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Anytime Optimal Control Strategy for Multi-Rate Systems

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ABSTRACT In this paper, we study a dual-rate system with fast-sampling at the input and propose a design to optimize the consecutive control signals. The objective of the optimization is to maximize the decay rate depending on the available resources to stabilize the control system faster. Stability conditions are enunciated in terms of linear matrix inequalities. The control solution is extended to time delays. A numerical example illustrates the benefits of the control proposal.

INDEX TERMS Multi-rate systems, anytime attention control, exponential stability, Lyapunov methods.

I. INTRODUCTION

Multi-rate systems [1]–[4] have been studied extensively during the last years due to the possibilities they offer in the field of Control Theory, for instance in terms of improved behavior or resource savings. The theory for multi-rate systems has been developed to deal with systems where sensors and actuators are sampled at different rates. This can occur either because the multi-rate scheme provides a more satisfactory response or a better performance than the single-rate one, or due to temporal constraints in the elements of the system, which make the multi-rate sampling a necessity. For example, there are chemical analyzers such as in cement/ceramic applications which require careful preparations and a specific measurement process [3]. The multi-rate approach is also useful in robotics systems where the output measurement is obtained from visual sensors and a large time is required to process the information [5], [6]. In other cases, multi-rate control techniques enable to reduce the number of measurements such as in the reader/writer header positioning in hard disk drive servo systems [7], [8]. Other important areas of application are flight control [9] or multivariable control [10].

Multi-rate control theory is also interesting from the point of view of networked control systems (NCSs) [11]–[13], because they enable to limit the transmission of information through the network. When several control loops share the same network, some conflicts may appear in the use of the communication resources. If the sampling period is sufficiently large, the activity of the network is reduced and each control action can be applied on time. It is well known that the performance of a system is degraded if the sampling period is excessively enlarged. For this reason, the multi-rate approach tries to preserve the performance obtained by the single-rate case but considering a reduced number of available data. In this regard, the performance of the multi-rate loop can be improved through the computation of optimal input signals. With this purpose, we make use of the so-called anytime attention control strategies [14]–[16]. The idea is to improve the control signals applied to the plant depending on the available computation resources. This improvement can be carried out in different ways. For example, computing more control actions [16], computing future control signals [17] or increasing the order of the controller [18]. To the best of the authors' knowledge, the consideration of anytime attention control strategies in multi-rate sampled-data systems is novel.

In the present work, we consider a dual-rate system. The controller generates n control signals for each sampling of the sensor. Hence, the actuator changes n times faster than the sensor. This is known as Multi-Rate Input Control (MRIC). Depending on the available resources, n_{opt} input signals with $n_{\text{opt}} \leq n$ are optimized to maximize the decay rate of the system. The multi-rate system is modeled through lifting techniques [19]. This approach results specially useful because it enables to convert a periodic linear time-varying discrete system in a multivariable linear time-invariant discrete system in which the properties of stability, reachability, controllability and observability of the original time-varying system are conserved. This novel scheme can provide a clear improvement in the stabilization problem in comparison with the classical control strategies for multi-rate systems. In addition, the extension for the time-delay case is proposed to face situations in which the computation of the control signals is computationally hard, or to use the scheme in NCSs, where network delays may appear. In summary, the main contribution of the work is the design of a new framework in the multi-rate scenario, which guarantees a certain optimization of the decay rate of the system even in presence of time delays.

The paper is structured as follows. Section 2 provides some useful mathematical preliminaries. In Section 3, the dual-rate sampled-data system is presented. In Section 4, the control strategy is designed. Section 5 describes the extension of the results for systems with delay. In Section 6, some simulations are presented to test the validity of the theory. Finally, conclusions are provided in Section 7.

II. PRELIMINARIES

We define the set of real numbers and the set of natural numbers as $\mathbb{R}$ and $\mathbb{N}$, respectively. The n-dimensional real space is defined by $\mathbb{R}^n$. We refer to the euclidean norm of vector $x \in \mathbb{R}^n$ as $\|x\| := \sqrt{x^T x}$. Let $M \in \mathbb{R}^{n\times m}$, M^T denotes the transpose matrix of M. In addition, if M is a symmetric real matrix, then the maximum and the minimum eigenvalue of M are denoted by $\lambda_{\mathrm{M}}(M)$ and $\lambda_{\mathrm{m}}(M)$, respectively. We further denote a symmetric positive-definite matrix $P \in \mathbb{R}^{n\times n}$ as $P > 0$, while $P \geq 0$, $P < 0$ and $P \leq 0$ refer to symmetric positive-semidefinite, negative-definite, and negative-semidefinite matrices, respectively. We denote the identity matrix $\mathbb{I} \in \mathbb{R}^{n\times n}$ by $\mathbb{I}_n$. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Being $\lambda(A)$ the set of eigenvalues of A, we define $\mu(A) = \max\left\{\mu | \mu \in \lambda\left((A + A^T)/2\right)\right\}$. The norm of the matrix exponential [20] can be bounded then such as

$$\|e^{A\theta}\| \leq e^{\mu(A)\theta} \leq e^{\mu(A)(\theta+\epsilon)}, \quad \forall \epsilon \geq 0. \tag{1}$$

