Ranges of operators and Derivatives

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Abstract

We show a unified method of proving the existence of C^1-Fréchet smooth and Lipschitz mappings which are surjective or whose range of the derivative contains the whole dual unit ball. As an application, under Martin Maximum axiom, we obtain a complete result for those spaces with density character ω_1.

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1. Introduction

In this paper we are concerned with the classical problem on existence of (non-linear) mappings —having a certain degree of differentiability— from a Banach space onto another. A variant of this problem consists of building a bump function on a given Banach space having a derivative whose range contains the entire dual unit ball. We provide here a unified treatment of those two questions and show that, in the case of Banach spaces of density ω_1 and under an additional hypothesis of set theory (the so called "Martin's Maximum axiom") both problems (the second in the unavoidable setting of Asplund spaces) have a positive solution. We strongly rely on a deep result of Todorčević on the existence of biorthogonal systems —under the mentioned axiom— for every Banach space of density ω_1. The situation is different under CH. This completely describes the picture in the case of Banach spaces of density ω_1.

The notation is standard. If X is a Banach space, then B_X denotes its closed unit ball, S_X its unit sphere. Given $x \in X$ and $r > 0$, we put $B(x;r) := x + rB_X$. A bump function on a Banach space is a real function on it with a bounded non-empty support. For unexplained concepts we refer to [**?**].

2. Technical results

Let X be Banach space endowed with a Fréchet differentiable norm $\|\cdot\|$ and which admits a normalized biorthogonal system $\{e_\gamma, e^*_\gamma\}_{\gamma\in\Gamma}$ such that $C := \sup\{\|e^*_\gamma\| : \gamma \in \Gamma\} < \infty$ and $\operatorname{card}\Gamma = \operatorname{dens} X$.

Since $\|\cdot\|$ is Fréchet, there exists a C^1-Fréchet smooth and K-Lipschitz function $b : X \to [0,1]$, such that $b(x) = 1$ for every $x \in (2/3)B_X$, and $b(x) = 0$ for every $x \in X \setminus B_X$. Indeed, take $h : \mathbb{R} \to [0,1]$ smooth, K-Lipschitz, such that $h(t) = 1$ for all $t \in [-2/3, 2/3]$, and $h(x) = 0$ for $|x| > 1$. The function b can be taken to be the composition $h \circ \|\cdot\|$.

For $n \in \mathbb{N}$ and $\tau \in (0,1)$, consider the set $\{e_\sigma : \sigma \in \Gamma^n\}$ $(\subset X)$, where

$$e_\sigma := \sum_{i=1}^{n} \frac{\tau^i}{3^{i-1}} e_{\sigma(i)}. \tag{1}$$

Note that $\|e_\sigma\| \le 3/2$ for every $\sigma \in \Gamma^n$.

Consider, too, the set $\{b_\sigma : \sigma \in \Gamma^n\}$, where $b_\sigma : X \to \mathbb{R}$ is the function defined by

$$b_\sigma(x) := \frac{\tau^{2n}}{3^n} b\left(\frac{2C3^n}{\tau^n}(x - e_\sigma)\right). \tag{2}$$

Obviously, for any $\sigma \in \Gamma^n$ we have $\operatorname{supp}(b_\sigma) \subset B\left(e_\sigma; \frac{\tau^n}{2C3^n}\right)$, $|b_\sigma(x)| \leq 3^{-n}\tau^{2n}$, and $|b'_\sigma(x)| \leq 2CK\tau^n$. Observe too that, for $\sigma \in \Gamma^n$,

$$b_\sigma(x) = \begin{cases} \frac{\tau^{2n}}{3^n}, & \text{if } \|x - e_\sigma\| \leq \frac{\tau^n}{C3^{n+1}}, \\ 0, & \text{if } \|x - e_\sigma\| \geq \frac{\tau^n}{2C3^n}. \end{cases} \tag{3}$$

Lemma 2.1. *The supports of b_σ and $b_{\sigma'}$ are disjoint if $\sigma \neq \sigma'$ in Γ^n.*

