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Disjointness in hypercyclicity

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Abstract

We introduce a notion of disjointness for finitely many hypercyclic operators acting on a common space, notion that is weaker than Furstenberg's disjointness of fluid flows. We provide a criterion to construct disjoint hypercyclic operators, that generalizes some well-known connections between the Hypercyclicity Criterion, hereditary hypercyclicity and topological mixing to the setting of disjointness in hypercyclicity. We provide examples of disjoint hypercyclic operators for powers of weighted shifts on a Hilbert space and for differentiation operators on the space of entire functions on the complex plane.

Keywords: Hypercyclic vectors; Hypercyclic operators

1. Introduction

The notion of disjointness was introduced by Harry Furstenberg in 1967 in his seminal paper [10] for dynamical systems (for measure-preserving transformations and for homeomorphisms of compact spaces). In the topological category, a *joining* of two dynamical systems (X, T) and (Y, S) (X, Y being compact spaces and T, S being homeomorphisms) is a pair $(\Delta, T \times S)$, where $\Delta \subset X \times Y$ is a closed $T \times S$-invariant subset whose coordinate projections $\pi_X : \Delta \to X$, $\pi_Y : \Delta \to Y$ are surjective. (X, T) and (Y, S) are *disjoint* in the sense

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of Furstenberg [10, Definition II.1] if $X \times Y$ is the unique joining of these two systems. Observe that, if $X = Y$ and $x \in X$ is an element whose orbits $\mathrm{Orb}(T, x) := \{x, Tx, T^2x, \ldots\}$ and $\mathrm{Orb}(S, x) := \{x, Sx, S^2x, \ldots\}$ are dense in X, then $\Delta := \overline{\mathrm{Orb}(T \times S, (x, x))}^{X \times X}$ is a joining of (X, T) and (X, S). Thus disjointness of (X, T) and (X, S) would imply $\Delta = X \times X$ and, therefore, (x, x) has a dense orbit in $X \times X$ under $T \times S$. The existence of such elements (x, x) is our notion of disjointness for hypercyclic operators T and S.

A continuous linear operator on a Fréchet space X is said to be hypercyclic provided there is some vector $z \in X$ whose orbit $\{z, Tz, T^2z, \ldots\}$ is dense in X. Such z is called a hypercyclic vector for T. The first examples of hypercyclic operators were on the space $X = H(\mathbb{C})$ of entire functions on the complex plane $\mathbb{C}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$. In 1929, Birkhoff [6] showed that the translation operator $T_a : H(\mathbb{C}) \to H(\mathbb{C}) T_a f(z) = f(z + a)$ $(z \in \mathbb{C}, f \in H(\mathbb{C}))$ is hypercyclic whenever $0 \neq a \in \mathbb{C}$. In 1952, MacLane [21] showed that the derivative operator $f \overset{D}{\mapsto} f'$ is hypercyclic. More recently, Godefroy and Shapiro [12] provided a comprehensive extension of these two results to all convolution operators (but scalar multiples of the identity).

Given $N \geqslant 2$ operators $T_1, T_2, \ldots, T_N$ on a Fréchet space X, it has been natural to study the cyclic properties that their direct sum $T_1 \oplus \cdots \oplus T_N$ may inherit from those of $T_1, T_2, \ldots, T_N$. A significant example of this is Herrero's [17] central question on hypercyclicity and weak mixing:

Must $T \oplus T$ be hypercyclic whenever T is?

(Recall that a direct sum $T_1 \oplus T_2$ may fail to be cyclic even if each of T_1 and T_2 is hypercyclic [25].) Another example is Salas's characterization of those weighted shifts $T_1, \ldots, T_N$ on $X = c_0$ or ℓ_p $(1 \leqslant p < \infty)$ for which $T_1 \oplus \cdots \oplus T_N$ is hypercyclic [25]. But while the dynamical systems $(X, T_1), \ldots, (X, T_N)$ may be independent of each other (i.e., for $i \neq j$ the action of T_i may not depend on T_j and vice-versa), they may still have a certain correlation (or lack of it) manifested in the behaviour of the orbits

$$\left\{(z, z, \ldots, z), (T_1z, T_2z, \ldots, T_Nz), \left(T_1^2z, T_2^2z, \ldots, T_N^2z\right), \ldots\right\} \quad (z \in X)$$

in X^N. In this paper we propose to study the situation in which such orbits are dense:

Definition 1.1. We say that $N \geqslant 2$ hypercyclic operators $T_1, \ldots, T_N$ acting on a Fréchet space X are *disjoint*, or *diagonally hypercyclic* (in short, d-hypercyclic), provided there is some vector $(z, z, \ldots, z)$ in the diagonal of $X^N = X \times X \times \cdots \times X$ such that

$$\left\{(z, z, \ldots, z), (T_1z, T_2z, \ldots, T_Nz), \left(T_1^2z, T_2^2z, \ldots, T_N^2z\right), \ldots\right\}$$

is dense in X^N. We call the vector $z \in X$ a d-hypercyclic vector associated to the operators $T_1, T_2, \ldots, T_N$.

While we focus on d-hypercyclicity here, Definition 1.1 admits natural extensions to notions of d-cyclicity, d-supercyclicity, and d-chaoticity. We were kindly informed by Bernal-González of his recent paper [2], in which he independently introduces the concept of d-hypercyclicity.

The paper is organized as follows: Section 2 is devoted to develop the basic theory of disjoint hypercyclic operators. We provide a generalization of the so-called Hypercyclicity Criterion, which we call the d-Hypercyclicity Criterion, and which allow us to construct d-hypercyclic operators. Just as the Hypercyclicity Criterion is related to the notions of hereditary hypercyclicity and topological mixing [5,13], we establish in Theorem 2.7 connections between the

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d-Hypercyclicity Criterion and the notions of hereditary d-hypercyclicity and d-topological mixing. In Section 3 we analyze examples of d-hypercyclic translation and differential operators on the space $H(\mathbb{C})$ of entire functions. In particular, we extend Birkhoff's mentioned result by showing that N translation operators $T_{a_1}, \ldots, T_{a_N}$ on $H(\mathbb{C})$ are d-hypercyclic if and only if the complex numbers $a_1, \ldots, a_N$ are all distinct and non-zero (Theorem 3.1). In Section 4 we give a characterization of the d-hypercyclicity of powers of unilateral (respectively bilateral) weighted shifts on $\ell_2(\mathbb{N})$ (respectively $\ell_2(\mathbb{Z})$) (Theorem 4.1). One consequence of it is a generalization of the mentioned classical result of Rolewicz: Given integers $1 \leqslant r_1 \leqslant r_2 \leqslant \cdots \leqslant r_N$ and scalars $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, the operators $\lambda_1 B^{r_1}, \ldots, \lambda_N B^{r_N}$ are d-hypercyclic on ℓ_2 if and only if $1 \leqslant r_1 < r_2 < \cdots < r_N$ and $1 < |\lambda_1| < \cdots < |\lambda_N|$ (Corollary 4.2). Another consequence is the existence, for each $N \geqslant 2$, of d-hypercyclic operators $T_1, \ldots, T_N$ on $\ell_2(\mathbb{Z})$ so that their Hilbert Adjoints $T_1^*, \ldots, T_N^*$ are also d-hypercyclic (Theorem 4.11).

2. Disjoint hypercyclic operators

Two d-hypercyclic operators must be substantially different, and in particular an operator can never be d-hypercyclic with a scalar multiple of itself. To see this, suppose to the contrary that T and cT have a d-hypercyclic vector $0 \neq z \in X$, where c is a scalar. Then there must exist sequences (n_k) and (m_k) for which $(T^{n_k}z, (cT)^{n_k}z) \underset{k\to\infty}{\longrightarrow} (z, 0)$ and $(T^{m_k}z, (cT)^{m_k}z) \underset{k\to\infty}{\longrightarrow} (0, z)$. But the first limit forces $|c| < 1$, while the second one forces $|c| > 1$, a contradiction.

Several remarkable advances have been made in understanding dense orbits of (linear) operators. Ansari [1] showed that an operator T and its iterates T^n $(n = 1, 2, \ldots)$ have the same set of hypercyclic vectors. The second author [22] and independently Costakis [8] showed that if the union of finitely many orbits of T is dense, then one of such orbits must already be dense. Bourdon and Feldman [7] generalized the previous two results by showing that a somewhere dense orbit must be everywhere dense. Also, León-Saavedra and Müller [20] showed that T and its rotations $e^{i\theta}T$ $(\theta \in \mathbb{R})$ have the same hypercyclic vectors. Each of these four results also holds for d-hypercyclicity. This follows from the fact that we study the hypercyclic behaviour of the orbit of $(z, \ldots, z) \in X^N$ under an operator of the form $T_1 \oplus \cdots \oplus T_N$.

In what follows, $L(X)$ denotes the space of linear and continuous operators on a separable, infinite dimensional Fréchet space X.

Definition 2.1. We say that $N \geqslant 2$ sequences of operators $(T_{1,j})_{j=1}^\infty, \ldots, (T_{N,j})_{j=1}^\infty$ in $L(X)$ are *d-topologically transitive* (respectively *d-mixing*) provided for every non-empty open subsets $V_0, \ldots, V_N$ of X there exists $m \in \mathbb{N}$ so that $\emptyset \neq V_0 \cap T_{1,m}^{-1}(V_1) \cap \cdots \cap T_{N,m}^{-1}(V_N)$ (respectively so that $\emptyset \neq V_0 \cap T_{1,j}^{-1}(V_1) \cap \cdots \cap T_{N,j}^{-1}(V_N)$ for each $j \geqslant m$). Also, we say that $N \geqslant 2$ operators $T_1, \ldots, T_N$ in $L(X)$ are d-topologically transitive (respectively d-mixing) provided $(T_1^j)_{j=1}^\infty, \ldots, (T_N^j)_{j=1}^\infty$ are d-topologically transitive sequences (respectively d-mixing sequences).