$B(\theta, A)$ denotes

$$B(\theta, A) = \begin{cases} 0 & \text{if } \theta \leq 0 \\ \int_0^\theta e^{As} B ds & \text{if } \theta > 0 \end{cases} \tag{2}$$

and $B_\mu(\theta, A)$

$$B_\mu(\theta, A) = \begin{cases} 0 & \text{if } \theta \leq 0 \\ \int_0^\theta e^{\mu(A)s} \|B\| ds & \text{if } \theta > 0. \end{cases}$$

Consequently,

$$\|B(\theta, A)\| \leq B_\mu(\theta, A) \leq B_\mu(\theta + \epsilon, A), \quad \forall \epsilon \geq 0. \tag{3}$$

Finally, we define the exponential stability of a system

$$\dot{x}(t) = f(t, x) \tag{4}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \in \mathbb{R}^n$ is a domain that contains the origin $x = 0$ such that

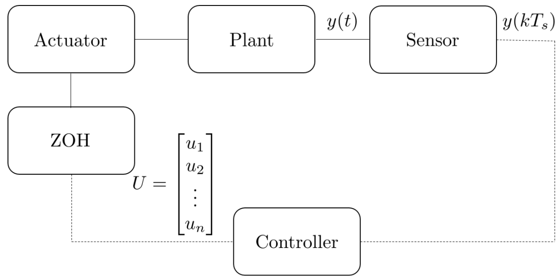

FIGURE 1. Block diagram of the control problem. The state of the plant is transmitted every T_s and is used to compute the n applied signals. (Solid line) Continuous signals. (Dashed line) Sampled signals.

Definition 1 [21]: The equilibrium point $x = 0$ of (4) *is exponentially stable if there exist positive constant ϵ, c and α such that*

$$\|x(t)\| \leq ce^{-\alpha(t-t_0)}\|x(t_0)\|, \quad \forall \|x(t_0)\| < \epsilon \tag{5}$$

and globally exponentially stable if (5) *is satisfied for any initial state $x(t_0)$.*

III. PROBLEM STATEMENT

In this Section, we formulate the control problem of a dual-rate system (fast control or MRIC configuration) using lifting techniques and propose an optimized control to improve the performance.

Let us consider a continuous linear time-invariant (LTI) plant denoted by

$$\begin{aligned} \dot{x}_p(t) &= A_p x_p(t) + B_p u(t) \\ y(t) &= C_p x_p(t) \end{aligned} \tag{6}$$

with $x_p(t) \in \mathbb{R}^{n_{xp}}$ the state vector of the plant, $u(t) \in \mathbb{R}^{n_u}$ the input vector, $y(t) \in \mathbb{R}^{n_y}$ the output vector, and $A_p \in \mathbb{R}^{n_{xp}\times n_{xp}}$, $B_p \in \mathbb{R}^{n_{xp}\times n_u}$, and $C_p \in \mathbb{R}^{n_y\times n_{xp}}$ constant matrices. We assume that the output of (6) is sampled with sampling period T_s as shown in Fig. 1. In contrast, the controller changes the input signal n times during T_s. Let us assume that each input u_i for $i = 1, \ldots, n$ remains constant using a zero-order-hold (ZOH) and is applied during the period of time T_i satisfying $T_s = \sum_{i=1}^{n} T_i$. Since the periods $\{T_i\}$ are not necessarily equal, a non-uniform multi-rate control system is obtained as shown in Fig. 2. Then

$$u(t) = \begin{cases} u_1 \text{ if } t \in [kT_s, kT_s + T_1) \\ u_2 \text{ if } t \in [kT_s + T_1, kT_s + T_1 + T_2) \\ \vdots \\ u_n \text{ if } t \in [kT_s + \sum_{i=1}^{n-1} T_i, kT_s + \sum_{i=1}^{n} T_i). \end{cases} \tag{7}$$

To achieve the objective of guaranteeing the exponential stability, we make use of a lifted model of the plant [22], [23]. First, let us observe that if (6) is discretized and $n = 1$ (the single-rate case), then

$$\begin{aligned} x_p(k+1) &= e^{A_p T_s} x_p(k) + B_p(T_s, A_p) u(k) \\ y(k+1) &= C_p x_p(k+1), \end{aligned}$$

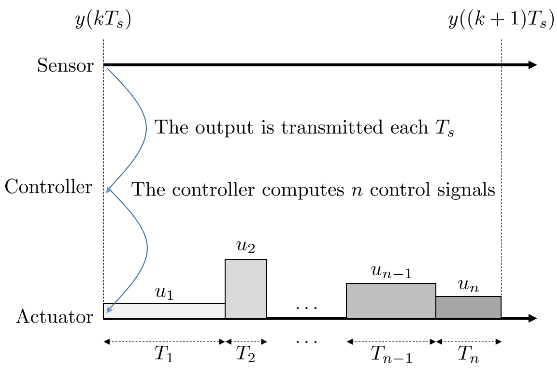

FIGURE 2. Scheme of the transmitted signals. n different signals are applied during the sampling period T_s.

