

On eigenvalues and eigenvectors of subdirect sums

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Abstract: Some new properties of the eigenvalues of the subdirect sums are presented for the particular case of 1-subdirect sums. In particular, it is shown that if an eigenvalue λ is associated with certain blocks of matrix A or matrix B then λ is also an eigenvalue associated with the 1-subdirect sum $A \oplus_1 B$. Some results concerning eigenvectors of the k -subdirect sum $A \oplus_k B$ for an arbitrary positive integer k are also given.

Key-Words: Matrices, subdirect sum, eigenvalues, eigenvectors, overlapping blocks, domain decomposition.

1 Introduction

The concept of k -subdirect sum of matrices, introduced in [4], appears naturally in several contexts related to square matrices: e.g., matrix completion problems, analysis of matrix classes, overlapping subdomains in domain decomposition methods, global stiffness matrix in FEM, etc; see [4], [2], [1] and the references therein.

The characterization of the invertibility of the subdirect sum of two nonsingular matrices have been studied recently [1]. In the particular case of a 1-subdirect sum it is known that the 1-subdirect sum of two singular matrices is a singular matrix [4]. We present some new properties of the eigenvalues of 1-subdirect sums, which has relevance in some instances; see [4] and the references therein for details. We also give some results concerning eigenvectors of the subdirect sums.

2 Subdirect sums

Let A and B be two square matrices of order n_1 and n_2 , respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned into 2×2 blocks as follows,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad (1)$$

where A_{22} and B_{11} are square matrices of order k . Following [4], we call the following square matrix of

order $n = n_1 + n_2 - k$,

$$C = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \quad (2)$$

the k -subdirect sum of A and B and denote it by $C = A \oplus_k B$.

2.1 Eigenvalues of 1-subdirect sums

The particular case of $k = 1$, i.e., a 1-subdirect sum, is important in certain applications (see [4] and the references therein). We denote as $\sigma(A)$ the spectrum (i.e., the set of eigenvalues) of a matrix A . Some first properties of the eigenvalues of 1-subdirect sums can be summarized as follows.

Theorem 1 *Let A and B be matrices of order n_1 and n_2 , respectively, partitioned as in (1) and let $k = 1$. Let $C = A \oplus_1 B$. Then any of the following statements*

- i) $\lambda \in \sigma(A) \cap \sigma(A_{11})$
- ii) $\lambda \in \sigma(A_{11}) \cap \sigma(B_{22})$
- iii) $\lambda \in \sigma(B) \cap \sigma(B_{22})$

implies that $\lambda \in \sigma(C)$.

Proof. Let us denote $a_{22} = A_{22}$ and $b_{11} = B_{11}$ to display that this quantities are matrices of order 1×1 . A direct computation shows that

$$\begin{aligned} \det(C - \lambda I) &= \\ &= \begin{vmatrix} A_{11} - \lambda I & A_{12} & O \\ A_{21} & a_{22} + b_{11} - \lambda & B_{12} \\ O & B_{21} & B_{22} - \lambda I \end{vmatrix} \end{aligned} \quad (3)$$

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$(A_{21}, a_{22} - \lambda, 0)$ and $(0, b_{11}, B_{12})$ we have

$$\det(C - \lambda I) = \det \begin{bmatrix} A_{11} - \lambda I & A_{12} & O \\ A_{21} & a_{22} - \lambda & 0 \\ O & B_{21} & B_{22} - \lambda I \end{bmatrix} + \det \begin{bmatrix} A_{11} - \lambda I & A_{12} & O \\ 0 & b_{11} & B_{12} \\ O & B_{21} & B_{22} - \lambda I \end{bmatrix},$$

which leads to

$$\det(C - \lambda I) = \det(A - \lambda I)\det(B_{22} - \lambda I) + \det(A_{11} - \lambda I)\det \begin{bmatrix} b_{11} & B_{12} \\ B_{21} & B_{22} - \lambda I \end{bmatrix}, \quad (4)$$

and from this expression we have that each of the statements (i) and (ii) implies $\lambda \in \sigma(C)$. To prove (iii) the technique is the same but now we expand the n_1 th-row of equation (3) as a sum of the rows $(A_{21}, a_{22}, 0)$ and $(0, b_{11} - \lambda, B_{12})$ to finally get

$$\det(C - \lambda I) = \det \begin{bmatrix} A_{11} - \lambda I & A_{12} \\ A_{21} & a_{22} \end{bmatrix} \det(B_{22} - \lambda I) + \det(A_{11} - \lambda I)\det(B - \lambda I), \quad (5)$$

from which we conclude that statement (iii) implies $\lambda \in \sigma(C)$. \square

Example 2 Given the matrices

$$A_{11} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}, \quad A = \left[\begin{array}{ccc|c} & & & 5 \\ & A_{11} & & 4 \\ & & & 3 \\ \hline 1 & 0 & 1 & 5 \end{array} \right]$$

$$B = \left[\begin{array}{c|ccc} 3 & 1 & 0 & 1 \\ \hline 2 & 6 & 3 & 0 \\ 1 & -2 & 5 & -1 \\ 1 & -1 & 1 & 4 \end{array} \right]$$

the spectra of A and A_{11} are

$$\sigma(A) = \{-2, 1, 1, 7\}, \quad \sigma(A_{11}) = \{-2, 1, 3\}$$

and we obtain that the 1-subdirect sum

$$C = A \oplus_1 B = \left[\begin{array}{ccc|c|ccc} 1 & -1 & 4 & 5 & 0 & 0 & 0 \\ 3 & 2 & -1 & 4 & 0 & 0 & 0 \\ 2 & 1 & -1 & 3 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 8 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 6 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 5 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 4 \end{array} \right]$$

has the eigenvalues

$$\sigma(C) \approx \{9.8, -2, 1, 1.7, 4.9 \pm 2.4i, 4.7\}$$

and according to theorem 1, since the eigenvalues -2 and 1 are common to A and A_{11} they are also eigenvalues of C .