Definition 2.2. We say that $N \geqslant 2$ sequences $(T_{1,j})_{j=1}^\infty, \ldots, (T_{N,j})_{j=1}^\infty$ in $L(X)$ are *d-universal* (respectively *densely d-universal*) if

$$\{(T_{1,j}z, T_{2,j}z, \ldots, T_{N,j}z):\ j \in \mathbb{N}\}$$

is dense in X^N for some vector $z \in X$ (respectively for each vector z in a given dense subset of X). We call such vector z a *d-universal vector* for $(T_{1,j})_{j=1}^\infty, \ldots, (T_{N,j})_{j=1}^\infty$. Also, we say that

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$(T_{1,j})_{j=1}^{\infty}, \ldots, (T_{N,j})_{j=1}^{\infty}$ are *hereditarily universal* (respectively *hereditarily densely universal*) provided for each increasing sequence of positive integers (n_k) the sequences $(T_{1,n_k})_{k=1}^{\infty}, \ldots, (T_{N,n_k})_{k=1}^{\infty}$ are d-universal (respectively densely d-universal).

The following mimics an observation by Grosse-Erdmann [14, Satz 1.2.2].

Proposition 2.3. *Let $(T_{1,n})_{n=1}^{\infty}, \ldots, (T_{N,n})_{n=1}^{\infty}$ be sequences of operators in $L(X)$. Then the following are equivalent*:

(a) *$(T_{1,n})_{n=1}^{\infty}, \ldots, (T_{N,n})_{n=1}^{\infty}$ are d-topologically transitive.*
(b) *The set of d-universal vectors for $(T_{1,n})_{n=1}^{\infty}, \ldots, (T_{N,n})_{n=1}^{\infty}$ is a dense G_δ.*

Proof. The implication (b) $\Rightarrow$ (a) is immediate. To see the converse, let $\{A_j\colon\ j \in \mathbb{N}\}$ be a basis for the topology of X. By (a), for each $J = (j_1, \ldots, j_N) \in \mathbb{N}^N$, $\bigcup_{m \geqslant k}(T_{1,m}^{-1}(A_{j_1}) \cap \cdots \cap T_{N,m}^{-1}(A_{j_N}))$ is both open and dense in the Baire space X. But the set of d-universal vectors for $(T_{1,n})_{n=1}^{\infty}, \ldots, (T_{N,n})_{n=1}^{\infty}$ is

$$\bigcap_{J \in \mathbb{N}^N} \bigcap_{k \in \mathbb{N}} \bigcup_{m \geqslant k} \bigl(T_{1,m}^{-1}(A_{j_1}) \cap \cdots \cap T_{N,m}^{-1}(A_{j_N})\bigr). \qquad \square$$

Godefroy and Shapiro [12] gave a sufficient condition for hypercyclicity, called blow-up/collapse condition by Grosse-Erdmann [16]. Inspired in this condition, we formulate an analogous one for disjoint hypercyclicity.

Proposition 2.4 *(Blow-Up/Collapse). Let $(T_{1,n})_{n=1}^{\infty}, \ldots, (T_{N,n})_{n=1}^{\infty}$ be $N \geqslant 2$ sequences of operators acting on a Fréchet space X. Suppose that for each open neighborhood of zero W of X and non-empty open subsets $V_0, V_1, \ldots, V_N \subset X$ there exists $m \in \mathbb{N}$ so that*

$$W \cap T_{1,m}^{-1}(V_1) \cap \cdots \cap T_{N,m}^{-1}(V_N) \neq \emptyset,$$
$$V_0 \cap T_{1,m}^{-1}(W) \cap \cdots \cap T_{N,m}^{-1}(W) \neq \emptyset.$$

Then $(T_{1,n})_{n=1}^{\infty}, \ldots, (T_{N,n})_{n=1}^{\infty}$ are d-topologically transitive.

The next definition is adapted from the so-called Hypercyclicity Criterion, due to Kitai [19] and to Gethner and Shapiro [11]; see also [4,5] and [13].

Definition 2.5. Let (n_k) be a strictly increasing sequence of positive integers. We say that $T_1, \ldots, T_N \in L(X)$ satisfy the *d-Hypercyclicity Criterion with respect to* (n_k) provided there exist dense subsets $X_0, X_1, \ldots, X_N$ of X and mappings $S_{l,k}: X_l \to X$ $(1 \leqslant l \leqslant N, k \in \mathbb{N})$ satisfying

$$\begin{aligned} T_l^{n_k} &\underset{k\to\infty}{\longrightarrow} 0 \quad \text{pointwise on } X_0, \\ S_{l,k} &\underset{k\to\infty}{\longrightarrow} 0 \quad \text{pointwise on } X_l, \quad \text{and} \\ \bigl(T_l^{n_k} S_{i,k} - \delta_{i,l}\,\mathrm{Id}_{X_l}\bigr) &\underset{k\to\infty}{\longrightarrow} 0 \quad \text{pointwise on } X_l\ (1 \leqslant i \leqslant N). \end{aligned} \tag{1}$$

In general, we say that $T_1, \ldots, T_N$ satisfy the d-Hypercyclicity Criterion if there exists some sequence (n_k) for which (1) is satisfied.

As happens with the Hypercyclicity Criterion for the case of $N = 1$ operator, we show in Proposition 2.6 below that the d-Hypercyclicity Criterion is closely related to the notion of topological mixing.

Proposition 2.6. *Let $T_1, \dots, T_N$ satisfy the d-Hypercyclicity Criterion with respect to a sequence (n_k). Then the sequences $\{T_1^{n_k}\}_{k=1}^\infty, \dots, \{T_N^{n_k}\}_{k=1}^\infty$ are d-mixing. In particular, $T_1, \dots, T_N$ are d-hypercyclic.*

Proof. Let $V_0, \dots, V_N$ be open and non-empty subsets of X. Pick $y_l \in V_l \cap X_l$ and $\epsilon > 0$ so that $B(y_l, (N+1)\epsilon) \subset V_l$ $(0 \leqslant l \leqslant N)$. By (1), there exists $k_0 \in \mathbb{N}$ so that $T_l^{n_k} y_0$, $S_{l,k} y_l$, and $(T_l^{n_k} S_{i,k} y_i - \delta_{i,l} y_i)$ belong to $B(0, \epsilon)$ for $k \geqslant k_0$ and $1 \leqslant i \leqslant N$. Then for each $k \geqslant k_0$ we have $z_k := y_0 + \sum_{i=1}^N S_{i,k} y_i \in V_0$ and $T_l^{n_k} z \in B(y_l, (N+1)\epsilon) \subset V_l$ $(1 \leqslant l \leqslant N)$. That is, $V_0 \cap T_1^{-n_k}(V_1) \cap \dots \cap T_N^{-n_k}(V_N) \neq \emptyset$ for each $k \geqslant k_0$. □

In [5, Theorem 2.3] we characterized the weakly mixing property in terms of the Hypercyclicity Criterion, and in terms of hereditarily hypercyclicity. A similar result can be established for d-hypercyclicity.

Theorem 2.7. *Let $T_l \in L(X)$ $(1 \leqslant l \leqslant N)$, where $N \geqslant 2$. The following are equivalent*:

(a) *$T_1, \dots, T_N$ satisfy the d-Hypercyclicity Criterion.*
(b) *$T_1, \dots, T_N$ are hereditarily densely d-hypercyclic.*
(c) *For each $r \in \mathbb{N}$, $\overbrace{T_1 \oplus \dots \oplus T_1}^{r}, \dots, \overbrace{T_N \oplus \dots \oplus T_N}^{r}$ are d-topologically transitive operators in $L(X^r)$.*

Proof. (a) $\Rightarrow$ (b). Suppose $T_1, \dots, T_N$ satisfy the d-Hypercyclicity Criterion with respect to a given sequence (n_k). If (n_{k_j}) is any subsequence of (n_k), $T_1, \dots, T_N$ clearly satisfies the d-Hypercyclicity Criterion with respect to it and hence by Proposition 2.6 $(T_1^{n_{k_j}})_{j=1}^\infty, \dots, (T_N^{n_{k_j}})_{j=1}^\infty$ are d-mixing. This gives that $T_1, \dots, T_N$ are hereditarily densely d-hypercyclic with respect to (n_k).

(b) $\Rightarrow$ (c). Suppose $T_1, \dots, T_N$ are hereditarily densely d-hypercyclic with respect to (n_k), say. Let $r \in \mathbb{N}$ be fixed, and for each $l = 0, \dots, N$ and $k = 1, \dots, r$ let $V_{l,k} \subset X$ be open and non-empty. We want to show that there exists $m \in \mathbb{N}$ so that

$$\emptyset \neq V_{0,k} \cap \bigcap_{l=1}^N T_l^{-m}(V_{l,k}) \quad (1 \leqslant k \leqslant r).$$

Since $(T_1^{n_k})_{k=1}^\infty, \dots, (T_N^{n_k})_{k=1}^\infty$ are densely d-hypercyclic, there exists a subsequence $(n_{1,k})$ of (n_k) so that $\emptyset \neq V_{0,1} \cap \bigcap_{l=1}^N T_l^{-n_{1,k}}(V_{l,1})$ $(k \in \mathbb{N})$. Next, since $(T_1^{n_{1,k}})_{k=1}^\infty, \dots, (T_N^{n_{1,k}})_{k=1}^\infty$ are densely d-hypercyclic, there exists a subsequence $(n_{2,k})$ of $(n_{1,k})$ so that $\emptyset \neq V_{0,2} \cap \bigcap_{l=1}^N T_l^{-n_{2,k}}(V_{l,2})$ $(k \in \mathbb{N})$. Proceeding this way, after r steps we obtain a chain of subsequences $(n_{r,k}) \subset \dots \subset (n_{1,k}) \subset (n_k)$ so that

$$\emptyset \neq V_{0,j} \cap \bigcap_{l=1}^N T_l^{-n_{r,k}}(V_{l,j}) \quad (1 \leqslant j \leqslant r)$$

for every $k \in \mathbb{N}$. Pick then $m := n_{r,1}$.

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(c) $\Rightarrow$ (a). By (c), the operators $T_1, \dots, T_N$ satisfy that for each $r \in \mathbb{N}$ and non-empty open subsets $V_{l,k}$ $(0 \leqslant l \leqslant N,\ 1 \leqslant k \leqslant r)$ of X, there exists $m \in \mathbb{N}$ arbitrarily large with

$$V_{0,k} \cap \bigcap_{l=1}^{N} T_l^{-m}(V_{l,k}) \neq \emptyset \quad (1 \leqslant k \leqslant r). \tag{2}$$

Let $\{A_{1,n} \times A_{2,n} \times \cdots \times A_{N,n}\}_{n=1}^{\infty}$ be a basis of non-empty sets for the product topology of X^N, and let $\{A_{0,n}\}_{n=1}^{\infty}$ be a basis for the topology of X.