with $k \in \mathbb{N}$ and $x(k) = x(kT_s)$. However, in the proposed problem (Fig. 1) the discretization leads to a periodic linear time-varying discrete system, because the input signal changes n times in each T_s. Consequently, it can be enunciated that

$$x_p(t) = e^{A_p h} x_p(k) + \sum_{i=1}^{n} B_p^i \left(h, A_p\right) u_i \quad (8)$$

with $h = t - kT_s$, $T_0 = 0$, and

$$B_p^i(h, A_p) = e^{A_p\left(h - \min\left(h, \sum_{j=0}^{i} T_j\right)\right)} \times B_p\left(\min\left(h - \sum_{j=0}^{i-1} T_j, T_i\right), A_p\right) \quad (9)$$

for $i = 1, \ldots, n$. Note that $B_p^i(h, A_p) = 0$ if $\sum_{j=0}^{i-1} T_j \geq h$ due to (2), and then the terms corresponding to times larger than h do not contribute in (8). Now, we can write the dual-rate sampling scenario in a lifted fashion as

$$x_p(k+1) = \tilde{A}_p x_p(k) + \tilde{B}_p U \quad (10)$$

with $U = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}^T$, $\tilde{A}_p = e^{A_p T_s}$, and $\tilde{B}_p = \begin{bmatrix} B_p^1(T_s, A_p) & \cdots & B_p^n(T_s, A_p) \end{bmatrix}$. Then, the control problem is to find the set of control signals $\{u_i\}$, which optimizes the performance of the plant (8). In this paper, we understand the optimization of the performance as the maximization of the decay rate in the local interval between kT_s and $(k+1)T_s$. Note that the gain c in (5) might become unacceptably large if the unique optimization objective is the maximization of the decay rate. Certain conditions should be satisfied in the optimization to avoid this problem.

IV. DECAY-RATE BASED OPTIMAL DESIGN

In this Section, we develop the conditions to find the set (7), which guarantees an acceptable value of gain c, while the decay rate is maximized. First of all, let us define the following lemma.

Lemma 1: The exponential stability of the discretized plant (10) *is guaranteed with decay rate $\hat{\alpha}$, if there exits a Lyapunov function $V(x_p(kT_s)) = V(k) : \mathbb{R}^{n_{xp}} \to \mathbb{R}$ and positive scalars λ_1, λ_2 and positive integer q such that*

$$\lambda_1 \|x_p(k)\|^q \leq V(k) \leq \lambda_2 \|x_p(k)\|^q \quad (11)$$

and

$$V(k+1) - e^{q\hat{\alpha}T_s} V(k) \leq 0. \quad (12)$$

Algorithm 1 Computation of Optimized Control Signals

Offline Computation

Step 1off We fix an auxiliary gain $K = \begin{pmatrix} -D_c C_p & C_c \end{pmatrix}$ and a number of signals n which provide the exponential stability of system (16), if control signals (20) would be applied. To guarantee it, conditions (21) based on LMI are developed.

Step 2off We prove that if (10) is exponentially stable, then (6) is also exponentially stable with the same decay rate α and with gain (22).

Step 3off Constraints in the LMI variable P are established to reach a desired agreement between α and the gain c.

Step 4off We develop the constraints (29) in which the set of actual control signals $\{u_i\}$ for $i = 1, \ldots, n$ should satisfy respect to the set of auxiliary control signals $\{\hat{u}_i\}$ for $i = 1, \ldots, n$ in order to maintain the exponential stability of (6).

Online Computation

Step 1on The output signal $x(kT_s)$ is received from the sensor in the controller at the instant T_s.

Step 2on The constraints of Step 4off are evaluated taking into account K and $x(kT_s)$, and the available computation resources decide the n_{opt} control signals that are going to be optimized.

Step 3on The control signals $\{u_i\}$ for $i = 1, \ldots, n$ are computed online to maximize the performance without being limited to the form of the auxiliary controller, and while the exponential stability is achieved.

Then, the cost function which is optimized is

$$J = -\alpha = -\frac{\log\left(\frac{V(k+1)}{V(k)}\right)}{qT_s} \leq -\hat{\alpha}. \quad (13)$$

The idea is to guarantee a certain behavior using auxiliary control signals (which are not actually applied to the plant) computed from the auxiliary LTI controller, and then use them as the initial values to obtain the optimized signals, which guarantee the same c and at least the same (or even greater) decay rate as the auxiliary LTI controller. Depending on the available computation resources, n_{opt} input signals with $1 \leq n_{\text{opt}} \leq n$ are computed to maximize α, while the rest of input signals $(n - n_{\text{opt}})$ are maintained in their initial values determined by the auxiliary controller. The procedure is synthesized in Algorithm 1. To develop Step 1off, let us define the auxiliary LTI controller as

