BIORTHOGONAL SYSTEMS IN BANACH SPACES
Petr Hájek ¹
Vicente Montesinos ²
Jon Vanderwerff ³
Václav Zizler ⁴

¹ Mathematical Institute of the Academy of Sciences of the Czech Republic, Žitná 25, Praha 1, 11567 Prague, Czech Republic. Email: hajek@math.cas.cz
² Departamento de Matemática Aplicada, E. T. S. I. Telecomunicación, Universidad Politécnica de Valencia, C/Vera, s/n. 46071 Valencia, Spain and Instituto de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, C/Vera, s/n. 46071 Valencia, Spain. Email: vmontesinos@mat.upv.es
³ Department of Mathematics, La Sierra University, Riverside, CA., U.S.A. Email: jvanderv@lasierra.edu
⁴ Department of Mathematical Sciences University of Alberta, 632 Central Academic Building, Edmonton, Alberta T6G 2G1, Canada, and Mathematical Institute of the Academy of Sciences of the Czech Republic, Žitná 25, Praha 1, 11567 Prague, Czech Republic. Email: zizler@math.cas.cz
To Paola, Danuta, Judith, and Jarmila
The main theme of this book is the relation between the global structure of Banach spaces and the various types of generalized “coordinate systems”—or “bases”—they possess. This subject is not new; in fact, it has been investigated since the inception of the study of Banach spaces. The existence of a nice basis in a Banach space is very desirable. Bases are not only very useful in many analytic calculations and various constructions but can also be used to classify Banach spaces. The long-standing hope of having such a system in every Banach space was shattered first by Enflo’s construction of a separable Banach space without a Schauder basis and, more recently, by the work of Argyros, Gowers, Maurey, Schlumprecht, Tsirelson, and others that has produced hereditarily indecomposable Banach spaces and, in particular, Banach spaces containing no unconditional Schauder basic sequence. In light of these results, the classical rich structural theory of various special classes of Banach spaces, such as $\mathcal{L}_p$ spaces, separable $C(K)$ spaces, or Banach lattices, to name a few separable classes, as well as nonseparable weakly compactly generated or weakly countably determined (Vašák) spaces—and the coordinate systems they possess—increases in value, importance, and complexity. Of course, in order to obtain more general results, one has to weaken the analytic properties of the desired systems.

In this book, we systematically investigate the concepts of Markushevich bases, fundamental systems, total systems, and their variants. The material naturally splits into the case of separable Banach spaces, as is treated in the first two chapters, and the nonseparable case, which is covered in the remainder of the book.

Our starting point is that every separable Banach space has a fundamental total biorthogonal system. This was proved by Markushevich, and hence today such systems are called Markushevich bases. However, there are now several significantly stronger versions of this result. Indeed, using Dvoretzky’s theorem combined with orthogonal transformation techniques, Pelczyński and, independently, Plichko, obtained $(1 + \varepsilon)$-bounded Markushevich bases in every separable Banach space. More recently, Terenzi has constructed several
versions of strong Markushevich bases in all separable spaces that, in particular, allow one to recover every vector from its coordinates using permutations and blockings. These results, together with some background material, are treated in Chapter 1.

In Chapter 2, we present some classical as well as some recent results on the universality of spaces. This includes basic material on well-founded trees, applications of the Kunen-Martin theorem, theorems of Bourgain and Szlenk, and a thorough introduction to the geometric theory of the Szlenk index.

Chapter 3’s material is preparatory in nature. In particular, it presents some results and techniques dealing with weak compactness, decompositions, and renormings that are useful in the nonseparable setting.

Chapter 4 focuses on the existence of total, fundamental, or, more generally, biorthogonal systems in Banach spaces. Among other things, we give Plichko’s characterization of spaces admitting a fundamental biorthogonal system, the Godefroy-Talagrand results on representable spaces, a version under the clubsuit axiom (♣) of Kunen’s example of a nonseparable $C(K)$ space without any uncountable biorthogonal system, and finally the recent result of Todorcević under Martin’s Maximum (MM) axiom on the existence of a fundamental biorthogonal system for every Banach space of density $\omega_1$. These latter results are typically obtained by using powerful infinite combinatorial methods—in the form of additional axioms in ZFC.

Many Banach spaces with nice structural and renormability properties can be classified according to the types of Markushevich bases they possess. Chapters 5 and 6 present, in detail, characterizations of several important classes of Banach spaces using this approach. This concerns spaces that are weakly compactly generated, weakly Lindelöf determined, weakly countably determined (Vasák), and Hilbert generated, as well as some others.

Chapter 7 deals with the class of spaces possessing long unconditional Schauder bases and their renormings. In particular, elements of the Pełczyński and Rosenthal structural theory of spaces containing $c_0(I)$, $\ell_\infty$, and operators that fix these spaces are discussed. The Pełczyński, Argyros, and Talagrand circle of results on the containment of $\ell_1(I)$ in dual spaces is also included.

The concluding chapter, Chapter 8, is devoted to some applications of biorthogonal and other weaker systems. Among other things, it presents some results on the existence of support sets, the theory of norm-attaining operators, and the Mazur intersection property.

It is our hope that the contents of this book reflect that nonseparable Banach space theory is a flourishing field. Indeed, this is a field that has recently attracted the attention of researchers not only in Banach space theory but also in many other areas, such as topology, set theory, logic, combinatorics, and, of course, analysis. This has influenced the choice of topics selected for this book. We tried to illustrate that the use of set-theoretical methods is, in some cases, unavoidable by showing that some important problems in the structural theory of nonseparable Banach spaces are undecidable in ZFC.
Given the breadth of this field and the diverse areas that impinge on the subject of this book, we have endeavored to compile a large portion of the relevant results into a streamlined exposition—often with the help of simplifications of the original proofs. In the process, we have presented a large variety of techniques that should provide the reader with a good foundation for future research. A substantial portion of the material is new to book form, and much of it has been developed in the last two decades. Several new results are included.

Unfortunately, for reasons of space and time, it has not been possible to include all relevant results in the area, and we apologize to all authors whose important results have been left out. Nevertheless, we believe that the present text, together with [ArTod05], [DGZ93a], [Fab97], [JoLi01h, Chap. 23 and 41], [McNe92], [Negr84], and the introductory [Fa01], will help the reader to gain a clear picture of the current state of research in nonseparable Banach space theory.

We especially hope that this book will inspire some young mathematicians to choose Banach space theory as their field of interest, and we wish readers a pleasant time using this book.

Acknowledgments

We are indebted to our institutions, which enabled us to devote a significant amount of time to our project. We therefore thank the Mathematical Institute of the Czech Academy of Sciences in Prague (Czech Republic), the Department of Mathematics of the University of Alberta (Canada), the Universidad Politécnica de Valencia (Spain) and La Sierra University, California (USA). For their support we thank the research grant agencies of Canada, the Czech Republic, and Spain (Ministerio de Universidades e Investigación and Generalitat Valenciana). In particular, this work was supported by the following grants: NSERC 7926 (Canada), Institutional Research Plan of the Academy of Sciences of the Czech Republic AV0Z10190503, IAA100190502, GA ČR 201/04/0090 and IAA100190610 (Czech Republic), and Projects BFM2002-01423, MTM2005-08210, and the Research Program of the Universidad Politécnica de Valencia (Spain).

We are grateful to our colleagues and students for discussions and suggestions concerning this text. We are especially indebted to Marián Fabian, Gilles Godefroy, Gilles Lancien, and Stevo Todoričević, who provided advice, support and joint material for some sections.