Also, for each $n \in \mathbb{N}$ and $l = 0, \dots, N$, let $A_{l,n,0} := A_{l,n}$, and $W_n := B(0, \frac{1}{n})$.

Step 1: Get $A_{l,1,1} \subset A_{l,1,0}$ $(1 \leqslant l \leqslant N)$ open, non-empty sets of diameter less than $\frac{1}{2}$ and so that $\overline{A_{l,1,1}} \subset A_{l,1,0}$. By (2), there exists $n_1 > 1$ so that

$$\begin{cases} \emptyset \neq A_{0,1,0} \cap \bigcap_{l=1}^{N} T_l^{-n_1}(W_1), \\ \emptyset \neq W_1 \cap T_l^{-n_1}(A_{l,1,1}) \cap \bigcap_{s \neq l} T_s^{-n_1}(W_1) \quad (1 \leqslant l \leqslant N). \end{cases} \tag{3}$$

Next, get $A_{0,1,1}$ non-empty open subset of $A_{0,1,0}$ of diameter less than $\frac{1}{2}$ with $\overline{A_{0,1,1}} \subset A_{0,1,0}$ and so that $T_l^{n_1}(\overline{A_{0,1,1}}) \subset W_1$ $(1 \leqslant l \leqslant N)$. Also by (3), we may pick $w_{s,1,1} \in W_1$ $(1 \leqslant s \leqslant N)$ so that for each $1 \leqslant l \leqslant N$

$$T_l^{n_1} w_{s,1,1} \in \begin{cases} A_{l,1,0} & \text{if } s = l, \\ W_1 & \text{if } s \neq l. \end{cases}$$

Step 2: For $k = 1, 2$, get $\emptyset \neq A_{l,k,3-k} \subset A_{l,k,2-k}$ open subsets of diameter less than $\frac{1}{3}$ so that $\overline{A_{l,k,3-k}} \subset A_{l,k,2-k}$ and $\overline{A_{l,2,1}} \cap \overline{A_{l,1,2}} = \emptyset$ $(1 \leqslant l \leqslant N)$. By (2), there exists $n_2 > n_1$ so that

$$\begin{cases} \emptyset \neq A_{0,k,2-k} \cap \bigcap_{l=1}^{N} T_l^{-n_2}(W_2), \\ \emptyset \neq W_2 \cap T_l^{-n_2}(A_{l,k,3-k}) \cap \bigcap_{s \neq l} T_s^{-n_2}(W_2) \quad (1 \leqslant l \leqslant N) \end{cases} \quad (k = 1, 2). \tag{4}$$

Next, for each $k = 1, 2$ and $l = 1, \dots, N$, get $w_{l,k,3-k} \in W_2$ and a non-empty open subset $A_{0,k,3-k}$ of $A_{0,k,2-k}$ satisfying that diameter $(A_{0,k,3-k}) < \frac{1}{3}$, $T_l^{n_2}(\overline{A_{0,k,3-k}}) \subset W_2$, and for $1 \leqslant s \leqslant N$

$$T_l^{n_2} w_{s,k,3-k} \in \begin{cases} A_{l,k,3-k} & \text{if } s = l, \\ W_2 & \text{if } s \neq l. \end{cases}$$

Continuing this process inductively by using (2) on each step, we obtain integers $1 < n_1 < n_2 < \cdots$ and, for each $l \in \{1, \dots, N\}$ and each $i \in \mathbb{N}$, non-empty open sets $A_{l,k,i+1-k}$ $(1 \leqslant k \leqslant i)$ of diameter less than $\frac{1}{i+1}$ and $w_{l,k,i+1-k} \in W_l$ satisfying

(i) $\overline{A_{l,k,i+1-k}} \subset A_{l,k,i-k} \subset A_{l,k}$.
(ii) Each collection $\{\overline{A_{l,k,i+1-k}}\colon 1 \leqslant k \leqslant i\}$ is pairwise disjoint.
(iii) $T_l^{n_i}(\overline{A_{0,k,i+1-k}}) \subset W_i$.
(iv) For $1 \leqslant s \leqslant N$

$$T_l^{n_i} w_{s,k,i+1-k} \in \begin{cases} A_{l,k,i+1-k} & \text{if } s = l, \\ W_i & \text{if } s \neq l. \end{cases}$$

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Now, for each fixed $0 \leqslant l \leqslant N$ and $m \in \mathbb{N}$ there exists a unique $a_{l,m} \in X$ so that $\{a_{l,m}\} = \bigcap_{j=m+1}^{\infty} \overline{A_{l,m,j-m}}$. Notice that by (ii) $a_{l,m} \neq a_{l,n}$ whenever $n \neq m$, and that $X_l = \{a_{l,m}\colon m \in \mathbb{N}\}$ is dense in X. So $S_{l,m} : X_l \to X$

$$S_{l,m} a_{l,k} := \begin{cases} w_{l,k,m+1-k} & \text{if } m \geqslant k, \\ 0 & \text{if } 1 \leqslant m < k \end{cases}$$

is well defined, and by (iv) $S_{l,k} \underset{k\to\infty}{\longrightarrow} 0$ pointwise on X_l $(1 \leqslant l \leqslant N)$. Also, by (iv)

$$T_s^{n_m} S_{l,m} a_{l,k} = T_s^{n_m} w_{l,k,m+1-k} \in \begin{cases} A_{l,k,m+1-k} & \text{if } s = l, \\ W_m & \text{if } s \neq l. \end{cases}$$

So $(T_s^{n_k} S_{l,k} - \delta_{s,l} \operatorname{Id}_{X_l}) \underset{k\to\infty}{\longrightarrow} 0$ pointwise on X_l $(1 \leqslant l \leqslant N)$, too. Finally, $T_l^{n_k} \underset{k\to\infty}{\longrightarrow} 0$ pointwise on X_0 $(1 \leqslant l \leqslant N)$, by (iii). So $T_1, \dots, T_N$ satisfies the d-Hypercyclicity Criterion for (n_k). $\square$

Remark 2.8. The same arguments may be used to obtain a d-Universality type version of Theorem 2.7: Let $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ be sequences in $L(X)$. Then the following statements are equivalent:

(a) The sequences $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ satisfy the d-Universality Criterion with respect to some (n_k). Namely, there exist dense subsets $X_0, \dots, X_N$ of X, a strictly increasing sequence of positive integers (n_k), and mappings $S_{l,k} : X_l \to X$ $(1 \leqslant l \leqslant N, k \in \mathbb{N})$ so that for each $1 \leqslant l \leqslant N$ we have
   (i) $T_{l,n_k} \underset{k\to\infty}{\longrightarrow} 0$ pointwise on X_0,
   (ii) $S_{l,k} \underset{k\to\infty}{\longrightarrow} 0$ pointwise on X_l, and
   (iii) $(T_{l,n_k} S_{i,k} - \delta_{l,i} \operatorname{Id}_{X_l}) \underset{k\to\infty}{\longrightarrow} 0$ pointwise on X_l.

(b) There exists a strictly increasing sequence of positive integers (n_k) so that $(T_{1,n_k})_{k=1}^{\infty}, \dots, (T_{N,n_k})_{k=1}^{\infty}$ are hereditarily densely d-universal sequences. Namely, that for each subsequence (n_{k_j}) of (n_k), there exists a dense set of vectors $z \in X$ for which $\{(T_{1,n_{k_j}} z, \dots, T_{N,n_{k_j}} z)\colon j \in \mathbb{N}\}$ is dense in X^N.

(c) For each $r \in \mathbb{N}$, the sequences $(\overbrace{T_{1,j} \oplus \cdots \oplus T_{1,j}}^{r})_{j=1}^{\infty}, \dots, (\overbrace{T_{N,j} \oplus \cdots \oplus T_{N,j}}^{r})_{j=1}^{\infty}$ are d-topologically transitive sequences in $L(X^r)$.

Remark 2.9. We also observe that $T_1, \dots, T_N$ satisfy the d-Hypercyclicity Criterion if and only if the sequence (L_n) in $L(X, X^N)$ given by $L_n(x) = (T_1^n x, \dots, T_N^n x)$ $(x \in X, n \in \mathbb{N})$ satisfies the so-called Universality Criterion [15, Theorem 2]. Indeed, if $T_1, \dots, T_N$ satisfy the d-Hypercyclicity Criterion with respect to (n_k), then the set $Y_0 := X_1 \times \cdots \times X_N$ is dense in X^N, and the mapping $S_k : Y_0 \to X$ given by $S_k(x_1, \dots, x_N) := \sum_{l=1}^{N} S_{l,k} x_l$ satisfies that S_k $(L_{n_k} S_k - I) \to 0$ pointwise on Y_0, while $L_{n_k} \to 0$ pointwise on X_0. On the other hand, the converse follows from Theorem 2.7 and [4, Remark 2.3(d)].

Proposition 2.10. *Suppose $T_1, \dots, T_N \in L(X)$ satisfy the d-Hypercyclicity Criterion. Then they have a d-hypercyclic linear manifold.*

Proof. By Proposition 2.6, there exist integers $1 \leqslant n_1 < n_2 < \cdots$ so that the sequences $(T_1^{n_k})_{k=1}^{\infty}, \dots, (T_N^{n_k})_{k=1}^{\infty}$ are hereditarily densely d-universal. This means that the sequence $(L_k)_{k=1}^{\infty} \subset$

$L(X, X^N)$ given by $L_k x = (T_1^{n_k} x, \dots, T_N^{n_k} x)$ $(x \in X)$, is hereditarily densely universal. By a result by Bernal-González [3, Theorem 1], $(L_k)_{k=1}^{\infty}$ has a universal linear manifold. But the universal vectors for the sequence $(L_k)_{k=1}^{\infty}$ are precisely the d-hypercyclic vectors for $T_1, \dots, T_N$. □

3. Disjoint hypercyclic translation and differentiation operators

Let $H(\mathbb{C})$ be the space of entire functions on the complex plane $\mathbb{C}$, and endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$. For each $a \in \mathbb{C}$, T_a denotes the translation operator on $H(\mathbb{C})$ given by $T_a(f)(z) = f(z + a)$ $(z \in \mathbb{C}, f \in H(\mathbb{C}))$.