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c e(t) \\ \hat{u}(t) &= C_c x_c(t) + D_c e(t), \end{aligned} \quad (14)$$

where $e(t) = r(t) - y(t)$ being the reference signal $r(t) = 0$ by simplicity, $x_c(t) \in \mathbb{R}^{n_{x_c}}$ is the state vector of the controller, $\hat{u}(t) \in \mathbb{R}^{n_u}$ is the auxiliary input vector, and $A_c \in \mathbb{R}^{n_{x_c} \times n_{x_c}}$, $B_c \in \mathbb{R}^{n_{x_c} \times n_y}$, $C_c \in \mathbb{R}^{n_u \times n_{x_c}}$, and $D_c \in \mathbb{R}^{n_u \times n_y}$ are constant matrices. The whole closed-loop system composed by (6) and (14) takes the following form

$$\dot{x}(t) = Ax(t) + B \begin{bmatrix} \hat{u}(t) \\ y(t) \end{bmatrix} \quad (15)$$

with $x(t) = \begin{bmatrix} x_p(t) & x_c(t) \end{bmatrix}^T$ and $A = \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}$, $B = \begin{bmatrix} B_p & 0 \\ 0 & -B_c \end{bmatrix}$. Note that $\hat{u}(t)$ can be rewritten as $\hat{u}(t) = Kx(t) = \begin{bmatrix} -D_c C_p & C_c \end{bmatrix} x(t)$. In a multi-rate scenario and using a lifted framework, (15) is transformed following (8) into

$$x(k+1) = \tilde{A}x(k) + \tilde{B} \begin{bmatrix} \hat{U} \\ Y \end{bmatrix} \quad (16)$$

with $\tilde{A} = e^{AT_s}$,

$$\tilde{B}^T = \begin{bmatrix} \left(B_p^1\right)^T (T_s, A_p) & 0 \\ \vdots & \vdots \\ \left(B_p^n\right)^T (T_s, A_p) & 0 \\ 0 & \left(B_c^1\right)^T (T_s, A_c) \\ \vdots & \vdots \\ 0 & \left(B_c^n\right)^T (T_s, A_c) \end{bmatrix},$$

where $B_c^i(T_s, A_c)$ with $i = 1, \ldots, n$ is analogous to (9), and $\hat{U} = \begin{bmatrix} \hat{u}_1 & \cdots & \hat{u}_n \end{bmatrix}^T$, $Y = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^T$ being

$$\hat{u}_i = Kx(kT_s + \sum_{j=0}^{i-1} T_j), \quad (17)$$

$$y_i = C_p x_p(kT_s + \sum_{j=0}^{i-1} T_j). \quad (18)$$

Note that the value of y_i for $i > 1$ is unknown. Hence, some estimation about y_i should be done to compute the auxiliary input signals $\{\hat{u}_i\}$ and, consequently be able to obtain the optimized $\{u_i\}$. Following [11], [13], we assume that a model of the plant is known and we can obtain an estimated value of the system output in the actuator update instants. Let us consider that the model coincides with (6). Therefore, since the extension for a different model is straightforward, this allows us to focus on the optimization problem without tarnishing the development due to the inclusion of new notation. Since $\hat{u}_i$ and y_i for $i = 1, \ldots, n$ can be written recursively in terms of $x(k)$ in view of (17)-(18), equation (16) can be replaced by

$$x(k+1) = \hat{\Pi}x(k), \quad (19)$$

where

$$\hat{\Pi} = \prod_{i=1}^{n} e^{AT_i} + B\left(T_i, A\right) \begin{bmatrix} K \\ C_p \ 0 \end{bmatrix}.$$

To prove the exponential stability of the system, let us state the following assumption.

Assumption 1: There exists an auxiliary matrix $K = \begin{bmatrix} D_c C_p & C_c \end{bmatrix}$ such that

$$\hat{u}_i = Kx\left(kT_s + \sum_{j=0}^{i-1} T_j\right) \quad (20)$$

and a symmetric positive definite matrix P such that the following LMI is satisfied

$$\hat{\Pi}^T P \hat{\Pi} - e^{-2\hat{\alpha}T_s} P \leq 0. \quad (21)$$

We can formulate now the following theorem to compute Step 2off.