Proof. Let j be the first index in $\{1, 2, \ldots, n\}$ such that $\sigma(j) \neq \sigma'(j)$. Put $f := e^*_{\sigma(j)}$. If $j < n$,

$$\begin{aligned} \|e_\sigma - e_{\sigma'}\| &\geq \frac{1}{\|f\|}\langle e_\sigma - e_{\sigma'}, f\rangle \\ &\geq \frac{1}{C}\left(\frac{\tau^j}{3^{j-1}}\right)\left(1 - \sum_{i=1}^{\infty}\left(\frac{\tau}{3}\right)^i\right) \geq \frac{1}{2C}\frac{\tau^j}{3^{j-1}}. \end{aligned}$$

In case $j = n$ we get

$$\|e_\sigma - e_{\sigma'}\| = \frac{\tau^n}{3^{n-1}}\|e_{\sigma(n)} - e_{\sigma'(n)}\| \geq \frac{\tau^n}{3^{n-1}}\frac{1}{C}.$$

Then, in both cases,

$$\|e_\sigma - e_{\sigma'}\| \geq \frac{1}{2C}\frac{\tau^n}{3^{n-1}}. \tag{4}$$

In particular, given σ and σ' in Γ^n such that $\sigma \neq \sigma'$,

$$B\left(e_\sigma; \frac{\tau^n}{2C3^n}\right) \cap B\left(e_{\sigma'}; \frac{\tau^n}{2C3^n}\right) = \emptyset.$$

□

Denote by $\Gamma^{\mathbb{N}}$ the set of all sequences of elements in Γ. Given $\sigma \in \Gamma^{\mathbb{N}}$ we set

$$e_\sigma = \sum_{j=1}^{\infty}\frac{\tau^i}{3^{i-1}}e_{\sigma(i)}.$$

Observe that $\|e_\sigma\| \leq 3/2$. Denote by $\sigma^n \in \Gamma^n$ the initial segment of length n of σ. Then,

$$\|e_\sigma - e_{\sigma^n}\| \leq \tau \sum_{i=n+1}^{\infty}\left(\frac{\tau}{3}\right)^{i-1} = \tau\frac{(\tau/3)^n}{1-(\tau/3)}.$$

Choose $0 < \tau < \frac{3}{1+9C}$. Then $\frac{3\tau}{3-\tau} \leq \frac{1}{3C}$, and this implies

$$(\|e_\sigma - e_{\sigma^n}\| \leq) \ \ \tau \frac{(\tau/3)^n}{(1-\tau/3)} \leq \frac{\tau^n}{C3^{n+1}}. \tag{5}$$

In view of (3) we get

$$b_{\sigma^n}(e_\sigma) = \frac{\tau^{2n}}{3^n}. \tag{6}$$

Denote by T_τ the set $\left\{e_\sigma : \sigma \in \Gamma^{\mathbb{N}}\right\}$ $(\subset (3/2)B_X)$. The estimate (5) implies that for $n \in \mathbb{N}$,

$$T_\tau \subset \bigcup_{\sigma\in\Gamma^n} B\left(e_\sigma; \frac{\tau^n}{C3^{n+1}}\right). \tag{7}$$

Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a family of continuous affine mappings from X into some Banach space Y, such that

$$\sup_{\gamma\in\Gamma}\sup_{x\in B_X}\{\|f_\gamma(x)\|, \|f'_\gamma(x)\|\} \leq 1. \tag{8}$$

Take $n \in \mathbb{N}$. Lemma 2.1 shows that in the following expression, the summands have mutually disjoint supports.

$$b_n(x) := \sum_{\sigma\in\Gamma^n} b_\sigma(x) f_{\sigma(n)}(x), \quad x \in X. \tag{9}$$

In particular, for $x \in X$,

$$\|b_n(x)\| \leq \frac{\tau^{2n}}{3^n}\|x\|, \quad \|b'_n(x)\| \leq \left(2CK\tau^n + \frac{\tau^{2n}}{3^n}\right)\|x\|. \tag{10}$$

Let

$$\widetilde{b} := \sum_{n=1}^{\infty} b_n. \tag{11}$$

Due to the estimations in (10), the series in (11) defines a Lipschitz and C^1-Fréchet smooth mapping $\widetilde{b}$ from X to Y, and $\mathrm{supp}(\widetilde{b}) \subset 2B_X$.

