emptyset$, see Corollary 3).
2. $F_m^{\lambda,\xi} \setminus \Delta_{\lambda,m} \subset \Gamma \setminus (\bigcup_{\alpha<\xi} B_{\lambda,\alpha} \cup \bigcup_{i\le j} A^i \cup \bigcup_{\lambda<\beta} A_\lambda)$

It is also easy to see that by a suitable choice of $\alpha_\lambda, \xi_\lambda \in R'$, for $\lambda \in L$, we may, without loss of generality, assume that putting for $m \in \{1, 2, \dots, N\}$

$$\hat{x}_{\lambda,m} := \tilde{x}_{\gamma_m^{\lambda,\alpha_\lambda}} - \tilde{x}_{\gamma_m^{\lambda,\xi_\lambda}},$$
$$\hat{f}_{\lambda,m} = h_{\gamma_m^{\lambda,\alpha_\lambda}},$$

we have, in addition, that $\operatorname{supp}(\hat{x}_{\lambda,m}) \cap \operatorname{supp}(\hat{x}_{\lambda',m'}) = \emptyset$ unless $\lambda = \lambda', m = m'$. Thus we have that

$$\{\hat{x}_{\lambda,m}, \hat{f}_{\lambda,m}\}_{m\in\{1,\dots,N\}}$$

is a biorthogonal $(2 + 2\varepsilon)$-bounded biorthogonal system such that vectors $\hat{x}_{\lambda,m}$, $m \in \{1, 2, \dots, N\}$, have disjoint supports with similar systems built previously in the inductive process. Next, we put

$$S_\lambda := B_{\lambda,\alpha_\lambda} \cup \bigcup_{m=1}^{N} \operatorname{supp}(\hat{x}_{\lambda,m}), \text{ for } \lambda \in L.$$

Again, we have $S_\lambda \cap S_{\lambda'} = \emptyset$, unless $\lambda = \lambda'$. Let $S_\lambda^1 = D_{\lambda,\alpha_\lambda} = \{\gamma_1^{\lambda,\alpha_\lambda}, \dots, \gamma_N^{\lambda,\alpha_\lambda}\}$. For every $\gamma = \gamma_m^{\lambda,\alpha_\lambda} \in S_\lambda^1$, we put $g_\gamma := \hat{f}_{\lambda,m}$, $y_\gamma := \hat{x}_{\lambda,m}$. This choice guarantees that conditions a., d., and e. are satisfied. It remains to use standard linear algebra in order to add elements g_γ, y_γ for $\gamma \in S_\lambda^2$, so that b. and c. will be satisfied.
To finish the inductive step, put $A^{j+1} := A^j \cup \bigcup_{\lambda\in L} S_\lambda$. For $\gamma \in C^j$, we let $x_\gamma^{j+1} := x_\gamma^j, f_\gamma^{j+1} := f_\gamma^j$. For $\lambda \in L$ put $\hat{x}_{\lambda,0} := x_\lambda$, $\hat{f}_{\lambda,0} := f_\lambda$. We have that $\{\hat{x}_{\lambda,m}; \hat{f}_{\lambda,m}\}_{m\in\{0,\dots,N\}}$ is a biorthogonal system. Let $W := (a_{i,j})_{i,j=0,\dots,N}$ be a matrix from Lemma 4. Put, for $k = 0, 1, 2, \dots, N$,

$$u_k^\lambda := \sum_{m=0}^{N} a_{k,m}\hat{x}_{\lambda,m}, \qquad v_k^\lambda := \sum_{m=0}^{N} a_{k,m}\hat{f}_{\lambda,m}.$$

Finally, define x_γ^{j+1} and f_γ^{j+1} for $\gamma \in A^{j+1}$ in the following way:

$$x_\gamma^{j+1} := \begin{cases} u_0^\lambda, & \text{if } \gamma = \lambda \in L, \\ u_m^\lambda, & \text{if } \gamma \in S_\lambda^1 \ (= D_{\lambda,\alpha_\lambda}),\ \gamma = \gamma_m^{\lambda,\alpha_\lambda}, \\ y_\gamma, & \text{if } \gamma \in S_\lambda^2. \end{cases}$$

$$f_\gamma^{j+1} := \begin{cases} v_0^\lambda, & \text{if } \gamma = \lambda \in L, \\ v_m^\lambda, & \text{if } \gamma \in S_\lambda^1 \ (= D_{\lambda,\alpha_\lambda}),\ \gamma = \gamma_m^{\lambda,\alpha_\lambda}, \\ g_\gamma, & \text{if } \gamma \in S_\lambda^2. \end{cases}$$

Since W is an orthonormal matrix, we obtain that $\{x_\gamma^{j+1}; f_\gamma^{j+1}\}_{\gamma\in\{\lambda\}\cup S_\lambda^1}$ is again a biorthogonal system, for every $\lambda \in L$.
It remains to estimate the norms of the new vectors (norms of functionals are not altered). By using the condition d., (2), and the orthonormality of W, we get the following estimate, whenever $\gamma \in \{\lambda\} \cup S_\lambda^1$:

$$\begin{aligned}\|x_\gamma^{j+1}\| &< 2^{-n/2}\|x_\lambda\| + (1+\sqrt{2}) \max_{1\le m\le N} \|\hat{x}_{\lambda,m}\| \\ &\le 2^{-n/2}C + (1+\sqrt{2})2(1+2\varepsilon) < 2(1+\sqrt{2}) + 13\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, these estimates imply conditions 4., 7., 8.. The remaining conditions follow from our construction by standard arguments.

□

Let us recall that Plichko in [Pl86] ([HMVZ], Example 5.19) has constructed an example of a WCG space which has no C-bounded M-basis, for every $C < 2$. On the other hand, in [Pl79] there is a generalization of the construction of $(1+\varepsilon)$-bounded M-basis in a separable space, to the case of WCG spaces, where one obtains $(2+\varepsilon)$-bounded M-bases. This result can be generalized to spaces with "many projections". In particular, one gets the following result.

Proposition 6. *Every Banach space belonging to a $\mathcal{P}$-class of nonseparable Banach space admits a $(2+\varepsilon)$-bounded M-basis for every $\varepsilon > 0$.*

Proof. Only formal changes in the proof in [Pl79] are needed. Let $\{P_\alpha\}_{\alpha\in\Gamma}$ be a projectional resolution of the identity in X, such that $P_\alpha(X)$ belong to $\mathcal{P}$ for all α. Each space $X_\alpha = (P_{\alpha+1} - P_\alpha)(X)$ contains a 1-complemented separable space Y_α, which is 2-complemented in the whole X. In each of Y_α, we can build an M-basis, $\{x_i^\alpha; f_i^\alpha\}_{i\in\mathbb{N}}$, such that $\{x_i^\alpha\}_{i=1,\ldots,N}$ is almost isometric to the unit basis of ℓ_2^N, for suitable values of N. Using complementability, it is possible to extend f_i^α, $i = 1,\ldots,N$, onto the whole X keeping the norm below $2+\varepsilon$. Using a standard device (see, e.g., [Fa97, Proposition 6.2.4]), we can glue all those partial biorthogonal systems into a full M-basis for X. This is the key ingredient in the proof, and the rest follows along the lines of [Pl79].

□

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