Theorem 3.1. *Let $a_1, \dots, a_N$ be pairwise distinct, non-zero complex numbers. Then the translation operators $T_{a_1}, \dots, T_{a_N}$ are d-topologically mixing. In particular, they are d-hypercyclic.*

Proof. Let $h, g_1, \dots, g_N \in H(\mathbb{C})$ and let $\epsilon, r > 0$ be given. Pick an integer

$$n_0 > \max_{i \neq j} \left\{ \frac{2r}{|a_i - a_j|} \right\} + \max_{1 \leqslant i \leqslant N} \left\{ \frac{2r}{|a_i|} \right\}.$$

It suffices to show that for each $n \geqslant n_0$ there exists $f = f_n \in H(\mathbb{C})$ satisfying

$$\begin{cases} \sup_{|z| \leqslant r} \{|f(z) - h(z)|\} < \epsilon, \\ \sup_{|z| \leqslant r} \{|T_{a_i}^n(f)(z) - g_i(z)|\} < \epsilon \quad (1 \leqslant i \leqslant N). \end{cases} \tag{5}$$

Now, let $n \geqslant n_0$ be fixed. Notice that the closed discs $D(0, r)$, $D(na_1, r)$, $\dots$, $D(na_N, r)$ are pairwise disjoint, and that their union has connected complement in the Riemann sphere. Hence by Runge's Approximation Theorem [18, Theorem 16.6.1] there exists $f = f_n \in H(\mathbb{C})$ so that

$$\begin{cases} |f(w) - h(w)| < \epsilon & \text{for each } w \in D(0, r), \\ |f(w) - g_i(w - na_i)| < \epsilon & \text{for each } w \in D(na_i, r) \ (1 \leqslant i \leqslant N). \end{cases} \tag{6}$$

But (6) gives (5). □

Notice that $T_a^{-1} = T_{-a}$ for each $a \in \mathbb{C}$. Hence Theorem 3.1 gives the following.

Corollary 3.2. *There exist invertible operators $T \in L(H(\mathbb{C}))$ so that T and T^{-1} are d-hypercyclic. Indeed, for each $N \in \mathbb{N}$ and $a \neq 0$, the translation operators T_a, $T_{2a}, \dots, T_{Na}$, $T_a^{-1}, \dots, T_{Na}^{-1}$ are d-hypercyclic.*

We also give examples of d-hypercyclic differentiation operators.

Proposition 3.3. *Let $1 \leqslant r_1 < \cdots < r_N$, and $\lambda_1, \dots, \lambda_N \in \mathbb{C} \setminus \{0\}$, where $N \geqslant 2$. Then $\lambda_1 D^{r_1}, \dots, \lambda_N D^{r_N}$ are d-mixing.*

Proof. $T_1 = \lambda_1 D^{r_1}, \dots, T_N := \lambda_N D^{r_N}$ satisfy the d-Hypercyclicity Criterion with respect to the sequence (n), if we let $X_0 = \cdots = X_N = \text{span}\{1, z, z^2, \dots\}$, and consider for each $1 \leqslant l \leqslant N$ and each $n \in \mathbb{N}$ the linear map $S_{l,n} : X_l \to X$ determined by $S_{l,n} z^k := \frac{1}{\lambda_l^n} \frac{z^{k+r_l n}}{(k+1)(k+2)\dots(k+r_l n)}$ for each $k = 0, 1, 2, \dots$. □

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Notice that (modulo the order of the indexes) strict inequality must occur with $r_1 < \cdots < r_N$ in Proposition 3.3, since an operator can never be d-hypercyclic with a scalar multiple of itself. However, the operators D and $I + D$ are d-hypercyclic. More generally, we have:

Proposition 3.4. *Let $\Phi_1, \ldots, \Phi_N \in H(\mathbb{C})$ be entire functions of exponential type, so that the set $U_0 = \{\lambda \in \mathbb{C}\colon \max_{1 \leqslant l \leqslant N}\{|\Phi_l(\lambda)|\} < 1\}$ and each of the sets*

$$U_l = \left\{\lambda \in \mathbb{C}\colon \left|\Phi_l(\lambda)\right| > 1 \text{ and } \max_{j \neq l}\left\{\left|\Phi_j(\lambda)\right|\right\} < \left|\Phi_l(\lambda)\right|\right\} \quad (1 \leqslant l \leqslant N)$$

are non-empty. Then the operators $\Phi_1(D), \ldots, \Phi_N(D)$ are d-mixing.

Proof. We verify that $\Phi_1(D), \ldots, \Phi_N(D)$ satisfy the d-Hypercyclicity Criterion with respect to the full sequence (n). Notice that for each $0 \leqslant j \leqslant N$ the set $X_j = \operatorname{span}\{e^{\lambda z}\colon \lambda \in U_j\}$ is dense in $H(\mathbb{C})$, since the interior of U_j contains an accumulation point [12]. Next, consider for each $m \in \mathbb{N}$ and each $1 \leqslant l \leqslant N$ the linear mappings $S_{l,m} : X_l \to H(\mathbb{C})$ determined by $S_{l,m}e^{\lambda z} := \frac{1}{\Phi_l(\lambda)^m} e^{\lambda z}$ $(\lambda \in U_l)$. The proposition now follows from Theorem 2.7. $\square$

Remark 3.5. If $N = 2$ and $\Phi_1(z) = c_1 z$ and $\Phi_2(z) = c_2 z^2$ for non-zero scalars c_1, c_2, then U_0 and U_2 in the above proposition are always non-empty, but $U_1 \neq \emptyset \Leftrightarrow 0 < |c_2| < |c_1|^2$. So it follows by Proposition 3.3 that the conditions in Proposition 3.4 for $\Phi_1(D), \ldots, \Phi_N(D)$ to satisfy the d-Hypercyclicity Criterion with respect to the full sequence (n) are sufficient, but not necessary.

4. Disjoint hypercyclic weighted shifts

4.1. Unilateral shifts

In what follows, $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ $(1 \leqslant p < \infty)$ over the real or complex scalar field $\mathbb{K}$ and, given a bounded sequence $a = (a_k)_k$ of non-zero weights, let $B_a : X \to X$ be the unilateral weighted shift

$$x = (x_0, x_1, \ldots) \xmapsto{B_a} (a_1 x_1, a_2 x_2, \ldots).$$

Salas [25] characterized the hypercyclicity of B_a in terms of the weight sequence a, and the first examples of hypercyclic shifts were given in 1969 by Rolewicz [23], who showed that if B is the unilateral unweighted backward shift then any scalar multiple λB is hypercyclic whenever $|\lambda| > 1$.

For disjoint hypercyclicity, if we consider $T_1 := \lambda_1 B$, $T_2 := \lambda_2 B^2$, $|\lambda_2| > |\lambda_1| > 1$, we have $(T_1^n x, T_2^n x) = (\lambda_1^n(x_n, x_{n+1}, \ldots, x_{2n}, x_{2n+1}, \ldots), \lambda_2^n(x_{2n}, x_{2n+1}, \ldots))$ for each $x \in X$. The blow-up/collapse condition of Proposition 2.4 is easily verified and the pair T_1, T_2 are d-hypercyclic. We can establish the following characterization for a finite family of (different!) powers of weighted shifts.

Theorem 4.1. *Let $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ $(1 \leqslant p < \infty)$, and let integers $1 \leqslant r_1 < r_2 < \cdots < r_N$ be given. For each $1 \leqslant l \leqslant N$, let $a_l = (a_{l,n})_{n=1}^{\infty}$ be a weight sequence and $B_{a_l} : X \to X$ be the corresponding unilateral backward shift*

$$x = (x_0, x_1, \ldots) \xmapsto{B_{a_l}} (a_{l,1} x_1, a_{l,2} x_2, \ldots).$$

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The following are equivalent:

(a) $B_{a_1}^{r_1}, \dots, B_{a_N}^{r_N}$ *are d-hypercyclic.*
(b) *For each* $\epsilon > 0$ *and* $q \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *satisfying, for each* $0 \leqslant j \leqslant q$,

$$|a_{l,j+1} \dots a_{l,j+r_l m}| > \frac{1}{\epsilon} \quad (1 \leqslant l \leqslant N),$$
$$\frac{|a_{l,j+1} \dots a_{l,j+r_l m}|}{|a_{s,j+(r_l-r_s)m+1} \dots a_{s,j+r_l m}|} > \frac{1}{\epsilon} \quad (1 \leqslant s < l \leqslant N).$$

(c) $B_{a_1}^{r_1}, \dots, B_{a_N}^{r_N}$ *satisfy the d-Hypercyclicity Criterion.*

Proof. (a) $\Rightarrow$ (b). Let $\epsilon > 0$ and $q \in \mathbb{N}$ be given. Pick $0 < \delta < 1$ with $\frac{\delta}{1-\delta} < \epsilon$, and let $x = (x_0, x_1, \dots)$ be a d-hypercyclic vector for $B_{a_1}^{r_1}, \dots, B_{a_N}^{r_N}$. Next, let $m \in \mathbb{N}$ $(m > q)$ so that

$$|x_k| < \delta \quad \text{for } k \geqslant r_1 m, \tag{7}$$
$$\left\| B_{a_l}^{r_l m} x - (e_0 + \dots + e_q) \right\| < \delta \quad (1 \leqslant l \leqslant N). \tag{8}$$

So for $l = 1, \dots, N$ we have

$$1 - \delta < |a_{l,i+1} \dots a_{l,i+r_l m}\, x_{i+r_l m}| < 1 + \delta \quad \text{if } 0 \leqslant i \leqslant q,$$
$$|a_{l,i+1} \dots a_{l,i+r_l m}\, x_{i+r_l m}| < \delta \quad \text{if } i > q. \tag{9}$$

Now, let $0 \leqslant j \leqslant q$ and $1 \leqslant l \leqslant N$ be fixed. By (7) and (9), $|a_{l,j+1} \dots a_{l,j+r_l m}| > \frac{1-\delta}{\delta} > \frac{1}{\epsilon}$. Also, for $1 \leqslant s < l$

$$\frac{|a_{l,j+1} \dots a_{l,j+r_l m}|}{|a_{s,j+(r_l-r_s)m+1} \dots a_{s,j+r_l m}|} > \frac{|a_{l,j+1} \dots a_{l,j+r_l m} x_{j+r_l m}|}{\delta} > \frac{1-\delta}{\delta} > \frac{1}{\epsilon},$$

where the first inequality follows taking $i = j + (r_l - r_s)m$ in (9), and the second inequality follows from (9).