We thank the Editorial Board of Springer-Verlag, in particular editors Karl Dilcher and Mark Spencer, for their interest in this project. Our gratitude extends to their staff for their help and efficient work in publishing this text. In particular, we thank the copyeditor for his/her excellent and precise work that improved the final version of the book.
Above all, we thank our wives, Paola, Danuta, Judith, and Jarmila, for their understanding, patience, moral support, and encouragement.
Prague, Valencia, and Riverside.
September, 2006

Petr Hájek
Vicente Montesinos
Jon Vanderwerff
Václav Zizler
Contents

Preface ................................................................. VII

Standard Definitions, Notation, and Conventions ........ XVI

1 Separable Banach Spaces ........................................ 1
1.1 Basics ........................................................... 2
Minimal systems; basic facts on biorthogonal systems (b.o.s.); fundamental minimal systems; Markushevich bases (M-bases); norming M-bases; shrinking b.o.s.; boundedly complete b.o.s.
1.2 Auerbach Bases .................................................. 5
Every finite-dimensional space has an Auerbach basis; Day's construction of a countable infinite Auerbach system
1.3 Existence of M-bases in Separable Spaces ................. 8
Markushevich theorem on the existence of M-bases in separable spaces; every space with a \( w^* \)-separable dual has a bounded total b.o.s.
1.4 Bounded Minimal Systems .................................... 9
Peà lczyński-Plichko \((1+\varepsilon)\)-bounded M-basis theorem; no shrinking Auerbach systems in \( C(K) \) (Plichko)
1.5 Strong M-bases ................................................... 21
Terenzi's theorem on the existence of a strong M-basis in every separable space; flattened perturbations; Vershynin’s proof of Terenzi’s theorem; strong and norming M-bases; strong and shrinking M-bases
1.6 Extensions of M-bases .......................................... 29
Gurarii-Kadets extension theorem; extension of bounded M-bases to bounded M-bases (Terenzi); extension of bounded M-bases and quotients; a negative result for extending Schauder bases; extension in the direction of quasicomplements (Milman)
1.7 \( \omega \)-independence ............................................. 38
Relation to biorthogonal systems; every \( \omega \)-independent system in a separable space is countable (Fremlin-Kalton-Sersouri)
1.8 Exercises .......................................................... 42
2 Universality and the Szlenk Index ................................. 45
  2.1 Trees in Polish Spaces ........................................... 46
     Review of some techniques on trees in Polish spaces; Kunen-Martin result
     on well-founded trees in Polish spaces
  2.2 Universality for Separable Spaces .............................. 49
     Complementable universality of (unconditional) Schauder bases (Kadets-
     Pelczyński); no complementable universality for superreflexive spaces
     (Johnson-Szankowski); if a separable space is isomorphically universal
     for all reflexive separable spaces, then it is isomorphically universal for
     all separable spaces (Bourgain); in particular, there is no separable
     Asplund space universal for all separable reflexive spaces (Szlenk); there
     is no superreflexive space universal for all superreflexive separable spaces
     (Bourgain); there is a reflexive space with Schauder basis complementable
     universal for all superreflexive spaces with Schauder bases (Prus); there
     is a separable reflexive space universal for all separable superreflexive spaces
     (Odell-Schlumprecht)
  2.3 Universality of M-bases ........................................... 57
     Iterated $w^*$-sequential closures of subspaces of dual spaces (Banach-
     Godun-Ostrovskij); no universality for countable M-bases (Plichko)
  2.4 Szlenk Index ....................................................... 62
     Szlenk derivation; properties of the Szlenk index ($\Sigma(X)$); for separable
     spaces, $\Sigma(X) = \omega$ if and only if $X$ admits a UKK* renorming (Knaust-
     Odell-Schlumprecht)
  2.5 Szlenk Index Applications to Universality ..................... 70
     Under GCH, if $\tau$ is an uncountable cardinal, there exists a compact space
     $K$ of weight $\tau$ such that every space of density $\tau$ is isometrically iso-
     morphic to a subspace of $C(K)$ (Yesenin-Volpin); for infinite cardinality $\tau$,
     there is no Asplund space of density $\tau$ universal for all reflexive spaces
     of density $\tau$; there is no WCG space of density $\omega_1$ universal for all WCG
     spaces of density $\omega_1$ (Argyros-Benyamini)
  2.6 Classification of $C[0, \alpha]$ Spaces ............................ 73
     Mazurkiewicz-Sierpiński representation of countable compacta; Bessaga-
     Pelczyński isomorphic classification of $C(K)$ spaces for $K$ countable
     compact as spaces $C[0,\omega^\alpha]$ for $\alpha < \omega_1$; Samuel’s evaluation of the Szlenk
     index of those spaces
  2.7 Szlenk Index and Renormings .................................... 77
     $w^*$-dentability index $\Delta(X)$ of a space $X$; Asplund spaces with $\Delta(X) < \omega_1$
     have a dual LUR norm; characterization of superreflexivity by $\Delta(X) \leq \omega_1$
     estimating $\Delta(X)$ from above by $\Psi(S\Sigma(X))$
  2.8 Exercises ......................................................... 82

3 Review of Weak Topology and Renormings ..................... 87
  3.1 The Dual Mackey Topology ...................................... 88
     Grothendieck’s results on the Mackey topology on dual spaces
  3.2 Sequential Agreement of Topologies in $X^*$ ..................... 92
Characterizations of spaces not containing $\ell_1$ by using the Mackey topology $\tau(X^*, X)$ in the dual; results of Emmanuele, Ørno and Valdivia

3.3 Weak Compactness in $ca(\Sigma)$ and $L_1(\lambda)$ ......................... 95
Weak compactness in $L_1(\mu)$; weak compactness in $ca(\Sigma)$; Grothendieck results on the Mackey topology $\tau(X^*, X)$ in $(L_1(\mu))^*$ and $(C(K))^*$; Josefson-Nissenzweig theorem

3.4 Decompositions of Nonseparable Banach Spaces ....................... 102
Corson and Valdivia compacta; weakly Lindelöf-determined spaces; projectional resolutions of the identity (PRI); projectional generators (PG); separable complementation property; every nonseparable space with a PG has a PRI; PG in WCG spaces

3.5 Some Renorming Techniques .............................................. 107
LUR renorming of a space with strong M-basis (Troyanski); LUR renorming of subspaces of $\ell_\infty$ closely related to $\sigma$-shrinkable M-bases; weak 2-rotund property of Day’s norm on $c_0(I)$; Troyanski’s results on uniform properties of the Day norm

3.6 A Quantitative Version of Krein’s Theorem ......................... 119
Closed convex hulls of $\varepsilon$-weakly relatively compact sets

3.7 Exercises ............................................................. 125

4 Biorthogonal Systems in Nonseparable Spaces .................. 131
4.1 Long Schauder Bases .................................................. 132
Uncountable version of Mazur’s technique; bounded total b.o.s. in every space

4.2 Fundamental Biorthogonal Systems ................................. 137
The existence of fundamental biorthogonal systems (f.b.o.s.) implies the existence of bounded f.b.o.s. lifting f.b.o.s. from quotients; f.b.o.s. in $\ell_\infty$; f.b.o.s. in Johnson-Lindenstrauss spaces (JL); $\ell_\infty(I)$ has an f.b.o.s. iff card $I \leq c$

4.3 Uncountable Biorthogonal Systems in ZFC ....................... 143
Uncountable b.o.s. in $C(K)$ when $K$ contains a nonseparable subset; b.o.s. in representable spaces; every nonseparable dual space contains an uncountable b.o.s.

4.4 Nonexistence of Uncountable Biorthogonal Systems ............ 148
Under axiom $\clubsuit$ there is a scattered nonmetrizable compact $K$ such that $(C(K))^*$ is hereditarily $w^*$-separable and $C(K)$ is hereditarily weakly Lindelöf; in particular, $C(K)$ does not contain an uncountable b.o.s.

4.5 Fundamental Systems under Martin’s Axiom ..................... 152
Under Martin’s axiom $\text{MA}_{\omega_1}$, any Banach space with density $\omega_1$ with a $w^*$-countably tight dual unit ball has an f.b.o.s.; under Martin’s Maximum (MM), any Banach space of density $\omega_1$ has an f.b.o.s. (Todorčević)

4.6 Uncountable Auerbach Bases ....................................... 158
Any nonseparable space with a $w'$-separable dual ball has a norm with no Auerbach basis (Godun-Lin-Troyanski)