Take $\sigma \in \Gamma^{\mathbb{N}}$. Due to (5) and (3) we get, for all $n \in \mathbb{N}$, $b_n(e_\sigma) = \frac{\tau^{2n}}{3^n} f_{\sigma(n)}(e_\sigma)$, hence

$$\widetilde{b}(e_\sigma) = \sum_{n=1}^{\infty} \frac{\tau^{2n}}{3^n} f_{\sigma(n)}(e_\sigma),$$

and

$$\widetilde{b}'(e_\sigma) = \sum_{n=1}^{\infty} \frac{\tau^{2n}}{3^n} f'_{\sigma(n)}(e_\sigma).$$

Lemma 2.2. *Let X be a Banach space endowed with a Fréchet smooth norm $\|\cdot\|$, which admits a biorthogonal system $\{e_\gamma, e^*_\gamma\}_{\gamma\in\Gamma}$ such that* $\operatorname{card}\Gamma = \operatorname{dens} X$. *Let Y be Banach space such that* $\operatorname{dens} Y = \operatorname{card}\Gamma$.

(i) *If $f_\gamma = y_\gamma$ is a family of constant functions on X, where $\{y_\gamma\}_{\gamma\in\Gamma}$ is dense in B_Y, then $\frac{\tau^2}{3}B_Y \subset \widetilde{b}(T_\tau)$.*
(ii) *If* $\operatorname{dens}\mathfrak{L}(X,Y) = \operatorname{dens} X$, *taking $\{f_\gamma\}_{\gamma\in\Gamma}$ a dense subset of $B_{\mathfrak{L}(X,Y)}$, then $\frac{\tau^2}{3}B_{\mathfrak{L}(X,Y)} \subset \widetilde{b}'(T_\tau)$.*
(iii) *If $Y = \mathrm{I\!R}$, and $\{f_\gamma\}_{\gamma\in\Gamma}$ is a dense subset of B_{X^*}, then $\frac{\tau^2}{3}B_{X^*} \subset \widetilde{b}'(T_\tau)$.*

Proof. The proofs of (i) and (ii) are similar, and clearly (iii) follows from (ii). It is then enough to show (i). Let $y \in \frac{\tau^2}{3}B_Y$. Choose $\gamma_1 \in \Gamma$ so that $\|y - \frac{\tau^2}{3}y_{\gamma_1}\| < \frac{\tau^4}{3^2}$. Then we can find $\gamma_2 \in \Gamma$ such that $\|y - \frac{\tau^2}{3}y_{\gamma_1} - \frac{\tau^4}{3^2}y_{\gamma_2}\| < \frac{\tau^6}{3^3}$. Proceed inductively, at each step choosing $\gamma_n \in \Gamma$ such that $\|y - \sum_{j=1}^n \frac{\tau^{2j}}{3^j}y_{\gamma_j}\| < \frac{\tau^{2(n+1)}}{3^{n+1}}$. Clearly, $y = \sum_{j=1}^\infty \frac{\tau^{2j}}{3^j}y_{\gamma_j}$ $(= \widetilde{b}(e_\sigma)$, where $\sigma := (\gamma_1, \gamma_2, \ldots))$, and this shows (i). □

3. Applications

In this section we give some applications of the results in the previous section, proving several known results on surjective mappings by using Lemma 2.2.

Theorem 3.1. (S.M. Bates, [**?** , Theorem 1]) *Let X, Y be separable infinite dimensional Banach spaces. Then there exists a C^1-Fréchet smooth and Lipschitz mapping from X onto Y.*

Proof. By a result of W.B Johnson and H.P. Rosenthal [**?** , Theorem 1.b.7], there exists a linear quotient mapping $Q : X \to Z$, where Z is a Banach space with a Schauder basis $\{e_n\}_{n=1}^\infty$. Assume, without loss of generality, that the basis is seminormalized and contained in the image $Q(B_X)$. Denote by $I : Z \to c_0$ the bounded linear operator $I(\sum_{i=1}^\infty a_i e_i) := (a_i)_{i=1}^\infty$. Fix $\tau = 3/7$ and the corresponding $T_\tau \subset c_0$. Note that $T_\tau \subset I \circ Q(B_X)$. By Lemma 2.2 there exist a C^1-Fréchet smooth and Lipschitz mapping $\widetilde{b}$ from c_0 to Y, with $\operatorname{supp}(\widetilde{b}) \subset 2B_{c_0}$ and a positive constant $c > 0$ such that $cB_Y \subset f(T_\tau)$. It is now easy to verify that $\widetilde{f}(x) = \sum_{n=1}^\infty n\widetilde{b}\big(\frac{4^n e_1 + I\circ Q(x)}{n}\big)$ is the sought surjective operator.