(b) $\Rightarrow$ (c). By (b), there exist integers $1 \leqslant n_1 < n_2 < n_3 < \cdots$ satisfying, for each $q \in \mathbb{N}$ and each $0 \leqslant j \leqslant q$, that

$$|a_{l,j+1} \dots a_{l,j+r_l n_q}| > q \quad (1 \leqslant l \leqslant N), \tag{10}$$
$$\frac{|a_{l,j+1} \dots a_{l,j+r_l n_q}|}{|a_{s,j+(r_l-r_s)m+1} \dots a_{s,j+r_l n_q}|} > q \quad (1 \leqslant s < l \leqslant N). \tag{11}$$

Now, let $X_0 = \operatorname{span}\{e_0, e_1, \dots\}$. Notice that X_0 is dense in X, and that $B_{a_l}^{r_l n_q} \underset{q\to\infty}{\longrightarrow} 0$ pointwise on X_0 $(1 \leqslant l \leqslant N)$. For each $1 \leqslant l \leqslant N$ and $q \in \mathbb{N}$, consider the mapping $S_{l,q} : X_0 \to X$ given by

$$S_{l,q}(x_0, x_1, \dots) = \Bigg(\overbrace{0, \dots, 0}^{r_l n_q}, \frac{x_0}{a_{l,1} a_{l,2} \dots a_{l,r_l n_q}}, \dots, \frac{x_j}{a_{l,j+1}, a_{l,j+2}, \dots a_{l,j+r_l n_q}}, \dots\Bigg).$$

So $B_{a_l}^{r_l n_q} S_{l,q} = \mathrm{Id}_{X_0}$ and $S_{l,q} \underset{q\to\infty}{\longrightarrow} 0$ pointwise on X_0 $(1 \leqslant l \leqslant N)$, by (10). Now, for $1 \leqslant s < l \leqslant N$ we have $B_{a_l}^{r_l n_q} S_{s,q} \underset{q\to\infty}{\longrightarrow} 0$ pointwise on X_0, since $r_s < r_l$. Also, $B_{a_s}^{r_s n_q} S_{l,q} \underset{q\to\infty}{\longrightarrow} 0$ pointwise on X_0, by (11). So $B_{a_1}^{r_1}, \dots, B_{a_N}^{r_N}$ satisfy the d-Hypercyclicity Criterion.

(c) $\Rightarrow$ (a). This is immediate from Proposition 2.6. □

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Corollary 4.2. *Let $N \geqslant 2$, and let $r_l \in \mathbb{N}$, $\lambda_l \in \mathbb{C}$ $(1 \leqslant l \leqslant N)$ with $1 \leqslant r_1 \leqslant r_2 \leqslant \cdots \leqslant r_N$. Then $\lambda_1 B^{r_1}, \dots, \lambda_N B^{r_N}$ are d-hypercyclic if and only if*

$$\begin{cases} 1 \leqslant r_1 < r_2 < \cdots < r_N, \\ 1 < |\lambda_1| < |\lambda_2| < \cdots < |\lambda_N|. \end{cases} \tag{12}$$

Proof. For each $1 \leqslant l \leqslant N$, let λ_l^{1/r_l} denote a fixed root of $z^{r_l} - \lambda_l$, and let B_{a_l} denote the unilateral backward shift with constant weight sequence $a_l = (a_{l,n})_{n=1}^{\infty} = (\lambda_l^{1/r_l})_{n=1}^{\infty}$. So $B_{a_l}^{r_l} = \lambda_l B^{r_l}$. Now, suppose $\lambda_1 B^{r_1}, \dots, \lambda_N B^{r_N}$ $(= B_{a_1}^{r_1}, \dots, B_{a_N}^{r_N})$ are d-hypercyclic, and let $1 \leqslant s < l \leqslant N$ be fixed. Since hypercyclic operators on normed spaces must have norm strictly larger than 1, $|\lambda_s| = \|\lambda_s B^{r_s}\| > 1$. Also, since $\lambda_s B^{r_s}$ and $\lambda_l B^{r_l}$ are d-hypercyclic and no operator can be d-hypercyclic with a scalar multiple of itself, $r_s < r_l$. Finally, by Theorem 4.1 (taking $\epsilon = 1$ in (b2)),

$$\frac{|\lambda_l|^m}{|\lambda_s|^m} = \frac{(|\lambda_l|^{\frac{1}{r_l}})^{r_l m}}{(|\lambda_s|^{\frac{1}{r_s}})^{r_s m}} = \frac{|a_{l,1} a_{l,2} \dots a_{l,r_l m}|}{|a_{s,1+(r_l - r_s)m} \dots a_{s,r_l m}|} > 1$$

for some $m \in \mathbb{N}$. Hence, $|\lambda_s| < |\lambda_l|$ and since $1 \leqslant s < l \leqslant N$ were arbitrary, the inequalities in (12) follow. Conversely, suppose that

$$\begin{cases} 1 \leqslant r_1 < r_2 < \cdots < r_N, \\ 1 < |\lambda_1| < |\lambda_2| < \cdots < |\lambda_N|, \end{cases}$$

and let $\epsilon > 0$ and $q \in \mathbb{N}$ be given. Pick $m \in \mathbb{N}$ large enough so that $|\lambda_1|^m > \frac{1}{\epsilon}$ and $(\frac{|\lambda_l|}{|\lambda_s|})^m > \frac{1}{\epsilon}$ for all $1 \leqslant s < l \leqslant N$. Then $B_{a_1}^{r_1} = \lambda_1 B^{r_1}, \dots, B_{a_N}^{r_N} = \lambda_N B^{r_N}$ satisfy for every $1 \leqslant l \leqslant N$ and $0 \leqslant j \leqslant q$ that

$$|a_{l,j+1} \dots a_{l,j+r_l m}| = \left(|\lambda_l|^{\frac{1}{r_l}}\right)^{r_l m} = |\lambda_l|^m > |\lambda_1|^m > \frac{1}{\epsilon}, \tag{13}$$

$$\frac{|a_{l,j+1} \dots a_{l,j+r_l m}|}{|a_{s,j+(r_l - r_s)m+1} \dots a_{s,j+r_l m}|} = \left|\frac{\lambda_l}{\lambda_s}\right|^m > \frac{1}{\epsilon} \quad \text{if } 1 \leqslant s < l \leqslant N. \tag{14}$$

By Theorem 4.1, the operators $\lambda_1 B^{r_1}, \dots, \lambda_N B^{r_N}$ are d-hypercyclic. □

Notice that the differentiation operator on $H(\mathbb{C})$ may be viewed as a unilateral backward weighted shift. Hence we may contrast Corollary 4.2 with Proposition 3.3, where the condition of the scalars $1 < |\lambda_1| < \cdots < |\lambda_N|$ is not needed.

We need the following proposition, a slight modification of [25, Theorem 2.8] by Salas, for Corollary 4.4.

Proposition 4.3. *Let $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ $(1 \leqslant p \leqslant \infty)$, and $2 \leqslant N \in \mathbb{N}$ be fixed. For each $1 \leqslant l \leqslant N$, let $r_l \in \mathbb{N}$ and let B_{a_l} be a unilateral backward shift on X, with weight sequence $a_l = (a_{l,n})_{n=1}^{\infty}$. The following are equivalent*:

(a) $B_{a_1}^{r_1} \oplus \cdots \oplus B_{a_N}^{r_N}$ *is hypercyclic on* X^N.
(b) $\sup_{m \in \mathbb{N}} \{\min\{|a_{l,1} a_{l,2} \dots a_{l,r_l m}|: 1 \leqslant l \leqslant N\}\} = \infty$.
(c) *For each $\epsilon > 0$ and $q \in \mathbb{N}$, there exists an integer $m > q$ so that for each $0 \leqslant j \leqslant q$,*

$$|a_{l,j+1} a_{l,j+2} \dots a_{l,j+r_l m}| > \frac{1}{\epsilon} \quad (1 \leqslant l \leqslant N).$$

(d) $B_{a_1}^{r_1} \oplus \cdots \oplus B_{a_N}^{r_N}$ *satisfies the Hypercyclicity Criterion on* X^N.

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The following corollary follows from Theorem 4.1 and Proposition 4.3.

Corollary 4.4. *Let $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ $(1 \leqslant p < \infty)$, let $0 = r_0 < 1 \leqslant r_1 < r_2 < \cdots < r_N$ $(N \geqslant 2)$, and let B_w be a unilateral backward shift on X, with weight sequence $w = (w_n)_{n=1}^\infty$. The following are equivalent*:

(a) $B_w^{r_1}, B_w^{r_2}, \ldots, B_w^{r_N}$ *are d-hypercyclic.*
(b) $B_w^{r_1}, B_w^{r_2}, \ldots, B_w^{r_N}$ *satisfy the d-Hypercyclicity Criterion.*
(c) $\sup_{m\in\mathbb{N}}(\min\{\prod_{i=1}^{m(r_l-r_s)} w_i\colon 0 \leqslant s < l \leqslant N\}) = \infty$.
(d) $\bigoplus_{0\leqslant s<l\leqslant N} B_w^{(r_l-r_s)}$ *is hypercyclic on* $X^{\frac{N(N+1)}{2}}$.

In particular, $B_w, B_w^2, \ldots, B_w^N$ are d-hypercyclic if and only if $B_w \oplus B_w^2 \oplus \cdots \oplus B_w^N$ is hypercyclic on X^N.