4.7 Exercises ............................................................. 161
5 Markushevich Bases .................................................. 165
  5.1 Existence of Markushevich Bases ............................... 165
      \( \mathcal{P} \)-classes; every space in a \( \mathcal{P} \)-class has a strong M-basis; spaces with M-bases and injections into \( c_0(\Gamma) \); spaces with strong M-bases failing the separable complementation property; the space \( \ell_\infty(\Gamma) \) for infinite \( \Gamma \) has no M-basis (Johnson)
  5.2 M-bases with Additional Properties ............................ 170
      Every space with an M-basis has a bounded M-basis; WCG spaces without a 1-norming M-basis and variations; \( C[0,\omega_1] \) has no norming M-basis; countably norming M-bases
  5.3 \( \Sigma \)-subsets of Compact Spaces ............................ 176
      Topological properties of Valdivia compacta; Deville-Godefroy-Kalenda-Valdivia results
  5.4 WLD Banach Spaces and Plichko Spaces .......................... 179
      WLD spaces and full PG; characterizations of WLD spaces in terms of M-bases; weakly Lindelöf property of WLD spaces; property C of Corson; Plichko spaces; Kalenda’s characterization of WLD spaces by PRI when \( \text{dens} X = \omega_1 \); spaces with \( w^* \)-angelic dual balls without M-bases
  5.5 \( C(K) \) Spaces that Are WLD ................................. 187
      Corson compacta with property M; under CH, example of a Corson compact \( L \) such that \( C(L) \) has a renorming without PRI; on the other hand, under Martin’s axiom \( \text{MA}_{\omega_1} \), every Corson compact has property M
  5.6 Extending M-bases from Subspaces ............................. 191
      Extensions from subspaces of WLD spaces; the bounded case for \( \text{dens} X < \aleph_\omega \); \( c_0(\Gamma) \) is complemented in every Plichko overspace if card \( \Gamma < \aleph_\omega \); on the other hand, under GCH, \( c_0(\aleph_\omega) \) may not be complemented in a WCG overspace; extensions of M-bases when the quotient is separable; under \( \clubsuit \), example of a space with M-basis having a complemented subspace with M-basis that cannot be extended to the full space
  5.7 Quasicomplements ................................................. 197
      M-bases and quasicomplements; the Johnson-Lindenstrauss-Rosenthal theorem that \( Y \) is quasicomplemented in \( X \) whenever \( Y^* \) is \( w^* \)-separable and \( X/Y \) has a separable quotient; every subspace of \( \ell_\infty \) is quasicomplemented; Godun’s results on quasicomplements in \( \ell_\infty \); quasicomplements in Grothendieck spaces; Josefson’s theorems on limited sets; quasicomplementation of Asplund subspaces in \( \ell_\infty(\Gamma) \) iff \( X^* \) is \( w^* \)-separable; in particular, \( c_0(\Gamma) \) is not quasicomplemented in \( \ell_\infty(\Gamma) \) whenever \( \Gamma \) is uncountable (Lindenstrauss); the separable infinite-dimensional quotient problem
  5.8 Exercises ......................................................... 203

6 Weak Compact Generating .......................................... 207
  6.1 Reflexive and WCG Asplund Spaces ......................... 207
      Characterization of reflexivity by M-bases and by 2R norms; characterization of WCG Asplund spaces
  6.2 Reflexive Generated and Vášáik Spaces ..................... 212
Reflexive- and Hilbert-generated spaces; Fréchet $M$-smooth norms; weakly compact and $\sigma$-compact $M$-bases; $M$-basis characterization of WCG spaces; weak dual 2-rotund characterization of WCG; $\sigma$-shrinkable $M$-bases; $M$-basis characterization of subspaces of WCG spaces; continuous images of Eberlein compacts (Benyamini, Rudin, Wage); weakly $\sigma$-shrinkable $M$-bases; $M$-basis characterization of Vašák spaces; $M$-basis characterization of WLD spaces

6.3 Hilbert Generated Spaces ........................................... 225
$M$-basis characterization of Hilbert-generated spaces; uniformly Gâteaux (UG) smooth norms; uniform Eberlein compacts; $M$-basis characterization of spaces with UG norms; characterizations of Eberlein, uniform Eberlein, and Gul’ko compacts (Farmaki); continuous images of uniform Eberlein compacts (Benyamini, Rudin, Wage)

6.4 Strongly Reflexive and Superreflexive Generated Spaces ...... 233
Metrizability of $B_{X^*}$ in the Mackey topology $\tau(X^*, X)$; weak sequential completeness of strongly reflexive generated spaces (Edgar-Wheeler); applications of strongly superreflexive generated spaces to $L_1(\mu)$; superreflexivity of reflexive subspaces of $L_1(\mu)$ (Rosenthal); uniform Eberlein compacta in $L_1(\mu)$; almost shrinking $M$-bases (Kalton)

6.5 Exercises .............................................................. 239

7 Transfinite Sequence Spaces ............................................ 241
7.1 Disjointization of Measures and Applications ................. 241
Rosenthal disjointization of measures; applications to fixing $c_0(\Gamma)$ and $\ell_\infty(\Gamma)$ in $C(K)$ (Pełczyński, Rosenthal); applications to weakly compact operators; subspaces of $c_0(\Gamma)$ and containment of $c_0(\Gamma)$ in $C(K)$ spaces

7.2 Banach Spaces Containing $\ell_1(\Gamma)$ ............................ 252
Characterization of spaces containing $\ell_1(\Gamma)$ by properties of dual unit balls and quotients (Pełczyński, Argyros, Talagrand, Haydon)

7.3 Long Unconditional Bases .......................................... 259
Characterization of Asplund, WCG (Johnson), WLD (Argyros-Mercourakis), strongly reflexive-generated (Mercourakis-Stamati), UG or URED renormable (Troyanski) spaces with unconditional bases

7.4 Long Symmetric Bases ............................................. 266
Troyanski’s characterizations of spaces with symmetric basis having a UG or URED norm; applications to separable spaces

7.5 Exercises .............................................................. 270

8 More Applications .......................................................... 273
8.1 Biorthogonal Systems and Support Sets ....................... 273
There are no separable support sets (Rolewicz); $X$ has a bounded support set if it has an uncountable semibiorthogonal system; if $K$ is a nonmetrizable scattered compact, then $C(K)$ has a support set; under MM, a space is separable iff there is no support set (Todorčević); consistency of the existence of nonseparable spaces without support sets
8.2 Kunen-Shelah Properties in Banach Spaces ................. 276
Representation of closed convex sets as kernels of nonnegative $C^\infty$ functions; representation of closed convex sets as intersections of countably many half-spaces and $w^*$-separability; $w_1$-polyhedrons; for a separable space $X$, $X^*$ is separable iff every dual ball in $X^{**}$ is $w^*$-separable; if $X^*$ is $w^*$-hereditarily separable (e.g., $X = C(L)$ for $L$ in Theorem 4.41), then $X$ has no uncountable $\omega$-independent system

8.3 Norm-Attaining Operators ........................................ 284
Lindenstrauss’ properties $\alpha$ and $\beta$; renormings and norm-attaining operators; renorming spaces to have property $\alpha$ (Godun, Troyanski)

8.4 Mazur Intersection Properties ................................. 289
Characterizations of spaces with the Mazur intersection property (MIP); Asplund spaces without MIP renormings (Jiménez-Sevilla, Moreno); equivalence of Asplundness and MIP in separable spaces; if $X$ admits a b.o.s. $\{x_\gamma; x_\gamma^*; \gamma \in \Gamma\}$ such that $\operatorname{span}\{x_\gamma^*; \gamma \in \Gamma\} = X^*$, then $X$ has an MIP renorming; the non-Asplund space $\ell_1 \times \ell_2(c)$ has an MIP renorming (Jiménez-Sevilla, Moreno)

8.5 Banach Spaces with only Trivial Isometries .................. 297
Every space can be renormed to have only $\pm$-identity as isometries

8.6 Exercises ............................................................... 300

References ................................................................. 303

Symbol Index ............................................................ 323

Subject Index ........................................................... 327

Author Index ............................................................ 335
References


[BGT82] M. Bell and J. Ginsburg, and S. Todorčević, Countable spread of exp$\mathcal{Y}$ and $\lambda\mathcal{Y}$, Topology Appl. 14 (1982), no. 1, 1–12.


References


References


[Ha77] R. Haydon, *On Banach spaces which contain $\ell_1(\tau)$ and types of measures on compact spaces*, Israel J. Math. **28** (1977), 313–324.


312 References


[John77] W.B. Johnson, On quotients of L_p which are quotients of \ell_p, Compositio Math. 34 (1977), 69–89.

References


References


[Kis75] S.V. Kislyakov, Classification of spaces of continuous functions on ordinals, Sibirsk. Mat. Z. 16 (1975), 293–300.


[Kosz05] P. Koszmider, A space \(C(K)\) where all nontrivial complemented subspaces have big densities, Studia Math. 168 (2005), 109–127.


[KuTr82] D. Kutzarova and S.L. Troyanski, Reflexive Banach spaces without equivalent norms which are uniformly convex or uniformly differentiable in every direction, Studia Math. 72 (1982), 91–95.