Theorem 1: Consider the discretized closed loop system (16). *Suppose that Assumption 1 holds. Then, the system* (6) *is exponentially stable with at least decay rate* $\hat{\alpha} > 0$ *and gain*

$$c = \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}} e^{\hat{\alpha}T_s} \prod_{i=1}^{n} \Big(e^{\mu(A)T_i} + B_\mu(T_i, A)\left(\|K\| + \|C_p\|\right) \Big). \quad (22)$$

Proof: Consider a Lyapunov function of the form

$$V(k) = x^T(k) P x(k) \quad (23)$$

where $P > 0$. Then, it satisfies that

$$\lambda_m(P)\|x(k)\|^2 \leq V(k) \leq \lambda_M(P)\|x(k)\|^2$$

and (11) is fulfilled. Consider also K, so that (20) is satisfied. The exponential decrease of the system (16) with control signals (20) with decay rate $\hat{\alpha}$ is achieved if (12) of Lemma 1 is satisfied, i.e., if

$$\begin{aligned} &V(k+1) - e^{-2\hat{\alpha}T_s} V(k) \\ &= x^T(kT_s) \left(\hat{\Pi}^T P \hat{\Pi} - e^{-2\hat{\alpha}T_s} P \right) x(kT_s) \leq 0, \end{aligned} \quad (24)$$

which is guaranteed by (21). To find c, we use (23) and (24)

$$\|x(k)\| \leq \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}} e^{-\hat{\alpha}kT_s} \|x(0)\|.$$

Consequently, the state of the plant in any moment $t = kT_s + h, \forall h \in [0, T_s]$ is also exponentially stable since it can be bounded in view of (8) as follows

$$\begin{aligned} \|x(t)\| \leq\ & \|e^{Ah}\| \|x(k)\| + \left(\|K\| + \|C_p\|\right) \\ & \times \sum_{i=1}^{n} \left(\|e^{A\left(h - \sum_{j=0}^{i-1} T_j\right)}\| \, \|x\left(kT_s + \sum_{j=0}^{i} T_i\right)\| \right. \\ & \left. \times \ \|B\left(\min\left(h - \sum_{j}^{i-1} T_j, T_i\right), A\right)\| \right). \end{aligned} \quad (25)$$

Since $\|x\left(kT_s+\sum_{j=0}^{i}T_j\right)\|$ can be bounded recursively until obtaining a bound which depends on $x(k)$, and using (1) and (3), we can write

$$\begin{aligned}\|x(t)\| &\leq \hat{c}\|x(kT_s)\| \leq \hat{c}\sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}}e^{-\hat{\alpha}kT_s}\|x(0)\| \\ &= ce^{-\hat{\alpha}(t)}\|x(0)\|. \end{aligned} \tag{26}$$

Then,

$$\sqrt{\|x_p(t)\|^2+\|x_c(t)\|^2} \leq ce^{-\hat{\alpha}(t)}\sqrt{\|x_p(0)\|^2+\|x_c(0)\|^2}. \tag{27}$$

Since the initial state of the controller can be considered zero, the state of the plant in every moment is exponentially bounded with at least decay rate $\hat{\alpha}$ and gain c. □

Remark 1: Since c depends on the relation $\lambda_M(P)/\lambda_m(P)$, we limit this ratio in the solutions of the LMI (21) to obtain more precise solutions. This corresponds to the Step 3off of Algorithm 1. Note also that c in (22) increases with the number of input signals. This is not a problem of the optimization algorithm in the sense that the value of c is large due to the lifted model itself, and not due to the optimized signals. However, it is possible to add constrains of the form $V(kT_s+\sum_{j=0}^{i}T_j) - e^{-2\hat{\alpha}T_s}V(kT_s+\sum_{j=0}^{i-1}T_j) \leq 0$ in order to guarantee that the Lyapunov function decreases with each input signal, and consequently the value of c is more limited.

Proved that there exists a controller able to exponentially stabilize (16), we make use of the cost function (13) to maximize the decay rate of the system

$$x((k+1)T_s) = \tilde{A}x(kT_s) + \tilde{B}\begin{bmatrix}U\\Y\end{bmatrix}. \tag{28}$$

As maximizing the decay rate enlarges the gain, we need to establish constrains over each control signal (Step 4off) to maintain c in acceptable values. Let us consider the following assumption.

Assumption 2: The set of control signals $\{u_i\}$ for $i = 1,\ldots,n$ is formed by n_{opt} control signals computed to minimize (13) and $n-n_{\text{opt}}$ control signals from (20) satisfying that

$$\|u_i\| \leq \|K\|\|x\left(kT_s+\sum_{j=0}^{i-1}T_j\right)\|. \tag{29}$$

Then, the following Theorem can be stated.

Theorem 2: Consider the discretized closed loop system (28) *and control signals* $\{u_i\}$ *for* $i = 1,\ldots,n$. *Suppose that Assumption 2 holds. Then, the system* (6) *is globally exponentially stable with at least decay rate* $\alpha \geq \hat{\alpha}$ *and gain* c *as in* (22) .

Proof: Consider the minimization function (13). From Theorem 1 it is guaranteed at least $\alpha = \hat{\alpha}$ if $u_i = \hat{u}_i$ for $i = 1,\ldots,n$. Equation (29) from Assumption 2 implies that (25) is satisfied and therefore (27) is also satisfied. Hence, Definition 1 is fulfilled (and the system is exponentially stable with at least decay rate α and gain c). □

With these considerations, we can carry out in each sampling period the online computations (Step1on-Step3on) to minimize the cost function (13) taking into account the constraints (29).