Example 4.5. Let $\ell_2 = \ell_2(\mathbb{N})$, and let $w = (w_n)_{n=1}^\infty$ be the weight sequence given by

$$w_k = \begin{cases} 2 & \text{if } k \in \bigcup_{n\in\mathbb{N}}\{2^{2n}+1, \ldots, 2^{2n}+n\} \cup \{3(2^{2n}+n)-n+1, \ldots, 3(2^{2n}+n)\}, \\ \frac{1}{2^n} & \text{if } k = 2^{2n}+n+1 \text{ or } 3(2^{2n}+n)+1, \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Then the corresponding unilateral backward shift $B_w : \ell_2 \to \ell_2$ satisfies that $B_w \oplus B_w^3$ is hypercyclic on $\ell_2 \oplus \ell_2$, but the operators B_w and B_w^3 are not d-hypercyclic.

Indeed,

$$\min\left\{\prod_{j=1}^{m} w_j, \prod_{j=1}^{3m} w_j\right\} = 2^n \quad \text{if } m = 2^{2n}+n,\ n \in \mathbb{N},$$

while

$$\min\left\{\prod_{j=1}^{m} w_j, \prod_{j=1}^{2m} w_j\right\} = 1$$

for every $m \in \mathbb{N}$. By Proposition 4.3, $B_w \oplus B_w^3$ is hypercyclic on $\ell_2 \oplus \ell_2$, while $B_w \oplus B_w^3 \oplus B_w^2$ is not hypercyclic on $\ell_2 \oplus \ell_2 \oplus \ell_2$. Thus by Corollary 4.4, B_w, B_w^3 are not d-hypercyclic.

Theorem 4.1 may suggest that weighted backward shifts raised to the same power can never be d-hypercyclic, but this is not the case:

Theorem 4.6. *Let (n_q) be a strictly increasing sequence of positive integers, let $2 \leqslant N \in \mathbb{N}$, and let $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ $(1 \leqslant p < \infty)$. Then there exist unilateral backward weighted shifts $B_{a_1}, \ldots, B_{a_N}$ on X so that the sequences*

$$\left(B_{a_1}^{n_q}\right)_{q=1}^\infty, \quad \ldots, \quad \left(B_{a_N}^{n_q}\right)_{q=1}^\infty$$

are densely d-universal. In particular, $B_{a_1}, \ldots, B_{a_N}$ are d-hypercyclic.

Proof. Let $\{z_{1,n} \oplus \cdots \oplus z_{N,n}\}_{n=0}^\infty$ be dense in X^N, and so that each $z_{l,n} = (z_{l,n,j})_{j=0}^\infty$ satisfies

$$z_{l,n,j} \neq 0 \quad \Leftrightarrow \quad j \in \{0, 1, \ldots, n\}. \tag{15}$$

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By Proposition 2.4, it suffices to find a sequence $(x_n)_{n=0}^{\infty}$ in X, a subsequence $(m_n)_{n=0}^{\infty}$ of $(n_q)_{q=1}^{\infty}$, and weight sequences $a_l = (a_{l,k})_{k=1}^{\infty}$ $(1 \leqslant l \leqslant N)$ satisfying

$$\begin{aligned} &\text{(i)} \quad 0 < |a_{l,k}| \leqslant 2, \\ &\text{(ii)} \quad \|x_n\| < \frac{1}{n+1}, \quad \text{and} \\ &\text{(iii)} \quad B_{a_l}^{m_n} x_n = z_{l,n} \end{aligned} \tag{16}$$

for all $1 \leqslant l \leqslant N$, all $k \in \mathbb{N}$, and all $n \in \mathbb{N} \cup \{0\}$. We will repeatedly use the following easy fact: Given non-zero scalars $0 \neq w_l \in \mathbb{K}$ $(1 \leqslant l \leqslant N)$, there exist unique non-zero scalars $0 \neq y_l \in \mathbb{K}$ $(1 \leqslant l \leqslant N)$ satisfying

$$\begin{cases} 0 < y_1, \\ \max\{|y_1|, |y_2|, \dots, |y_N|\} = 1, \quad \text{and} \\ \frac{w_1}{y_1} = \frac{w_2}{y_2} = \dots = \frac{w_N}{y_N}. \end{cases} \tag{17}$$

Now, let $m_0 \in \{n_q \colon q \in \mathbb{N}\}$ be large enough so that

$$2^{m_0 - 1} > \max\bigl\{|z_{l,0,0}| \colon 1 \leqslant l \leqslant N\bigr\}. \tag{18}$$

For each $(l,k) \in [1,N] \times [1,m_0)$, set $a_{l,k} := 2$. Also, let

$$(a_{1,m_0}, a_{2,m_0}, \dots, a_{N,m_0}) = (y_1, y_2, \dots, y_N)$$

be the solution of (17) corresponding to the weights

$$w_l := \frac{z_{l,0,0}}{a_{l,1} a_{l,2} \dots a_{l,m_0-1}} \quad (1 \leqslant l \leqslant N).$$

So $x_0 = (x_{0,j})_{j=1}^{\infty} \in X$ given by

$$x_{0,j} = \begin{cases} \frac{z_{l,0,0}}{a_{l,1} a_{l,2} \dots a_{l,m_0}} & \text{if } j = m_0, \\ 0 & \text{if } j \neq m_0, \end{cases}$$

satisfies (regardless on how we later define the weights $a_{l,j}$ for $j > m_0$) that

$$B_{a_l}^{m_0} x_0 = z_{l,0} \quad (1 \leqslant l \leqslant N).$$

Also, by (18) and the fact that $\max\{|a_{1,m_0}|, \dots, |a_{N,m_0}|\} = 1$, it follows that $\|x_0\| = |x_{0,m_0}| < 1$. Inductively, suppose we have selected, for each $k = 0, \dots, n$, an integer $m_k \in \{n_q \colon q \in \mathbb{N}\}$, a vector $x_k \in X$, and weight scalars $a_{l,j} \in \mathbb{K}$ for $(l,j) \in [1,N] \times [1, m_n + n]$ so that

$$\begin{aligned} &0 < |a_{l,j}| \leqslant 2 \quad \text{for } (l,j) \in [1,N] \times [1, m_n + n], \\ &\|x_k\| < \frac{1}{k+1} \quad \text{for } k \in [0,n], \\ &B_{a_l}^{m_k} x_k = z_{l,k} \quad \text{for } (l,k) \in [1,N] \times [0,n]. \end{aligned} \tag{19}$$

After $(n+2)$ successive applications of (17), we may obtain N-tuples $(y_{1,j}, y_{2,j}, \dots, y_{N,j}) \in \mathbb{K}^N$ $(0 \leqslant j \leqslant n+1)$ so that for each $0 \leqslant j \leqslant n+1$,

$$\begin{cases} 0 < \min\{|y_{l,j}| \colon 1 \leqslant l \leqslant N\} \leqslant \max\{|y_{l,j}| \colon 1 \leqslant l \leqslant N\} = 1 \quad \text{and} \\ z_{N,n+1,j} \bigl(\prod_{i=j+1}^{m_n+n} a_{N,i}\bigr)^{-1} \bigl(\prod_{k=0}^{j} y_{N,k}\bigr)^{-1} = \cdots \\ \quad = z_{1,n+1,j} \bigl(\prod_{i=j+1}^{m_n+n} a_{1,i}\bigr)^{-1} \bigl(\prod_{k=0}^{j} y_{1,k}\bigr)^{-1}. \end{cases} \tag{20}$$

Next, let $m_n + n < m_{n+1} \in \{n_q\}_{q\in\mathbb{N}}$ be large enough so that

$$\frac{2^{m_{n+1}-1-(m_n+n)}}{(n+2)(n+1)} > \max\left\{\left|\frac{z_{N,n+1,j}}{(\prod_{i=j+1}^{m_n+n} a_{N,i})(\prod_{k=0}^{j} y_{N,k})}\right|: 0 \leqslant j \leqslant n+1\right\}. \tag{21}$$

Also, define

$$a_{l,k} := \begin{cases} 2 & \text{if } (l,k) \in [1,N] \times (m_n + n, m_{n+1}), \\ y_{l,j} & \text{if } k = m_{n+1} + j \ (0 \leqslant j \leqslant n+1). \end{cases} \tag{22}$$

Finally, let $x_{n+1} := (x_{n+1,k})_{k=0}^{\infty}$ be given by

$$x_{n+1,k} := \begin{cases} \frac{z_{N,n+1,j}}{a_{N,j+1}a_{N,j+2}\cdots a_{N,m_{n+1}+j}} & \text{if } k = m_{n+1} + j \text{ and } 0 \leqslant j \leqslant n+1, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

By (20) and (22), $B_{a_l}^{m_{n+1}} x_{n+1} = z_{l,n+1}$ $(1 \leqslant l \leqslant N)$. Also, by (20)–(23),

$$\|x_{n+1}\| \leqslant \sum_{j=0}^{n+1} |x_{n+1,m_{n+1}+j}| < \sum_{j=0}^{n+1} \frac{1}{(n+2)(n+1)} = \frac{1}{n+1}.$$

Finally, by (22) and (20), $0 < |a_{l,j}| \leqslant 2$ for $(l,j) \in [1,N] \times [1, m_{n+1} + n + 1]$. The proof of Theorem 4.6 is now complete. □

4.2. Bilateral shifts

Theorem 4.7. *Let $X = c_0(\mathbb{Z})$ or $\ell_p(\mathbb{Z})$ $(1 \leqslant p < \infty)$. For $l = 1, \ldots, N$, let $a_l = (a_{l,j})_{j\in\mathbb{Z}}$ be a bounded bilateral sequence of non-zero scalars, and let B_{a_l} be the associated backward shift on X given by $B_{a_l} e_k = a_{l,k} e_{k-1}$ $(k \in \mathbb{Z})$. For any integers $1 \leqslant r_1 < r_2 < \cdots < r_N$, the following are equivalent*:

(a) $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \ldots, B_{a_N}^{r_N}$ *have a dense set of d-hypercyclic vectors.*
(b) *For each $\epsilon > 0$ and $q \in \mathbb{N}$, there exists $m \in \mathbb{N}$ so that for $|j| \leqslant q$ we have*:

$$\textit{If } 1 \leqslant l \leqslant N, \quad \begin{cases} |\prod_{i=j+1}^{j+r_l m} a_{l,i}| > \frac{1}{\epsilon}, \\ |\prod_{i=j-r_l m+1}^{j} a_{l,i}| < \epsilon. \end{cases} \tag{24}$$

$$\textit{If } 1 \leqslant s < l \leqslant N, \quad \begin{cases} |\prod_{i=j+1}^{j+r_l m} a_{l,i}| > \frac{1}{\epsilon} |\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{s,i}|, \\ |\prod_{i=j-(r_l-r_s)m+1}^{j+r_s m} a_{l,i}| < \epsilon |\prod_{i=j+1}^{j+r_s m} a_{s.i}|. \end{cases} \tag{25}$$

(c) $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \ldots, B_{a_N}^{r_N}$ *satisfy the d-Hypercyclicity Criterion.*

Proof. (a) ⇒ (b). Pick $0 < \delta < 1$ so that $\frac{\delta}{1-\delta} < \epsilon$. Let $x = (x_j)_{j\in\mathbb{Z}}$ be a d-hypercyclic vector with

$$\left\| x - \sum_{|j|\leqslant q} e_j \right\| < \delta. \tag{26}$$

Let $m \in \mathbb{N}$ $(m > 2q)$ so that

$$\begin{aligned}&|x_k| < \delta \quad \text{for } |k| \geqslant r_1 m,\\ &\left\| B_{a_l}^{r_l m} x - \sum_{|j| \leqslant q} e_j \right\| < \delta.\end{aligned} \tag{27}$$

By (26),

$$\begin{cases} |x_j - 1| < \delta & \text{if } |j| \leqslant q, \\ |x_i| < \delta & \text{if } |i| > q. \end{cases} \tag{28}$$

By (27),

$$\begin{aligned}&\left| \left(\prod_{i=j+1}^{j+mr_l} a_{l,i} \right) x_{j+mr_l} - 1 \right| < \delta \quad \text{if } |j| \leqslant q,\\ &\left| \left(\prod_{i=k+1}^{k+mr_l} a_{l,i} \right) x_{k+mr_l} \right| < \delta \quad \text{if } |k| > q.\end{aligned} \tag{29}$$

Now, let $|j| \leqslant q$ be fixed. By (29)(i) and (28)(ii),

$$\frac{1}{\epsilon} < \frac{1-\delta}{\delta} < \left| \prod_{i=j+1}^{j+mr_l} a_{l,i} \right|.$$

Also (since $|j - mr_l| > q$), by (29)(ii) and (28)(i) we have

$$\left| \prod_{i=j-mr_l+1}^{j} a_{l,i} \right| < \frac{\delta}{1-\delta} < \epsilon.$$

Also, if $1 \leqslant s < l \leqslant N$

$$\frac{|\prod_{i=j+1}^{j+mr_l} a_{l,i}|}{|\prod_{i=j+(r_l-r_s)m+1}^{j+mr_l} a_{s,i}|} > \frac{|(\prod_{i=j+1}^{j+mr_l} a_{l,i}) x_{j+mr_l}|}{\delta} > \frac{1-\delta}{\delta} > \frac{1}{\epsilon},$$

where we could use (29)(ii) in the first inequality because $j + (r_l - r_s)m > q$. Similarly, if $1 \leqslant s < l \leqslant N$ we have, for each $|j| \leqslant q$,

$$\begin{aligned}\frac{|\prod_{i=j+(r_s-r_l)m+1}^{j+r_s m} a_{l,i}|}{|\prod_{i=j+1}^{j+r_s m} a_{s,i}|} &= \frac{|(\prod_{i=j+(r_s-r_l)m+1}^{j+r_s m} a_{l,i}) x_{j+r_s m}|}{|(\prod_{i=j+1}^{j+r_s m} a_{s,i}) x_{j+r_s m}|} < \frac{\delta}{|(\prod_{i=j+1}^{j+r_s m} a_{s,i}) x_{j+r_s m}|}\\ &< \frac{\delta}{1-\delta} < \epsilon,\end{aligned}$$

by (29)(ii) and (29)(i).

(b) $\Rightarrow$ (c). By (b), we may get integers $1 < n_1 < n_2 < n_3 < \cdots$ so that for $|j| \leqslant q$ we have: For each $1 \leqslant l \leqslant N$,

$$\begin{cases} |\prod_{i=j+1}^{j+r_l n_q} a_{l,i}| > q, \\ |\prod_{i=j-r_l n_q+1}^{j} a_{l,i}| < \frac{1}{q}, \end{cases} \tag{30}$$

and for $1 \leqslant s < l \leqslant N$,

$$\begin{cases} |\prod_{i=j+1}^{j+r_l n_q} a_{l,i}| > q|\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}|, \\ |\prod_{i=j-(r_l-r_s)n_q+1}^{j+r_s n_q} a_{l,i}| < \frac{1}{q}|\prod_{i=j+1}^{j+r_s n_q} a_{s.i}|. \end{cases} \tag{31}$$

Now, let $X_0 := \operatorname{span}\{e_k\colon k \in \mathbb{Z}\}$. By (30), $(B_{a_l}^{r_l})^{n_q} \underset{q\to\infty}{\longrightarrow} 0$ pointwise on X_0. Also, for each $1 \leqslant l \leqslant N$ and $q \in \mathbb{N}$ let $S_{l,q} : X_0 \to X$ be the linear map given by

$$S_{l,q} e_k = \frac{1}{a_{l,k+1}a_{l,k+2}\dots a_{l,k+r_l n_q}} e_{k+r_l n_q} \quad (k \in \mathbb{Z}).$$

By (30), $S_{l,q} \underset{q\to\infty}{\longrightarrow} 0$ pointwise on X_0. Also, $(B_{a_l}^{r_l})^{n_q} S_{l,q} = \mathrm{Id}_{X_0}$. Next, notice that for $1 \leqslant s < l \leqslant N$,

$$\left(B_{a_l}^{r_l}\right)^{n_q} S_{s,q} e_k = \frac{\prod_{i=k+(r_s-r_l)n_q+1}^{k+(r_s-r_l)n_q+1} a_{l,i}}{\prod_{i=k+1}^{k+r_s n_q} a_{s,i}} e_{k+r_s n_q - r_l n_q} \quad (k \in \mathbb{Z}).$$

So $(B_{a_l}^{r_l})^{n_q} S_{s,q} \underset{q\to\infty}{\longrightarrow} 0$, by (31)(ii). Finally, for $1 \leqslant s < l \leqslant N$,

$$\left(B_{a_s}^{r_s}\right)^{n_q} S_{l,q} e_k = \frac{\prod_{i=k+(r_l-r_s)n_q+1}^{k+r_l n_q} a_{s,i}}{\prod_{i=k+1}^{k+r_l n_q}} e_{k+(r_l-r_s)n_q} \quad (k \in \mathbb{Z}),$$

and so by (31)(i) $(B_{a_s}^{r_s})^{n_q} S_{l,q} \underset{q\to\infty}{\longrightarrow} 0$ pointwise on X_0. That is, $B_{a_1}^{r_1}, \dots, B_{a_N}^{r_N}$ satisfy the d-Hypercyclicity Criterion.

(c) $\Rightarrow$ (a) is immediate from Theorem 2.7. □

The following proposition is a slight modification of [25, Theorem 2.5] by Salas and of [9, Theorem 4.2] by Feldman.

Proposition 4.8. *Let $2 \leqslant N \in \mathbb{N}$, and $r_1, \dots, r_N \in \mathbb{N}$. For each $1 \leqslant l \leqslant N$, let B_{a_l} be a bilateral backward shift on $X = c_0(\mathbb{Z})$ or $\ell_p(\mathbb{Z})$ $(1 \leqslant p < \infty)$, with weight sequence $a_l = (a_{l,j})_{j\in\mathbb{Z}}$. The following are equivalent*:

(a) $B_{a_1}^{r_1} \oplus \dots \oplus B_{a_N}^{r_N}$ *is hypercyclic on* X^N.

(b) *For each $\epsilon > 0$ and $q \in \mathbb{N}$, there exists $n = n(\epsilon, q) \in \mathbb{N}$ so that for $|j| \leqslant q$ and $(1 \leqslant l \leqslant N)$ we have*

$$\begin{cases} |\prod_{i=j+1}^{j+r_l n} a_{l,i}| > \frac{1}{\epsilon}, \\ |\prod_{i=j+1-r_l n}^{j} a_{l,i}| < \epsilon. \end{cases}$$

(c) $B_{a_1}^{r_1} \oplus \dots \oplus B_{a_N}^{r_N}$ *satisfies the Hypercyclicity Criterion on* X^N.

Moreover, if in addition each B_{a_l} $(1 \leqslant l \leqslant N)$ is invertible, any of conditions (a)–(c) *is equivalent to*

(d) *There exists a strictly increasing sequence of positive integers (n_k) satisfying*

$$\begin{cases} \lim_{s\to\infty}|\prod_{i=1}^{r_l n_s} a_{l,i}| = \infty, \\ \lim_{s\to\infty}|\prod_{i=1}^{r_l n_s} a_{l,-i}| = 0 \end{cases} \quad (1 \leqslant l \leqslant N).$$

Corollary 4.9. *Let $X = c_0(\mathbb{Z})$ or $\ell_p(\mathbb{Z})$ $(1 \leqslant p < \infty)$, let $r_0 = 0 < 1 \leqslant r_1 < r_2 < \cdots < r_N$ $(N \geqslant 2)$, and let $B_w : X \to X$ be a bilateral backward shift with weight sequence $w = (w_j)_{j\in\mathbb{Z}}$. The following are equivalent*:

(a) $B_w^{r_1}, \ldots, B_w^{r_N}$ *satisfy the d-Hypercyclicity Criterion.*
(b) $B_w^{r_1}, \ldots, B_w^{r_N}$ *are densely d-hypercyclic.*
(c) $\bigoplus_{0\leqslant s<l\leqslant N} B_w^{(r_l - r_s)}$ *is hypercyclic on* $X^{\frac{N(N+1)}{2}}$.

In particular, $B_w, B_w^2, \ldots, B_w^N$ are densely d-hypercyclic if and only if $B_w \oplus B_w^2 \oplus \cdots \oplus B_w^N$ is hypercyclic on X^N.