W. Marciszewski, On Banach spaces $C(K)$ isomorphic to $c_0(\Gamma)$, Studia Math. 156, 3 (2003), 295–302.


J.P. Moreno, Geometry of Banach spaces with $(\alpha, \epsilon)$-property or $(\beta, \epsilon)$-property, Rocky Mount. J. Math. 27 (1997), 241–256.


316 References


A. Pełczyński, All separable Banach spaces admit for every $\epsilon > 0$ fundamental and total biorthogonal sequences bounded by $1 + \epsilon$, Studia Math. 55 (1976), 295–304.


References


[Raja] M. Raja, Dentability indices with respect to measures of noncompactness, preprint.


References 321


References


<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-SCP, 1-separable complementation property</td>
<td>105</td>
<td></td>
</tr>
<tr>
<td>2R, 2-rotund</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>$\aleph$, a cardinal number</td>
<td>XVII</td>
<td></td>
</tr>
<tr>
<td>$\aleph_0$, the cardinal of $\mathbb{N}$</td>
<td>XVII</td>
<td></td>
</tr>
<tr>
<td>$\aleph_1$, the first uncountable cardinal</td>
<td>XVII</td>
<td></td>
</tr>
<tr>
<td>$A^{&lt;\omega}$, union of the finite powers of a set</td>
<td>46</td>
<td></td>
</tr>
<tr>
<td>$A'$, the power of a set</td>
<td>46</td>
<td></td>
</tr>
<tr>
<td>$ba(\Sigma)$, bounded finitely additive scalar-valued measures</td>
<td>95</td>
<td></td>
</tr>
<tr>
<td>$B^+_\alpha$, intermediate open derivation of a set $B$</td>
<td>62</td>
<td></td>
</tr>
<tr>
<td>BAP, bounded approximation property</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>$\beta(E, F)$, the strong topology on $E$ associated to the pair $(E, F)$</td>
<td>88</td>
<td></td>
</tr>
<tr>
<td>$\beta T$, the Čech-Stone compactification of a completely regular topological space</td>
<td>XVII</td>
<td></td>
</tr>
<tr>
<td>$B_X$, the closed unit ball of a space $X$</td>
<td>XVII</td>
<td></td>
</tr>
<tr>
<td>$c$, the cardinal of the continuum</td>
<td>XVII</td>
<td></td>
</tr>
<tr>
<td>$C(K)$, the space of continuous functions on a compact space $K$</td>
<td>19</td>
<td></td>
</tr>
<tr>
<td>$c_0(\Gamma)$, the space of all finitely supported vectors with coordinates in $\Gamma$</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>$c_0(\Gamma)$, the space of all finitely supported vectors with coordinates in $\Gamma$</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>CAP, compact approximation property</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>$ca^+(\Sigma)$, the positive members of $ca(\Sigma)$</td>
<td>95</td>
<td></td>
</tr>
<tr>
<td>$ca(\Sigma)$, countable additive scalar-valued measures</td>
<td>95</td>
<td></td>
</tr>
<tr>
<td>CH, continuum hypothesis</td>
<td>153</td>
<td></td>
</tr>
<tr>
<td>$\chi_A$, the characteristic function of a set $A$</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>$\chi(K)$, the maximum nonempty Cantor derivative</td>
<td>71</td>
<td></td>
</tr>
<tr>
<td>♠, clubsuit axiom</td>
<td>148</td>
<td></td>
</tr>
<tr>
<td>$c_0$, the space of all null sequences</td>
<td>XVII</td>
<td></td>
</tr>
<tr>
<td>cof $\tau$, the cofinality of a cardinal $\tau$</td>
<td>63</td>
<td></td>
</tr>
<tr>
<td>$\text{span}(A)$, the closed linear span</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{C}(X)$, the space of all compact linear operators in $X$</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>$D$, the Cantor discontinuum</td>
<td>168</td>
<td></td>
</tr>
<tr>
<td>$\delta_{n, \beta}$, Kronecker delta</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$\Delta_n^v(B)$, intermediate slice derivation of a set $B$</td>
<td>77</td>
<td></td>
</tr>
<tr>
<td>$\Delta_\ell(B)$, intermediate slice derivation of a set $B$</td>
<td>77</td>
<td></td>
</tr>
<tr>
<td>$\delta f$, the subdifferential of a function $f$</td>
<td>81</td>
<td></td>
</tr>
<tr>
<td>dens $(T)$, the density of a topological space $T$</td>
<td>XVII</td>
<td></td>
</tr>
<tr>
<td>det, the determinant of a matrix</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>dist $(\cdot, \cdot)$, the distance between two objects</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>DP, the Dunford-Pettis property</td>
<td>92</td>
<td></td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>$\Delta(X)$</td>
<td>the $w^*$-dentability index of $X$</td>
<td>77</td>
</tr>
<tr>
<td>$e + X$</td>
<td>the coset or equivalence class of an element $e$ in $E/X$</td>
<td>30</td>
</tr>
<tr>
<td>$\ell_1^q$</td>
<td>finitely supported vectors in $\ell_1$ with rational coordinates</td>
<td>79</td>
</tr>
<tr>
<td>$\ell_p(I)$</td>
<td>the space of all $p$-summable sequences with indices in $I$</td>
<td>258</td>
</tr>
<tr>
<td>$(E, T)$</td>
<td>a topological vector space</td>
<td>89</td>
</tr>
<tr>
<td>$\eta(K)$</td>
<td>the height of a scattered space</td>
<td>71</td>
</tr>
<tr>
<td>$\epsilon$-WRK</td>
<td>$\epsilon$-weakly relatively compact</td>
<td>119</td>
</tr>
<tr>
<td>$F$</td>
<td>Fréchet differentiable</td>
<td>108</td>
</tr>
<tr>
<td>$G$</td>
<td>Gâteaux differentiable</td>
<td>108</td>
</tr>
<tr>
<td>$GCH$</td>
<td>the Generalized Continuum Hypothesis</td>
<td>70</td>
</tr>
<tr>
<td>$JL_0$</td>
<td>the space of Johnson and Lindenstrauss</td>
<td>129</td>
</tr>
<tr>
<td>$JL_2$</td>
<td>the space of Johnson and Lindenstrauss</td>
<td>140</td>
</tr>
<tr>
<td>$JT$</td>
<td>the James tree space</td>
<td>144</td>
</tr>
<tr>
<td>$K^{(\alpha)}$</td>
<td>the Cantor-Bendixson derivation of a scattered topological space</td>
<td>71</td>
</tr>
<tr>
<td>$L_1(\lambda)$</td>
<td>the space of all equivalence classes of functions absolutely integrable</td>
<td>95</td>
</tr>
<tr>
<td>$\ell_\infty$</td>
<td>the space of all bounded sequences</td>
<td>92</td>
</tr>
<tr>
<td>$\ell_c(I)$</td>
<td>the space of countably supported vectors in $\ell_\infty(I)$</td>
<td>111</td>
</tr>
<tr>
<td>$\ell_\infty(I)$</td>
<td>the space of all bounded functions on $I$</td>
<td>92</td>
</tr>
<tr>
<td>$\ell_p$</td>
<td>the space of all $p$-summable sequences</td>
<td>XVII</td>
</tr>
<tr>
<td>LUR</td>
<td>locally uniformly rotund</td>
<td>108</td>
</tr>
<tr>
<td>$\mathcal{L}(X)$</td>
<td>the space of all continuous linear operators in $X$</td>
<td>50</td>
</tr>
<tr>
<td>$\text{MA}_{\omega_1}$</td>
<td>Martin’s axiom</td>
<td>152</td>
</tr>
<tr>
<td>$M^\circ$</td>
<td>the polar set of a set $M$</td>
<td>88</td>
</tr>
<tr>
<td>MM</td>
<td>Martin’s Maximum axiom</td>
<td>158</td>
</tr>
<tr>
<td>$\mathcal{M}(\Sigma, \mu)$</td>
<td>the space of $\mu$-equivalence classes of $\Sigma$-measurable functions</td>
<td>97</td>
</tr>
<tr>
<td>$N$</td>
<td>the set of natural numbers</td>
<td>XVII</td>
</tr>
<tr>
<td>$\eta(K)$</td>
<td>the cardinality of $K^{(X(K))}$</td>
<td>71</td>
</tr>
<tr>
<td>$N$</td>
<td>the space $\mathbb{N}^\omega$</td>
<td>47</td>
</tr>
<tr>
<td>$| \cdot |_2$</td>
<td>Day’s norm</td>
<td>111</td>
</tr>
<tr>
<td>$\omega$</td>
<td>the ordinal of $\mathbb{N}$ under its natural order</td>
<td>XVII</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>the first uncountable ordinal</td>
<td>XVII</td>
</tr>
<tr>
<td>$\oplus$</td>
<td>the topological direct sum</td>
<td>2</td>
</tr>
<tr>
<td>$\langle \cdot, \cdot \rangle$</td>
<td>the orthogonal in the space of a set in the dual space</td>
<td>3</td>
</tr>
<tr>
<td>$\langle \cdot, \cdot \rangle^+$</td>
<td>the orthogonal in the dual of a set in the space</td>
<td>3</td>
</tr>
<tr>
<td>$o(T)$</td>
<td>ordinal index of a tree</td>
<td>46</td>
</tr>
<tr>
<td>$\mathcal{P}$-class</td>
<td>having a PRI and stable by differences</td>
<td>107</td>
</tr>
<tr>
<td>$PG$</td>
<td>projectional generator</td>
<td>104</td>
</tr>
<tr>
<td>$\mathcal{P}X$</td>
<td>family of all precompact sets</td>
<td>89</td>
</tr>
<tr>
<td>PRI</td>
<td>projectional resolution of the identity</td>
<td>103</td>
</tr>
<tr>
<td>$\mathbb{Q}$</td>
<td>the set of rational numbers</td>
<td>79</td>
</tr>
<tr>
<td>$\mathbb{Q}$-linear</td>
<td>closed under rational-linear combinations</td>
<td>104</td>
</tr>
<tr>
<td>$\mathbb{Q}$-span$(\cdot)$</td>
<td>the span with rational coefficients</td>
<td>104</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>the set of real numbers</td>
<td>6</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>rotund</td>
<td>108</td>
</tr>
<tr>
<td>$r(Y)$</td>
<td>the Dixmier characteristic of a subspace $Y$ in $X^*$</td>
<td>58</td>
</tr>
<tr>
<td>$\text{rca}(\mathcal{B})$</td>
<td>the space of regular countably additive measures</td>
<td>98</td>
</tr>
<tr>
<td>$\text{rca}^+(\mathcal{B})$</td>
<td>the positive elements of $\text{rca}(\mathcal{B})$</td>
<td>141</td>
</tr>
<tr>
<td>$(r_n)$</td>
<td>the sequence of Rademacher functions</td>
<td>93</td>
</tr>
<tr>
<td>$S^c$</td>
<td>the complement of a subset $S$</td>
<td>99</td>
</tr>
<tr>
<td>SCP</td>
<td>separable complementation property</td>
<td>105</td>
</tr>
<tr>
<td>$\sigma(X, Y)$</td>
<td>the topology on $X$ of the pointwise convergence on points of $Y$</td>
<td>20</td>
</tr>
</tbody>
</table>
Symbol Index