□

Theorem 3.2. (D. Azagra, R. Deville, [**?** , Theorem 1.3]) *Let X be a separable infinite dimensional Asplund space. Then there exists a C^1-Fréchet* *smooth and Lipschitz bump function whose range of the derivative contains the whole B_{X^*}.*

Proof. It follows from Lemma 2.2, by using the fact that the X contains a Schauder basic sequence, it admits a C^1-Fréchet smooth and K-Lipschitz bump function, and X^* is separable. □

The above theorem could be compared with James' characterization of reflexivity. Giles' characterization of spaces with the Mazur intersection property implies that the condition of having a nonempty interior for the range of the derivative of a convex and C^1-smooth function is equivalent to the reflexivity of the underlying space. Thus Theorem 3.2 emphasizes the crucial role of the convexity assumption. Note also that the Asplundness condition is necessary.

We provide a proof of the following result when the space X is DENS, i.e., dens(X)=w^*-dens(X^*).

Theorem 3.3. (D. Azagra, M. Jiménez-Sevilla, R. Deville, [**?** , Theorem 2.3]) *Let X be a Banach space with a Fréchet smooth bump and Y a Banach space so that* $\operatorname{dens} X = \operatorname{dens} \mathfrak{L}(X,Y)$. *Then, there exists a Fréchet smooth function $g : X \to Y$ so that g has bounded support and $g'(X) = \mathfrak{L}(X,Y)$.*

Proof. By [**?** , Theorem 5.3], the space X is Asplund. Using a result of M. Valdivia [**?**], there exists a biorthogonal system $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ in $X \times X^*$ such that $\overline{\operatorname{span}}^{w^*}\{x_\gamma;\ \gamma \in \Gamma\} = X^*$ (hence $\operatorname{dens}(X) = \operatorname{card}(\Gamma)$) and such that $\{x_\gamma; x^*_\gamma\big|_E\}$ is a shrinking Markushevich basis in $E \times E^*$, where $E := \overline{\operatorname{span}}\{x_\gamma;\ \gamma \in \Gamma\}$. By [**?** , Theorem 11.23], E has a C^1-Fréchet smooth equivalent norm. By Lemma 2.2 there exists a Lipschitz and C^1-Fréchet smooth function with the properties listed there. The rest of the proof is similar to the proof of Theorem 3.2. □

Remark. A complete proof of Theorem 3.3 can be given along the same lines by reformulating Lemma 2.2 in terms of maximal separated subsets of X.

3.1. New results

S. M. Bates identified in [**?**] some situations when smooth surjections exist. We are going to obtain a complete result for density ω_1, subject to a strong additional axiom of set theory.

Theorem 3.4. *(Martin's Maximum axiom)*

Let X, Y be infinite dimensional Banach spaces of density ω_1. Then there exists a C^1-Fréchet smooth and Lipschitz mapping from X onto Y.

Proof. By Todorčević's results [**?** , Theorems 4.48, 4.49] there exists a linear quotient mapping $Q : X \to Z$, where Z is a Banach space with a long Schauder basis $\{e_\alpha\}_{\alpha<\omega_1}$. Assume, without loss of generality, that the basis is seminormalized and contained in the image $Q(B_X)$. Denote by $I : Z \to c_0(\omega_1)$ the bounded linear operator $I(\sum_\alpha a_\alpha e_\alpha) = (a_\alpha)_{\alpha<\omega_1}$. Fix $\tau = 3/7$ and the corresponding $T_\tau \subset c_0(\omega_1)$. Note that $T_\tau \subset I \circ Q(B_X)$. By Lemma 2.2 there exist a C^1-Fréchet smooth and Lipschitz mapping f from $c_0(\omega_1)$ to Y, with $\mathrm{supp}(f) \subset 2B_{c_0(\omega_1)}$ and a positive constant c such that $cB_Y \subset f(T_\tau)$. It is now easy to verify that $\tilde{f}(x) = \sum_{n=1}^{\infty} nf\big(\frac{4^n e_1 + I\circ Q(x)}{n}\big)$ is the sought surjective operator. □

Similarly, we can generalize Azagra and Deville theorem. However, note that we are not assuming the existence of a C^1 smooth bump function. This is important, since the existence of such bumps is an open problem for Asplund spaces of density ω_1.