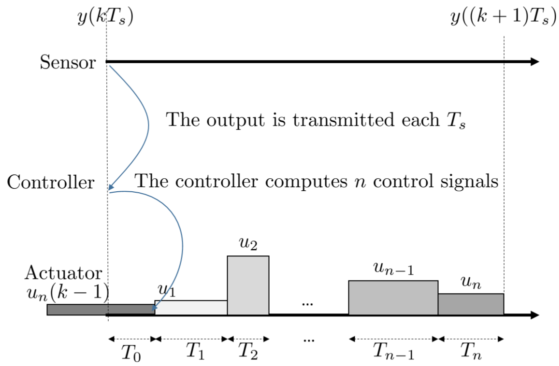

FIGURE 3. Scheme of the transmitted signals in the presence of delays. *n* different signals are applied during the sampling period T_s.

V. EXTENSION FOR TIME-DELAY SYSTEMS

In this Section, the results from Section 4 are formalized for the case with delays. Delays might appear in the control problem in several ways. It might be intrinsic of the plant, it might be caused by a network placed between the elements of the plant, it might be a consequence of the computation effort of Algorithm 1, etc. We assume that there exists a maximum delay τ and that the available optimization time is the difference between τ and the network time-varying delay. We follow the description in [11] and [13] to deal with delays in multi-rate systems. Let us assume that during the delay, the last input signal of the previous output sampling period is held (Fig. 3). Then,

$$\begin{aligned}x(t) &= \tilde{A}x(k) + \tilde{B}\begin{bmatrix}U\\Y\end{bmatrix} \\ &\quad + e^{A(h-\min(h,T_0))}B\left(\min\left(h,T_0\right),A\right)\begin{bmatrix}\psi(k)\\y_1\end{bmatrix}\end{aligned}$$

with $t = kT_s + h$, and where $T_0 = \tau$ and $\psi(k)$ corresponds to the last input signal of the previous period, i.e., $\psi(k) = u_n(k-1)$. Hence, we can define the following augmented state

$$\xi(t) = \begin{pmatrix}x(t)\\ \psi(k)\end{pmatrix}, \quad \forall t \in [kT_s,(k+1)T_s] \tag{30}$$

such that

$$\begin{aligned}\xi(k+1) &= \begin{bmatrix}\tilde{A}+A_1 & A_2\\ 0 & 0\end{bmatrix}\xi(k) \\ &\quad + \begin{bmatrix}\tilde{B}\\ B_d\end{bmatrix}\begin{bmatrix}U\\Y\end{bmatrix}\end{aligned} \tag{31}$$

where

$$A_1 = \begin{bmatrix}e^{AT_0} - B_c^0(T_0,A)C_p & 0\\ 0 & 0\end{bmatrix}$$

$$A_2 = \begin{bmatrix}0 & B_p^0(T_0,A)\\ 0 & 0\end{bmatrix}$$

and

$$\begin{bmatrix}\tilde{B}\\ B_d\end{bmatrix} = \begin{bmatrix} \left(B_p^1\right)^T(T_s,A_p) & 0 & 0\\ \vdots & \vdots & \vdots \\ \left(B_p^n\right)^T(T_s,A_p) & 0 & \mathbb{I}_{n_u}\\ 0 & \left(B_c^1\right)^T(T_s,A_c) & 0\\ \vdots & \vdots & \vdots\\ 0 & \left(B_c^n\right)^T(T_s,A_c) & 0\end{bmatrix}^T.$$

To study the exponential stability, we consider again the control signals $\{\hat{u}_i\}$ computed by the auxiliary controller (14) and replace $\{u_i\}$ by them in (31). We obtain after a recursive process that

$$\xi(k+1) = \hat{\Pi}_d \xi(k), \tag{32}$$

with

$$\hat{\Pi}_d = \begin{bmatrix} \hat{\Pi}A_1 & \hat{\Pi}A_2\\ K\hat{\Pi}_{n-1}A_1 & K\hat{\Pi}_{n-1}A_2\end{bmatrix}, \tag{33}$$

being

$$\hat{\Pi}_{n-1} = \prod_{i=1}^{n-1} e^{AT_i} + B(T_i,A)\begin{bmatrix} & K\\ C_p & 0\end{bmatrix}.$$

Then, we can use again Lemma 1 to guarantee the exponential stability of (32). Let us recall Assumption 1 for the augmented state (30).

Assumption 3: There exists an auxiliary matrix $K = \begin{bmatrix} D_cC_p & C_c\end{bmatrix}$ satisfying (20) and a symmetric positive definite matrix P such that

$$\hat{\Pi}_d^T P \hat{\Pi}_d - e^{-2\hat{\alpha}T_s}P \leq 0. \tag{34}$$

The following Theorem is formulated.