\[ \Sigma(\Gamma) \], vectors in \( \mathbb{R}(\Gamma) \) with only a countable number of nonzero coordinates \[ 177 \]
\( \sigma_C(x) \), the supremum of \( x \) on a set \( C \) \[ 80 \]
\( \sigma_C^\ast \), the conjugate function of \( \sigma_C \) \[ 80 \]
\( \text{SI}(\{X_i\}_{i=1}^\infty, \varepsilon) \), the index of an FDD \[ 67 \]
\( \text{span}(\cdot) \), the linear span \[ 22 \]
SPRI, separable projectional resolution of the identity \[ 107 \]
\( [s, t] \), an interval in a tree \[ 46 \]
\( s \succ t \), concatenation of two nodes \[ 46 \]
\( \text{supp}(\mu) \), the support of a measure \[ 188 \]
\( \text{supp}(x^* \in X^* \text{ (of } x \in X)) \) on a system in \( X \) \[ 44 \]
\( s|m \), the initial segment of a node \[ 46 \]
\( S_X \), the unit sphere of a space \( X \) \[ XVII \]
\( \text{Sz}_\varepsilon(B) \), the \( \varepsilon \)-Szlenk index of a set \[ 62 \]
\( \text{Sz}_\varepsilon(X) \), the \( \varepsilon \)-Szlenk index of a space \( X \) \[ 62 \]
\( \text{Sz}(X) \), the Szlenk index of \( X \) \[ 62 \]
\( \tau^+ \), the follower cardinal \[ 63 \]
\( T_{ACWK} \), the topology of the uniform convergence on all absolutely convex and weakly compact sets \[ 89 \]
\( \tau(E, F) \), the Mackey topology on \( E \) associated to the pair \( (E, F) \) \[ 89 \]
\( T_M(F, E) \), the topology of uniform convergence on sets of a family \( M \) \[ 88 \]
\( T_p \), the topology of the pointwise convergence \[ 180 \]
\( T_{pzc} \), the topology of the uniform convergence on the precompact sets \[ 90 \]
\( T_x \), branches starting in \( x \) \[ 46 \]
\( T(X, \{y_n\}_{n=1}^\infty, \varepsilon) \), tree in \( X \) defined by a sequence \( \{y_n\} \) in \( Y \) \[ 51 \]
UEC, uniform Eberlein compact \[ 72 \]
UF, uniformly Fréchet differentiable \[ 108 \]
UG, uniformly Gâteaux differentiable \[ 108 \]
UKK*, \( w^\ast \)-uniformly Kadets-Klee \[ 66 \]
\( \mid \), restriction to \[ 11 \]
\( \approx \), two symbols close up to \( \varepsilon \) \[ 25 \]
UR, uniformly rotund \[ 108 \]
\( w \), the weak topology on a space \( XVII \)
\( w-2R \), weakly 2-rotund \[ 108 \]
WCG, weakly compactly generated \[ 4 \]
\( w(E, F) \), the weak topology on \( E \) associated to the pair \( (E, F) \) \[ 88 \]
WLD, weakly Lindelöf determined \[ 103 \]
WRK, weakly relatively compact \[ 119 \]
\( w^\ast \), the weak\(^*\)-topology on a dual space \( XVII \)
\( w^\ast \)-uniformly rotund \[ 108 \]
\( \{x_m; x_m^*\}_{m=1}^\infty \), a biorthogonal system in \( X \times X^* \) \[ 2 \]
\( \{x_m; x_m^*\}, \{y_m; y_m^*\} \) or \( \{x_m, y_m; x_m^*, y_m^*\} \), extension of an M-basis \[ 30 \]
\( \{x_n; x_n^*\}_{n=1}^\infty \), countable biorthogonal system \[ 4 \]
\( X^\ast \), the topological dual of a space \( X \) \[ XVII \]
\( \langle x, x^\ast \rangle \), action of a functional \( x^\ast \) on a vector \( x \) \[ 2 \]
\( Y_\beta \), the iterated \( w^\ast \)-sequential closure of a subspace \( Y \hookrightarrow X^\ast \) \[ 58 \]
\( Y \hookrightarrow X \), \( Y \) is a subspace of \( X \) \[ XVII \]
Subject Index

Entries in bold typeface correspond to the pages where the corresponding concepts are defined.

admissible system of intervals of a tree
  see tree, admissible system of intervals
Auerbach basis  5, 6
in finite-dimensional spaces  5
uncountable  158, 161
Auerbach system  5
and Schauder basic sequence  8
  countable infinite  7
  in $C(K)$  19
  shrinking  21
axiom
  ♣  148, 149–152, 196, 284
  Martin Maximum  158, 276, 284
  $\text{MA}^{\omega_1}$  152, 153–156, 158, 191, 254

$ba(\Sigma)$, bounded finitely additive scalar-valued measures  95
basis
  Auerbach  see Auerbach basis
  Markushevich  see Markushevich basis, 30
  Schauder  see Schauder basis
basis constant  134, 135
unconditional  261
biorthogonal system  2
block perturbation  24
  bounded  9
  boundedly complete  5
  convex strong  44
flattened perturbation  24
functional coefficients  3
fundamental  3, 11, 23, 25, 137–140,
  142, 143, 152, 153, 156, 158, 196,
  204, 288, 289, 292, 295, 297
  and bounded  12, 137–139, 288
  extension  142
  lifting  139
  no extension  142
  not M-basis  39
  $\lambda$-norming  4
  $\lambda$-bounded  9
minimal  2
$w^*$-strong  43
no M-basis  4
normalized  44
normalized bounded  44
norming  4
shrinking  5
total  4, 8, 13, 20, 35, 38, 144, 196,
  199, 297
  and bounded  12, 135, 299
block partition  24, 27
boundary of a $w^*$-compact set  80
caliber  189, 190, 191, 256
Cantor-Bendixon derivation  71
cardinal
  regular  63, 64, 77, 194, 257, 258
Subject Index