Theorem 3.5. *Let X be a Asplund space of density ω_1. Then there exists a C^1-Fréchet smooth and Lipschitz function whose range of the derivative contains the whole B_{X^*}.*

Proof. Again, by Todorčević's results [**?** , Theorems 4.48, 4.49] there exists a linear quotient mapping $Q : X \to Z$, where Z is a Banach space with a seminormalized long Schauder basis $\{e_\alpha\}_{\alpha<\omega_1}$ contained in the image $Q(B_X)$. Denote by $I : Z \to c_0(\omega_1)$ the bounded linear operator $I(\sum_\alpha a_\alpha e_\alpha) = (a_\alpha)_{\alpha<\omega_1}$. Fix $\tau = 3/7$ and the corresponding $T_\tau \subset c_0(\omega_1)$. Note that $T_\tau \subset I \circ Q(B_X)$. By Lemma 2.2 there exist a C^1-Fréchet smooth and Lipschitz mapping f from $c_0(\omega_1)$ to Y, with $\mathrm{supp}(f) \subset 2B_{c_0(\omega_1)}$ and a positive constant c such that $cB_{X^*} \subset f'(T_\tau)$. The function $\tilde{f} : X \to Y$ defined by $f \circ I \circ Q$ is the function we were looking for. □

Remark 3.6. *Note that the function $\tilde{f}$ defined in the previous proof is not necessarily a bump function on X.*

Finally, we generalize Hájek's result on C^2 smooth surjections, see [**?**].

Theorem 3.7. *(Martin's Axiom, MA_{ω_1})*

There exists no surjective C^1-smooth operator, with locally uniformly continuous derivative, from $c_0(\omega_1)$ onto ℓ_2 (or more generally any separable space which admits a noncompact operator into some ℓ_p, $p < \infty$).

Proof. If we assume that there exists a surjective C^1-smooth operator $T : c_0(\omega_1) \to \ell_2$, with locally uniformly continuous derivative, by P. Hájek's result [? , Corollary 8], such operator is locally compact, which means that ℓ_2 could be covered by a countable collection of norm compact sets. However, MA_{ω_1} implies that the sigma ideal of all compact sets in a Polish space is closed for ω_1 unions, [? , Corollary 22J]. This is a contradiction. □

Remark 3.8. *Under MM, both Theorems 3.4, 3.5 and 3.7 hold, since Martin Maximum Axiom implies Martin's Axiom, see [? , pag. 53]. It should be pointed out that both axioms contradict the continuous hypothesis and that assuming CH changes the situation.*

Proposition 3.9. *(CH) Every Banach space of density $c = \omega_1$ is a range of a C^∞ smooth and Lipschitz operator from $c_0(c)$.*

Proof. Since $\operatorname{dens} X = \omega_1 = c$, then $\operatorname{card} X = c$. Taking $\{e_\gamma\}_\gamma \in c$ the standard basis of $c_0(c)$, define the operators $T_\gamma : c_0(c) \to c_0(c)$ defined by $T_\gamma(x) = 2(x - e_\gamma)$. Consider a C^∞ smooth and Lipschitz bump function $b : c_0(c) \to \mathbb{R}$ such that $b\big(\frac{2}{3}B_{c_0(c)}\big) = 1$ and $b\big(c_0(c) \setminus B_{c_0(c)}\big) = 0$. Consider the family of smooth bumps $\{b_\gamma = b \circ T_\gamma : \gamma \in c\}$. Define the function $\tilde{b}(x) = \sum_{\gamma \in c} b_\gamma(x)\varphi(\gamma)$, where $\varphi : c \to X$ is bijective. It is clear that $\tilde{b}$ is a surjective C^∞ smooth and Lipschitz map. □

Let us estate some open questions in the direction of our results.

Question 3.10. *Is it true that given density α, all spaces of this density are C^1 smooth images of each other?*

Question 3.11. *Is it true that for density c, question 3.10 holds even for C^∞ smooth operators?*