Theorem 3: Consider the discretized closed loop system (31). *Suppose that Assumption 3 holds. Then, the system* (30) *is exponentially stable with at least decay rate* $\hat{\alpha} > 0$ *and gain*

$$\begin{aligned} c = &\sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}}e^{\hat{\alpha}T_s}\\ &\times\left(B^0_{c\mu}(T_0,A_c)\|C_p\| + B^0_{p_\mu}(T_0,A_c) + e^{\mu(A)T_0}\right)\\ &\times \prod_{i=1}^n\left(e^{\mu(A)T_i} + B_\mu(T_i,A)\left(\|K\| + \|C_p\|\right)\right). \end{aligned} \tag{35}$$

Proof: $\|\xi(kT_s)\|$ could be exponentially bounded such that

$$\|\xi(kT_s)\| \leq \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}}e^{-\hat{\alpha}(kT_s)}\|\xi(0)\|$$

due to (34). In addition, if we observe that $\|\psi(0)\| = 0$ because at the initial conditions there is not a previous input signal, then

$$\|x(kT_s)\|^2 + \|\psi(kT_s)\|^2 \leq \frac{\lambda_M(P)}{\lambda_m(P)}e^{-2\hat{\alpha}(kT_s)}\|x(0)\|^2$$

and trivially

$$\|x(kT_s)\| \leq \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}}e^{-\hat{\alpha}(kT_s)}\|x(0)\|$$

Finally, we can guarantee anytime the exponential stability of the closed-loop system similarly to (26) and use the same arguments to ensure the stability of the plant.

$$\begin{aligned}\|x(t)\| = &e^{\mu(A)T_s}\|x(kT_s)\| + e^{\mu(A)T_0}B_\mu(T_0,A)\|\psi(k)\|\\ &+\sum_{i=1}^n\left(e^{\mu(A)\left(T_s-\sum_{j=0}^{i}T_j\right)}B_\mu(T_i,A)\right.\\ &\left.\times\left(\|K\|+\|C_p\|\right)\|x\left(kT_s+\sum_{j=0}^{i-1}T_j\right)\|\right),\end{aligned}$$

Thus, recalling (26), (27), c in (35) is obtained. □

Trivially, Theorem 2 is still valid for the time-delay case. Hence, we can use our knowledge about the delay to improve the control signals in terms of maximizing the decay rate.

VI. NUMERICAL EXAMPLE

In this Section, we provide a numerical example to test the validity of the developed strategy. Consider the plant proposed in [13]

$$\mathcal{P}(s) = \frac{4}{(s+1)(s+4)} \tag{36}$$

with sampling period $T_s = 1.25$s. Consider also the following auxiliary controller

$$\mathcal{C}(s) = K_p\left(1 + K_i\frac{1}{s} + K_d\frac{s}{fs+1}\right) \tag{37}$$

where $K_p = 1.05$, $K_i = 0.92$, $K_d = 0.15$ and $f = 0.1$. To apply our optimized control, let us describe the system (36), (37) in state-space. Then,

$$\mathcal{P}(s): \begin{cases}\dot{x}_p(t) = A_px_p(t) + B_pu(t)\\ y(t) = C_px_p(t) + D_pu(t)\end{cases}$$

$$\mathcal{C}(s): \begin{cases}\dot{x}_c(t) = A_cx_c(t) + B_ce(t)\\ \hat{u}(t) = C_cx_c(t) + D_ce(t),\end{cases}$$

where $e(t) = r(t) - y(t)$ and we take the reference $r(t) = 0$ by simplicity. Therefore, the whole continuous control loop formed by the plant and the auxiliary controller is described by

$$\begin{aligned}\dot{x}(t) = \begin{bmatrix}\dot{x}_p\\ \dot{x}_c\end{bmatrix} &= Ax(t) + B\begin{bmatrix}\hat{u}(t)\\ y(t)\end{bmatrix}\\ &= \begin{bmatrix}A_p & 0\\ 0 & A_c\end{bmatrix}x(t) + \begin{bmatrix}B_p & 0\\ 0 & -B_c\end{bmatrix}\begin{bmatrix}\hat{u}(t)\\ y(t)\end{bmatrix},\end{aligned}$$

where $\hat{u}(t) = Kx(t) = \begin{pmatrix}-D_cC_p & C_c\end{pmatrix}x(t)$. The numerical values of the matrices are

$$A_p = \begin{pmatrix}-5 & -4\\ 1 & 0\end{pmatrix},\quad B_p = \begin{pmatrix}1\\ 0\end{pmatrix},$$
$$C_p = \begin{pmatrix}0 & 4\end{pmatrix},\quad D_p = 0,$$

$$A_c = \begin{pmatrix} 0 & 0 \\ 0 & -10 \end{pmatrix}, \quad B_c = \begin{pmatrix} 1 \\ -10 \end{pmatrix},$$
$$C_c = \begin{pmatrix} 0.966 & 1.575 \end{pmatrix}, \quad D_c = 2.625.$$

Consider a multi-rate strategy with a delay $\tau = 0.2$ s and where two different input signals are transmitted to the actuator during the sampling period with $T_1 = 0.425$ s and $T_2 = 0.625$ s so that (33) is

$$\hat{\Pi}_d = \begin{pmatrix} -0.09 & -0.32 & 0.03 & -0.02 & -0.04 \\ -0.01 & 0.01 & 0.09 & 0.00 & -0.00 \\ -0.43 & -3.50 & 0.89 & -0.02 & -0.08 \\ 0.36 & 1.76 & 0.18 & 0.04 & 0.09 \\ -0.38 & -1.01 & 0.49 & -0.10 & -0.18 \end{pmatrix}$$