CCC  see property, countable chain condition
characteristic  $\mathfrak{z}(X)$  285
cofinality of an ordinal  63
compact
admitting a strictly positive measure  188
Corson  102, 178, 185, 187–191, 232, 240
not continuous image of a countably tight compact space  73
Eberlein  102, 195, 216, 219, 220, 223, 231, 232, 251
continuous image  223
not uniform Eberlein  233
scattered  251
scattered not uniform Eberlein  251
Gul’ko  231, 232
Rosenthal  144, 146
strong Eberlein  72
totally disconnected  70, 190
uniform Eberlein  72, 227, 230, 233, 237, 251, 265, 270
continuous image  232, 237, 270
universal  70, 72
Valdivia  102, 166, 177, 178, 184–188, 191, 204
zero-dimensional  70
compactification
Čech-Stone  XVII, 141, 147, 248, 257
continuum hypothesis  71, 148, 152, 153, 189–191, 254, 284
convex right-separated $\omega_1$-family  281, 283
countable tightness  103
countably closed  177, 189
density  XVII
dentability index  77, 79
and dual LUR renorming  78
and superreflexivity  77
Dixmier characteristic  58, 285
dual pair  88, 90
$\varepsilon$-weakly relatively compact  119, 125
family

of projections, shrinking  296
$\ell_1$-independent  38
of equicontinuous sets  89
$\omega$-independent  38, 39, 40, 281, 283
and $\omega_1$-convex right-separated family  283
$\omega_1$-polyhedron  see $\omega_1$-polyhedron
right-separated  151
saturation of a  88
uniformly independent  140
FDD  see finite-dimensional decomposition
finite-dimensional decomposition  53, 55–57, 69
bimonotone  53, 54, 67
block of a vector  53
blocking of a vector  53
boundedly complete  55, 68, 69
dual  53, 55, 67
index  67
lower $p$-estimate  54, 55, 56
$(p, q)$-estimate  54, 55–57, 66, 67
refinement of a blocking  53
shrinking  55, 66, 67, 69
skipped blocks  53
skipped $(p, q)$-estimate  54
support of a vector  53
upper $q$-estimate  54, 55, 56
fundamental system
no extension  142
Generalized Continuum Hypothesis  70, 72, 193–195
height of a scattered space  71, 83, 251
index
dentability  see dentability index
Szlenk  see Szlenk index
interchange of limits  122
James tree space  see space, James tree
Kunen-Shelah properties  see property, Kunen-Shelah
$\lambda$-norming
biorthogonal system  4
subset of $X^*$  4
map
regular 46, 48
Markushevich basis 4
bounded 14, 15, 21, 33, 35, 134, 170, 173, 192, 193
boundedly complete 208
characterization 4
countably 1-norming 192
countably norming 175, 185
countably supporting \(X^*\) 230
countably \(\lambda\)-norming 175
existence in separable spaces 8
extension 30, 35, 192, 205, 216, 220
if separable quotient 195
of bounded 192
no extension 196
no universal countable 62
norming 4, 8, 26, 27, 29, 173, 175, 211, 239, 270
not norming 5, 175, 176
not Schauder 22
shrinking 8, 29, 162, 208, 211, 295–297
\(\sigma\)-weakly compact 212, 213, 214, 259
\(\sigma\)-shrinkable 216, 219, 220, 239
strong 22
unbounded 10
weakly compact 212, 213
weakly \(\sigma\)-shrinkable 223, 224, 225
Mazur intersection property 289
and Asplund spaces 292
and density of \(X^*\) 291
and Fréchet differentiable norms 291
and Kunen-Shelah properties 291
and the Johnson-Lindenstrauss space 294
characterization of 289
in separable spaces 292
isomorphic embeddings in spaces with 294
measure
countably additive 95
Dirac 39, 188, 189, 275
finitely additive 95, 241, 242, 262
Haar 204
Lebesgue 190, 271
Radon 141, 188–190, 205
regular 99
support of a 188
total variation 20
node 46
compatible 46
concatenation 46
extension 46
initial segment 46
interval 46
norm
2-rotund 108, 211
2R see norm, 2-rotund
Day 111, 114, 115, 208, 214, 267, 268, 293
dual locally uniformly rotund 81, 211
dual LUR see norm, dual locally uniformly rotund
dually \(M\)-2-rotund 212, 213, 214
dually \(M\)-2R see norm, dually \(M\)-2-rotund
dually 2-rotund 212
dually 2R see norm, dually 2-rotund
F see norm, Fréchet differentiable
Fréchet differentiable 108, 211, 212, 259, 269, 277, 278, 291, 292, 294, 295
Fréchet \(M\)-smooth 212, 213, 215, 231
\(G\) see norm, Gâteaux differentiable
Gâteaux differentiable 108, 126, 204, 239, 258, 263, 267, 269
Lipschitz UKK* 83
locally uniformly rotund 79, 108, 109–111, 128, 167, 174, 175, 204, 206, 269, 293, 301
LUR see norm, locally uniformly rotund
pointwise locally uniformly rotund 111, 225
pointwise LUR see norm, pointwise locally uniformly rotund
\(R\) see norm, rotund
smooth see norm, Gâteaux differentiable
strictly convex \( \text{see norm, rotund} \)
symmetric \( 266, 267-269 \)
symmetrized type \( 209 \)
UF \( \text{see norm, uniformly Fréchet differentiable} \)
UG \( \text{see norm, uniformly Gâteaux differentiable} \)
uniformly Fréchet differentiable \( 108, 236, 237 \)
uniformly Gâteaux differentiable \( 108, 227, 233, 263-265, 267, 268 \)
uniformly rotund \( 78, 108 \)
uniformly rotund in every direction \( 263, 264, 265, 268 \)
and unconditional basis \( 268 \)
dual \( 265 \)
renorming \( 263 \)
symmetric \( 267 \)
uniformly smooth \( \text{see norm, uniformly Gâteaux differentiable} \)
uniformly \( w^*-\text{Kadets-Klee} \) \( 66, 67, 83-85 \)
UR \( \text{see norm, uniformly rotund} \)
URED \( \text{see norm, uniformly rotund in every direction} \)
w-2R \( \text{see norm, } \omega-2\text{-rotund} \)
w-2-rotund \( 108 \)
w-\text{uniformly rotund} \( 108, 264, 268, 269 \)
W-UR \( \text{see norm, } \omega^*-\text{uniformly rotund} \)

norm-attaining operator \( \text{see operator, norm-attaining} \)
norming
hyperplane \( 119 \)
subset of \( X^* \) \( 4, 253 \)
subspace of \( X^* \) \( 5, 57-59, 119, 160, 196 \)

\( \omega \)-independence \( \text{see family } \omega \)-independent
\( \omega_1 \)-polyhedron \( 278, 281 \)
\( \omega \)-sequence \( 148 \)

operator
attaining its norm \( 284 \)
compact \( 51 \)
completely continuous \( 93, 125 \)
Dunford-Pettis \( 93, 94, 242-244 \)
norm-attaining \( 284, 285 \)

strictly singular \( 244 \)
weakly compact \( 93, 242-244 \)
ordinal
cofinality \( 63 \)
orthogonal \( 6, 34 \)
complement \( 12 \)
subspace \( 6 \)

\( \mathcal{P} \)-class \( 107, 166, 192 \)
has strong M-basis \( 166, 167 \)
partial order
analytic \( 48, 81, 83 \)
compatible elements \( 152 \)
dense subset \( 152 \)
filter \( 152 \)
r extends \( p, q \) \( 152 \)
well-founded \( 48 \)

perturbation of a biorthogonal system flattened \( 24 \)

power of a set \( 46 \)

PRI \( \text{see projectional resolution of the identity} \)

projectional generator \( 104, 105, 106, 183, 192, 196 \)
full \( 104, 182 \)

projectional resolution of the identity \( 103, 105, 165, 167, 185, 191, 193, 196, 215-219, 221, 307, 316, 317 \)
and M-bases \( 165, 230 \)
and \( \mathcal{P} \)-classes \( 107 \)
and Plichko spaces \( 184 \)
and projectional generators \( 105, 192, 205 \)
and strong M-bases \( 167 \)
and WCG spaces \( 106, 216, 219 \)
and WLD spaces \( 204 \)
in every equivalent norm \( 185, 228 \)
nonexistence under renorming \( 169 \)