Remark 4.10. All hypercyclicity results presented here for bilateral backward shifts have their corresponding counterparts for bilateral forward shifts hold. In particular, if $Ae_k = w_k e_{k+1}$ $(k \in \mathbb{Z})$ is an invertible bilateral forward shift on $\ell_2(\mathbb{Z})$ and $2 \leqslant N \in \mathbb{N}$, the following are equivalent:

(a) $A, A^2, \ldots, A^N$ are densely d-hypercyclic.
(b) $A \oplus A^2 \oplus \cdots \oplus A^N$ is hypercyclic on X^N.
(c) There exists a strictly increasing sequence of positive integers (n_s) so that

$$\lim_{s\to\infty} \prod_{i=1}^{ln_s} w_i = \lim_{s\to\infty} \prod_{i=1}^{ln_s} \frac{1}{w_{-i}} = 0 \quad (1 \leqslant l \leqslant N).$$

In [24], Salas constructed a hypercyclic operator T on a Hilbert space so that it is Hilbert Adjoint T^* is also hypercyclic. Notice that it is not possible that T and T^* be d-hypercyclic. Namely, if x is a d-hypercyclic vector for T and T^*, there must exist integers $1 < n_1 < n_2 < \cdots$ so that $(T^{n_k}x, T^{*n_k}x) \to (x, 0)$, what forces $\|x\|^2 = \lim_{k\to\infty}\langle T^{n_k}x, x\rangle = \lim_{k\to\infty}\langle x, T^{*n_k}x\rangle = 0$, a contradiction. That is, no operator can be d-hypercyclic with its Hilbert Adjoint. However, we have the following.

Theorem 4.11. *Let $2 \leqslant N \in \mathbb{N}$ be given. There exist d-hypercyclic operators $T_1, \ldots, T_N$ on $\ell_2(\mathbb{Z})$ so that their Hilbert Adjoints $T_1^*, \ldots, T_N^*$ are also d-hypercyclic.*

Proof. Without loss of generality, we may assume $N \geqslant 10$. Let $Ae_j = w_j e_{j+1}$ $(j \in \mathbb{Z})$ be the forward bilateral shift with weight sequence $(w_j)_{j\in\mathbb{Z}}$ defined as follows.

For $0 \leqslant j \leqslant N$, let $w_j := 1$. Also, for each $s \in \mathbb{N}$ and $N^{2s-1} \leqslant j \leqslant N^{2s+1}$, define

$$w_j := \begin{cases} \frac{1}{5} & \text{if } j \in \{N^{2s-1}+1, \ldots, N^{2s-1}+s\} \cup \{N^{2s+1}-s, \ldots, N^{2s+1}-1\}, \\ 5 & \text{if } 1 \leqslant |j - 5N^{2s}| \leqslant s, \\ 1 & \text{otherwise.} \end{cases} \tag{32}$$

Finally, for $j \in \mathbb{Z} \setminus \mathbb{N}$ let $w_j := \frac{1}{w_{-j}}$. Notice that A is invertible, since $\frac{1}{5} \leqslant |w_j| \leqslant 5$ $(j \in \mathbb{Z})$. Also, by (32) we have

$$\lim_{s\to\infty} \prod_{i=1}^{ln_s} w_i = \lim_{s\to\infty} \prod_{i=1}^{ln_s} \frac{1}{w_{-i}} = \lim_{s\to\infty} \frac{1}{5^s} = 0 \quad (1 \leqslant l \leqslant N),$$

where $(n_s) = (N^{2s-1} + s)$. It follows from Remark 4.10 that $A, A^2, \dots, A^N$ are d-hypercyclic. So it suffices to verify that $A^*, A^{*2}, \dots, A^{*N}$ are d-hypercyclic. Now, notice that A^* is the invertible backward shift $A^* e_k = a_k e_{k-1}$ $(k \in \mathbb{Z})$, where $a_k = w_{k-1}$ $(k \in \mathbb{Z})$. Also, from (32) it follows that

$$\begin{cases} \prod_{i=1}^{lm_s} a_i = \prod_{i=0}^{lm_s-1} w_i = 5^s \underset{s\to\infty}{\longrightarrow} \infty, \\ \prod_{i=1}^{lm_s} a_{-i} = \prod_{i=0}^{lm_s-1} w_{-i} = 5^{-s} \underset{s\to\infty}{\longrightarrow} 0 \end{cases}$$

for every $(1 \leqslant l \leqslant N)$, where $(m_s) = (5N^{2s} + s + 1)$. Hence, by Corollary 4.9 and Proposition 4.8(d), $A^*, A^{*2}, \dots, A^{*N}$ are d-hypercyclic. □

5. Final comments and problems

In the previous sections we gave properties of d-hypercyclicity that are satisfied in general, and some others that we verified within some particular classes of operators. For instance, if T is a hypercyclic invertible bilateral shift on $\ell_p(\mathbb{Z})$, or if T is a translation on $H(\mathbb{C})$, $T^{-1}, T^{-2}, \dots, T^{-N}$ are d-hypercyclic whenever $T, T^2, \dots, T^N$ are. Also, if $a \neq 0$, then the translations $T_a, T_a^2, \dots, T_a^N, T_a^{-1}, \dots, T_a^{-N}$ on $H(\mathbb{C})$ (cf. Section 3) are d-hypercyclic. Hence it is natural to ask:

Problem 1. Let $T_1, \dots, T_N$ $(N \geqslant 2)$ be d-hypercyclic and invertible. Must $T_1^{-1}, \dots, T_N^{-1}$ be d-hypercyclic? If so, must $T_1, \dots, T_N, T_1^{-1}, \dots, T_N^{-1}$ be d-hypercyclic?

For unimodular multiples of d-hypercyclic operators we may also ask:

Problem 2. Let $T_1, \dots, T_N$ be d-hypercyclic, and let $\lambda_1, \dots, \lambda_N$ be scalars of modulus 1, where $N \geqslant 2$. Must $\lambda_1 T_1, \dots, \lambda_N T_N$ be d-hypercyclic? Must $T_1, \dots, T_N$ and $\lambda_1 T_1, \dots, \lambda_N T_N$ have the same d-hypercyclic vectors?

Notice that the answer is affirmative whenever $T_1, \dots, T_N$ are unilateral weighted shifts on $\ell_2(\mathbb{N})$. It is also affirmative whenever $\lambda_1 = \cdots = \lambda_N$, by a result of León-Saavedra and Müller [20]. It is also affirmative if $\lambda_1, \dots, \lambda_N$ are all roots of unity; this follows because $A_1, \dots, A_N$ and $A_1^r, \dots, A_N^r$ must have the same d-hypercyclic vectors, as $A_1 \oplus \cdots \oplus A_N$ and $(A_1 \oplus \cdots \oplus A_N)^r$ have the same hypercyclic vectors by a result of Ansari [1].

Finally, we do not know about the existence of a pair (or more) d-hypercyclic operators on any separable infinite dimensional Banach space.

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References

[1] S.I. Ansari, Hypercyclic and cyclic vectors, J. Funct. Anal. 128 (1995) 374–383.

[2] L. Bernal-González, Disjoint hypercyclic operators, Studia Math., in press.

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[3] L. Bernal-González, M.C. Calderón-Moreno, Densely hereditarily hypercyclic sequences and large hypercyclic manifolds, Proc. Amer. Math. Soc. 127 (1999) 3279–3285.
[4] L. Bernal-González, K.G. Grosse-Erdmann, The Hypercyclicity Criterion for sequences of operators, Studia Math. 157 (2003) 17–32.
[5] J. Bès, A. Peris, Hereditarily hypercyclic operators, J. Funct. Anal. 167 (1999) 92–112.
[6] G.D. Birkhoff, Démonstration d'un théoreme élémentaire sur les fonctions entières, C. R. Math. Acad. Sci. Paris 189 (1929) 473–475.
[7] P. Bourdon, N.S. Feldman, Somewhere dense orbits are everywhere dense, Indiana Univ. Math. J. 52 (3) (2003) 811–819.
[8] G. Costakis, On a conjecture of D. Herrero concerning hypercyclic operators, C. R. Math. Acad. Sci. Paris Ser. I 330 (2000) 179–182.
[9] N. Feldman, Hypercyclicity and supercyclicity for invertible bilateral weighted shifts, Proc. Amer. Math. Soc. 131 (2) (2002) 479–485.
[10] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation, Math. Systems Theory 1 (1967) 1–49.
[11] R.M. Gethner, J.H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, Proc. Amer. Math. Soc. 100 (2) (1987) 281–288.
[12] G. Godefroy, J.H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal. 98 (1991) 229–269.
[13] S. Grivaux, Hypercyclic operators, mixing operators, and the Bounded Steps Problem, J. Operator Theory 54 (2005) 147–168.
[14] K.G. Grosse-Erdmann, Holomorphe Monster und universelle Funktionen, Mitt. Math. Sem. Giessen 176 (1987).
[15] K.G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. 36 (1999) 345–381.
[16] K.G. Grosse-Erdmann, Recent developments on hypercyclicity, Rev. R. Acad. Cienc. Ser. A Mat. 97 (2003) 273–286.
[17] D.A. Herrero, Hypercyclic Operators and Chaos, J. Operator Theory 28 (1992) 93–103.
[18] E. Hille, Analytic Function Theory II, Ginn and Company, Boston, 1962.
[19] C. Kitai, Invariant Closed Sets for Linear Operators, PhD thesis, Univ. of Toronto, 1982.
[20] F. León-Saavedra, V. Müller, Rotations of hypercyclic operators, Integral Equations Operator Theory 50 (3) (2004) 385–391.
[21] G.R. MacLane, Sequences of derivatives and normal families, J. Anal. Math. 2 (1952) 72–87.
[22] A. Peris, Multi-hypercyclic operators are hypercyclic, Math. Z. 236 (2001) 779–786.
[23] S. Rolewicz, On orbits of elements, Studia Math. 32 (1969) 17–22.
[24] H. Salas, A hypercyclic operator whose adjoint is also hypercyclic, Proc. Amer. Math. Soc. 112 (1991) 765–770.
[25] H. Salas, Hypercyclic weighted shifts, Trans. Amer. Math. Soc. 347 (3) (1995) 993–1004.

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