Then, the inequality $\hat{\Pi}_d^T P \hat{\Pi}_d - P < 0$ is solved to obtain the matrix P which maximize the decay rate with a restrained difference between the maximum and minimum eigenvalues of P. Hence, P results

$$P = \begin{pmatrix} 2.35 & 0.07 & -0.13 & 0.26 & 0.01 \\ 0.07 & 2.95 & -0.29 & 0.01 & -0.05 \\ -0.13 & -0.29 & 0.47 & 0.17 & -0.76 \\ 0.26 & 0.01 & 0.17 & 0.30 & -0.38 \\ 0.01 & -0.05 & -0.76 & -0.38 & 1.88 \end{pmatrix}.$$

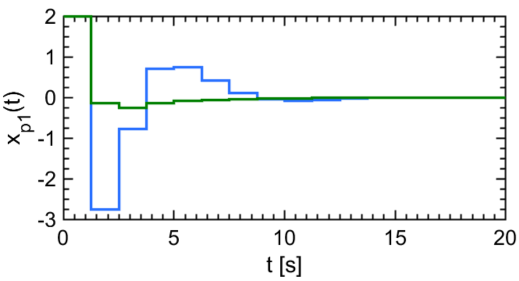

FIGURE 4. Evolution of state x_1. (Green) Optimal PID controller. (Blue) Auxiliary PID controller.

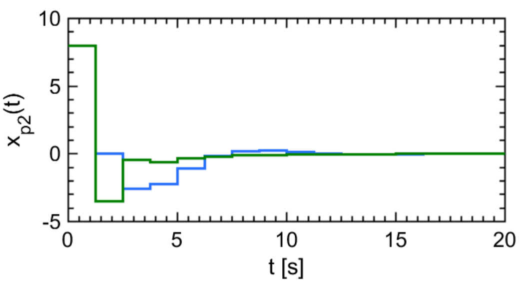

FIGURE 5. Evolution of state x_2. (Green) Optimal PID controller. (Blue) Auxiliary PID controller.

and the minimum decay is $\hat{\alpha} = 0.10$. From Theorem 3, we obtain the maximum gain $c = 9.9 \cdot 10^4$. If we simulate the system with initial conditions $x_0^T = \begin{pmatrix} 2 & 8 & 0 & 0 \end{pmatrix}$, Fig. 4-7 are obtained. The states of the plant (Fig. 4-5) reach the equilibrium faster with the optimization control than with

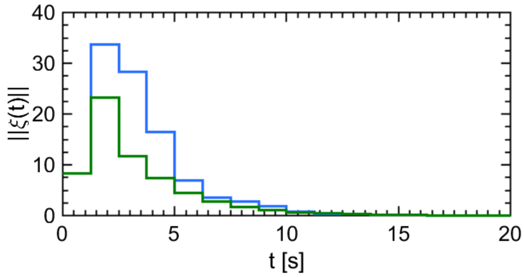

FIGURE 6. Norm of $\xi(t)$. (Green) Optimal PID controller. (Blue) Auxiliary PID controller.

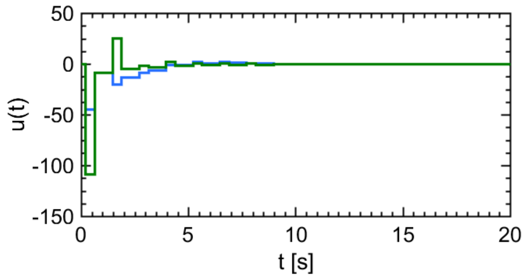

FIGURE 7. Input signal. (Green) Optimal PID controller. (Blue) Auxiliary PID controller.

the auxiliary PID controller. Naturally, the norm of the augmented state formed by x_p, x_c and ψ (Fig. 6) decreases also more quickly. The input signal (Fig. 7) in the optimization control requires a larger impulse at the beginning to reach this decay rate but always limited by constrains (29).

VII. CONCLUSIONS

In this paper, an algorithm based on the resolution of some LMI conditions to maximize the decay rate of a dual-rate system is introduced. The output of the system is sampled with slow rate, while the control input is computed and updated with the fast rate. This provides an improvement in the performance of the system and gives the opportunity to optimize the control signals depending on the available resources. The extension of the results to the time-delay case enlarges considerably the possible applications of the design, since the computation time or network delays can be considered in the optimization. The proposed example shows the benefits of the scheme.

Future work may be related with improvements in the optimization to reduce the computation time and waste of resources, application to experimental plants and the extension to distributed controllers.

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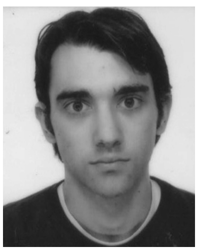

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