PRI’ \( 217 \)

separable \( 107, 226 \)
\( \sigma \)-shrinkable \( 216, 218, 219 \)
space without \( 167, 185, 191 \)
subordinated to a set \( 105, 217 \)

projections
uniformly bounded \( 133 \)
proper support point \( 274 \)

property
1-SCP \( \text{see property, } 1\text{-separable} \)
complementation
1-separable complementation 105, 167
and WCG 173
if PRI 167
1-separable complementation 105
A 284
α 284, 285, 286
and biorthogonal systems 287
(α, λ) 284
B 285
BAP see property, bounded approximation
β 284, 285, 286
(β, λ) 284
bounded approximation 49, 56
C 140, 144, 147, 180
CAP see property, compact approximation
compact approximation 50
countable chain condition 145, 152, 188, 190, 191, 249
and strictly positive measures 188
for a partial order 152
for a system of finite sets 153–155, 157
Dunford-Pettis 92, 93, 94, 201, 242–244, 253
Grothendieck 92, 170, 200
and M-bases 169, 170
ℓ∞(Γ) 92, 170, 201, 248
Kunen-Shelah 278, 281, 300
(M) 189
Radon-Nikodym 167
RNP see property, Radon-Nikodym
Schur 92, 102, 127, 128, 244
and C(K) spaces 127
and ℓ1 127
and limited sets 92
and WCG 240
characterization 127, 128
not ℓ1(Γ) 127
SCP see property, separable complementation
separable complementation 105
and no PRI renormable 184
and RNP 167
failure of 187
skipped SI 67
strict (α, λ) 284
strict α 284
strong (β, λ) 284
strong β 284, 285
Suslin 145
pseudobase 190
Q-span 104
quasicomplemented 36, 197
and M-basis 198
extension of bases in directions 36
from M-basis 197
subspace of WLD 197
Rademacher functions 93, 94, 101, 141, 201, 253
root lemma 153
Schauder basic sequence 7, 8, 60, 186
long 135
monotone 155
seminormalized 198
Schauder basis XVIII, 25, 44, 50, 52, 55, 60, 62
and FDD 53
as a strong M-basis 22
complementable universality 49, 50
in a nonsuperreflexive space 84
in C[0,1] 51, 158
interpolating 159
long 132, 133–136, 143, 162, 270
associated functionals 134
in a quotient 137, 138, 156
in C[0, Γ] 134
in C[ωω] 135
monotone 134, 136, 138
normalized 134
projections 134
reordering 134
symmetric 266
Mazur technique 198
monotone 6, 51
nonexistence in some spaces 6
no extension 35, 36
projections 135
rearrangement 135
shrinking 36, 69, 128, 259, 269, 313, 317
transfinite 132
unconditional 44, 49, 213, 259
implies norming  270
universality  49, 57
universal  49
universal  53
semibiorthogonal system  274, 275
separable projectional resolution of the
identity  107
series
subseries convergent  247
transfinite
convergence  132
weakly subseries convergent  247
set
absolutely convex  88
analytic  48, 83, 145–147
bounded  88
countably supported by another  105
countably supporting $X^*$  226, 230, 232
free for $f$  255
fundamental  88
$\lambda$-norming  4
limited  91, 201
norming  4
overfilling  42
perfect  73
polar  88
precompact  89
$\sigma$-shrinkable  216, 217, 219, 220
$\sigma$-weakly compact  212
subordinated to PRI  105
totally bounded  89
uniformly integrable  97
V-small  89
weakly $\sigma$-shrinkable  223
$\Sigma$-subset  177
$\Sigma(\Gamma)$  177
$\Sigma$-subspace  179
Sokolov
subspace of $\ell_\infty(\Gamma)$  111
space
angelic  92, 102, 147, 177, 178, 187, 189, 205, 206, 240, 253
not Corson  187
and bos  297
and DENS  295
and dentability index  77
and dual LUR renorming  78, 81
and Fréchet norm  211, 295
and hereditability  211, 259
and LUR renorming  204
and M-bases  295
and MIP  292, 294
and quasicomplementation  202
and Szlenk derivation  62
and Szlenk index  63–65, 72, 79
and universality  72
and WCG  211
and WLD  211
characterization  62, 63, 280
dual of  105, 166
Jayne-Rogers selection  295
no universal separable for separable
reflexive  53
operator from  128
quotient of space with basis  69
subspaces of  294
with unconditional basis  259
Ciesielski-Pol  167
complementably universal  49, 50, 56, 57
DENS  181, 294
Dunford-Pettis  92
Erdős  190
extremely disconnected  248
Fréchet-Urysohn  102, 178, 189
Gelfand-Phillips  201
Grothendieck  92
hereditarily indecomposable  162
Hilbert generated  212, 230
characterization  225
subspace of  227, 230
injective  93
isometrically universal  49
James tree  187, 265
$JL_0$  140
$JL_2$  140
locally convex  89
nonquasireflexive  60
not-WLD  261
Plichko  105, 166, 184, 185, 196
and Sobczyk  193
as a $\mathfrak{P}$-class  192
$C(K)$  188
Characterization 192
Extension of M-bases 192
Non-WLD 185
Not hereditary 184
Quasireflexive 59
Characterization 59
Quotient of a 59
Reflexive 51–53, 55–57, 72, 81, 142, 169, 170, 173, 200, 212, 233
And complemented 200
And metrizability 128
And norming M-basis 173
And separable quotient 205
And Szlenk index 72
And the Mackey topology 128
And weakly sequential completeness 240
Characterization 59, 208
With Schauder basis universal for separable superreflexive with Schauder basis 57
Without UG norm 233
Reflexive generated 212, 213
Representable 145
Retreactive 73
Schur 92
Stone 190
Strongly generated by a weakly compact set 233
Strongly reflexive generated 233
Strongly superreflexive generated 233
Subspace of WCG 240
Superreflexive and CAP 50
And renorming 78
And universality 53, 57
Characterization 77, 78
No universal separable for superreflexive separable 53
Universality for separable and 50
With BAP 57
With FDD 55
Uniformizable 89
Universal 49
Vasák 103, 166, 221, 223, 225, 228, 240
And PRI 228
$C(K)$ 231, 232
Renorming 225
WCG see space, weakly compactly generated without M-basis 174
Weakly compactly generated 4, 106, 162, 166, 173, 184, 211, 214, 223, 239, 251, 260, 270, 271
And Asplund 211
And DENS 295
And density 219
And M-bases 140, 173, 212, 216, 295
And MIP 294
And norming M-bases 173, 175
And PRI 219
And renorming 258
And Schur 240
And SCP 173
And universality 72
As P-classes 166
$C(K)$ 220, 233, 250
Characterization 213
dual 233
dual ball of 102
Factorization 233
Nonhereditary 212, 225
Not Asplund 292
PG for 106
Subspace of 119, 216, 219, 220, 265
With unconditional basis 259
Weakly countably determined see space, Vasák
Weakly Lindelöf 181
Weakly Lindelöf determined 103, 105, 140, 166, 173, 184, 192, 204, 263
And M-smooth norm 213
And Asplund 211
And dual LUR norm 211
And Fréchet norm 211
And M-basis 211
And quasicomplements 197
Are Plichko 184
Characterization 180, 185
Characterization when unconditional basis 261
Hereditability 211, 261
Is DENS 181
Of type $C(K)$ 189, 191
Subject Index

quotient 140
subspace 192
without G norm 263
with a reflexive quotient 142
with countable tightness 179
WLD  see space, weakly Lindelöf determined, 240
spaces
  totally incomparable 198, 200
SPRI  see separable projectional resolution of the identity
strong vertex point 302
subset
  countably closed 177, 189
  Q-linear 104
subspace
  quasicomplemented  see quasicomplemented, 197
  reflexive 198–200
  Σ  see Σ-subspace
Sokolov 111
transfinite sequence of w*-sequential limits 58
support cone 274
support of a measure 188, 189
support set 274
  and quotients 301
  and separable spaces 276
  and subspaces of finite codimension 274
  and support cone 274
  characterization of 275
  in C(K) 275, 276
  nonexistence in separable spaces 274
  nonseparable spaces without 276
system
  α-system 284
  β-system 284
Auerbach  see Auerbach system
biorthogonal  see biorthogonal system
fundamental  see biorthogonal system, fundamental
independent 252
M-basic 42
minimal 2, see biorthogonal system of intervals
  admissible  see tree, admissible system of intervals
semibiorthogonal  see semibiorthogonal system
trigonometric 44
uniformly minimal 9
Szczenk
  derivation 62, 73, 75, 82, 83
  index 53, 62, 63, 72–74, 76, 77, 79, 82
  and universality 45, 70
weak index 72
topological weight 70
topology
  compatible with a dual pair 89
  convergence in measure 97, 98
  Mackey τ(E, F) 89
  Mackey τ(X*, X) 92, 94, 127, 128, 238, 260
  order on ordinals 132
  strong β(E, F) 88, 91, 126
  uniform convergence on a family 88
weak 89
weak w(E, F) 88
tree 46
  admissible system of intervals 52
  analytic 47, 48
  branch 46
  closed 47
  isomorphic to a subtree 46
  ordinal index 46
  partial order 46
  root 46
  well-founded 46, 52
two arrow space 144
WLD  see space, weakly Lindelöf determined
  w*-compact
  fragmented by the dual norm 80
ZFC XVIII
Author Index

Acosta, M.D. 300, 303
Aguirre, F. 303
Alexandrov, G. 175, 303
Alspach, D.E. 303
Amir, D. 87, 102, 303
Angosto, C. 121, 125, 303
Archangelskii, A.V. 191
Azagra, D. 276, 304
Bachelis, G.F. 304
Banach, S. 45, 49, 60, 304
Bell, M 71, 72, 251, 273, 304
Bellenot, S.F. 297, 304
Benyamini, Y. 72, 223, 230, 232, 251, 303–305
Bessaga, C. 45, 73, 74, 82, 205, 305
Bohnenblust, F. 6, 305
Borwein, J.M. 237, 273, 274, 300, 305
Bosard, B. 46, 77, 79, 305
Bourgain, J. VIII, 45, 51, 53, 83, 147, 187, 240, 300, 303, 305
Cascales, B. 121, 125, 303, 305
Castillo, J.F. 193, 195, 245, 303, 305
Chen, D. 300, 305
Corson, H.H. 256, 305, 306
Courage, W. 306
Dashiel, F.K. 262, 306
Davie, A.M. 50

Davis, W.J. 9, 10, 22, 60, 69, 72, 140, 213, 216, 233, 234, 297, 306
Day, M.M. 5, 7, 12, 43, 306
Dellacherie, C. 48, 306
Diestel, J. 94, 101, 111, 112, 190, 205, 306
Dieudonné, J. 241, 246
Dilworth, S.J. 69, 306
Dodos, P. 72, 304, 306
Dowling, P.N. 307
Dugundji, J. 307
Dunford, N. 87, 95, 97, 98, 307
Dutrieux, Y. 307
Eberlein, W.F. 122, 307
Edgar, G.A. 307
Emmanuele, G. 87, 93, 307
Enflo, P. VII, 1, 36, 50, 252, 258, 307
Engelking, R. 70, 79, 149, 150, 152, 307
Erdős, P. 193, 194, 307
336 Author Index

Farmaki, V.  220, 223, 237, 304, 308
Ferenczi, V.  72, 306
Ferrera, J.  276, 304
Figiel, T.  69, 72, 213, 216, 233, 234, 306
Finet, C.  144, 175, 285, 289, 308
Fitzpatrick, S.  237, 305
Foreman, M.  158, 308
Frankiewicz, R.  252, 308
Fremlin, D.H.  38, 39, 147, 187, 191, 240, 305, 308
Georgiev, P.G.  300, 308
Giles, J.R.  237, 289, 300, 308, 313
Gibson, J.  273, 304
Girardi, M.  69, 306
Godun, B.V.  57–60, 131, 139, 142, 158, 197, 200, 204, 284, 285, 287–289, 309
González, A.  182, 310
González, M.  305
Gowers, W.T.  VII, 162
Granero, A.S.  121, 125, 193, 195, 245, 249, 273, 275, 278, 281, 283, 300, 303, 308, 310
Gregory, D.A.  289, 308
Grothendieck, A.  87–90, 92, 95, 97, 98, 122, 241, 248, 260, 310
Gruenhage, G.  310
Grünbaum, B.  82, 310
Ghraslewicz, R.  82, 310
Grzech, M.  252, 308
Gułko, S.P.  77, 223, 310
Gurarii, P.I.  55
Gurarii, V.I.  2, 29, 30, 44, 55, 159, 310
Hagler, J.  101, 128, 205, 253, 254, 311
Hajnal, A.  193, 194, 255, 307
Haydon, R.  254, 304, 311
Hollácky, P.  311
Howard, J.  127, 311
Huff, R.  311
James, R.C.  55, 78, 207, 269, 311, 312
Jameson, G.J.O.  312
Jaramillo, J.A.  277
Jarosz, K.  297, 312
Jech, T.  63, 64, 140, 194, 312
Johanis, M.  114, 208, 214, 311, 312
John, K.  87, 104, 211, 312
Joséflson, B.  87, 95, 101, 201, 202, 204, 313
Judd, R.  303
Juhász, I.  157, 313
Köthe, G.  89, 314
Kadets, M.I.  2, 29, 30, 44, 49, 142, 309, 310, 313
Kadets, V.  300, 313
Kalenda, O.  177, 185, 187, 188, 313
338 Author Index


Pettis, B.J. 87, 95, 97, 247

Petunin, Y. 315

Phelps, R.R. 81, 120, 292, 300, 310, 317

Phillips, R.S. 241, 246

Plans, A 23, 317


Pol, R. 318

Pospíšil, B. 140

Prochážka, A. 82, 311

Prus, S. 45, 53, 55–57, 69, 84, 318

Pták, V. 122, 123, 318

Rado, R. 193, 194, 307

Raja, M. 82, 121, 125, 305, 318

Reif, J. 220, 318

Reyes, A. 23, 317

Ribarska, N.K. 318

Rodríguez-Salinas, B. 251, 318

Rodelicz, S. 273, 274, 300, 318


Ruckel, W.H. 22

Rudin, M.E. 223, 230, 232, 304, 315

Rychtář, J. 265, 289, 292, 295, 311, 312, 319

Samuel, C. 45, 73–75, 319

Śapirovskii, B.E. 191

Schachermayer, W. 285, 286, 289, 308, 319

Schechtman, G. 49, 305, 319

Schlichter, G. 128, 233, 234, 319

Schlumberch, T. VII, 45, 57, 66, 208, 314, 316

Schwartz, J.T. 97, 98, 307

Sciffer, S. 237, 308

Semadeni, Z. 76, 159, 319

Sersouri, A. 38, 39, 273, 276, 283, 300, 308, 319

Shelah, S. 70, 71, 148, 158, 278, 308, 319

Sierpiński, W. 73

Sims, B. 289, 308

Singer, I. XVIII, 6, 9, 12, 22, 35, 43, 301, 306–308, 319, 320

Smídek, M. 311

Sobekci, D. 320

Sokolov, G.A. 314, 320

Stamati, E. 236, 260, 261, 271, 315

Starbird, T. 251, 305

Stlement, C. 147, 254, 311, 320

Szankowski, A. 50, 313

Szlenk, W. VIII, 45, 53, 62, 63, 317, 320


Tang, W.K. 81, 320

Terenzi, P. VII, 1, 22, 24, 27, 29, 30, 33, 35, 320

Todorčević, S. VIII, IX, 72, 131, 145, 153, 156, 158, 273, 276, 284, 300, 304, 320

Toulas, A. 162, 304


Tsarpalias, A. 223, 304, 316

Tsirelson, B.S. VII

Turett, B. 307

Tzafriri, L. XVIII, 44, 50, 57, 128, 135, 142, 199, 200, 205, 268, 269, 295, 315

Vašák, L. 87, 104, 321
Vanderwerff, J. 4, 174, 180, 192, 195, 196, 274, 300, 305, 321
van Dulst, D. 301, 307
Vechichko, N.V. 151, 321
Vershynin, R. 22, 24, 27, 321
Wage, M. 223, 230, 232, 304
Walker, R.C. 70, 190, 321
Whitfield, J.H.M. 4, 174, 180, 300, 321
Wilkins, D. 322
Williams, N.H. 255, 322
Wojtaszczyk, P. XVIII, 317, 322
Yesenin-Volpin, A.C. 70, 322
Yost, D. 167, 305, 318, 322
Zachariades, T. 303
Zajicek, L. 311
Zenor, P. 151, 322
Zippin, M. 36, 50, 246, 249, 313, 322