Example 3 Given the matrix A of example 2 and matrix

$$B = \left[\begin{array}{c|ccc} 4 & 1 & 2 & -1 \\ \hline 2 & & & \\ -2 & & B_{22} & \\ 1 & & & \end{array} \right]$$

with $B_{22} = \begin{bmatrix} 3 & 5 & -5 \\ -1 & -3 & 7 \\ -1 & -1 & 5 \end{bmatrix}$, the spectra of B and

B_{22} are $\sigma(B) \approx \{-1.5, 4.8, 2.8 \pm 1.0i\}$, $\sigma(B_{22}) = \{-2, 3, 4\}$, and we obtain that the 1-subdirect sum

$$C = A \oplus_1 B = \left[\begin{array}{ccc|c|ccc} 1 & -1 & 4 & 5 & 0 & 0 & 0 \\ 3 & 2 & -1 & 4 & 0 & 0 & 0 \\ 2 & 1 & -1 & 3 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 9 & 1 & 2 & -1 \\ \hline 0 & 0 & 0 & 2 & 3 & 5 & -5 \\ 0 & 0 & 0 & -2 & -1 & -3 & 7 \\ 0 & 0 & 0 & 1 & -1 & -1 & 5 \end{array} \right]$$

has the eigenvalues

$$\sigma(C) \approx \{9.9, -2, -1.7, 3.8, 1, 1, 3\},$$

and according to theorem 1, since the eigenvalues -2 and 3 are common to A_{11} and B_{22} they are also eigenvalues of C .

It is easy to find examples such that $\lambda \in \sigma(A) \cap \sigma(B)$ but $\lambda \notin \sigma(A \oplus_1 B)$. Indeed, although $\sigma(A) = \sigma(B)$ we can not ensure that $A \oplus_1 B$ shares eigenvalues with A and B , as we show in the next example.

Example 4 Given the matrices

$$A = \left[\begin{array}{cc|c} 2 & 1 & 1 \\ -8 & 3 & 9 \\ \hline 8 & 1 & -5 \end{array} \right], \quad B = \left[\begin{array}{c|cc} -3 & -13 & 10 \\ -5 & -11 & 10 \\ \hline -6 & -14 & 14 \end{array} \right]$$

$$C = A \oplus_1 B = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 & 0 \\ -8 & 3 & 9 & 0 & 0 \\ \hline 8 & 1 & -8 & -13 & 10 \\ 0 & 0 & -5 & -11 & 10 \\ 0 & 0 & -6 & -14 & 14 \end{array} \right]$$

has not even one eigenvalue in common with A and B , since $\sigma(C) \approx \{-10.1, 0, 0.8, 4.6 \pm 1.3i\}$.

There is a particular case in which $\lambda \in \sigma(A) \cap \sigma(B)$ implies $\lambda \in \sigma(A \oplus_1 B)$. This happens when $\lambda = 0$ as we state in the following result.

Corollary 5 Let A and B be matrices of order n_1 and n_2 , respectively, partitioned as in (1) and let $k = 1$. If A and B are singular matrices then $C = A \oplus_1 B$ is also a singular matrix.

Proof. From equation (4), or (5), making $\lambda = 0$ we have

$$\det(C) = \det(A_{11})\det(B) + \det(A)\det(B_{22}) \quad (6)$$

and the proof follows. \square

Remark. Expression (6) was already known in [4] but we have just obtained it as a particular case of the more general expressions (4) or (5).

The following examples show that theorem 1 does not hold when $k > 1$.

Example 6 Given the matrices

$$A = \left[\begin{array}{ccc|cc} & & & 1 & 5 \\ & & & 1 & 4 \\ & & & -1 & 3 \\ \hline 1 & 2 & 3 & 9 & 2 \\ 2 & -1 & 1 & 2 & 1 \end{array} \right]$$

with $A_{11} = \left[\begin{array}{ccc} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{array} \right]$, and

$$B = \left[\begin{array}{cc|cc} 3 & 1 & 0 & 1 \\ 2 & 6 & 3 & 0 \\ \hline 1 & -2 & 5 & -1 \\ 1 & -1 & 1 & 4 \end{array} \right],$$

we have $\sigma(A) \approx \{10.4, 3, -2, -0.4, 1\}$ and $\sigma(A_{11}) = \{-2, 1, 3\}$, and therefore

$$\sigma(A_{11}) \cap \sigma(A) = \sigma(A_{11}) = \{1, -2, 3\},$$

$$C = A \oplus_2 B = \left[\begin{array}{ccc|cc} 1 & -1 & 4 & 1 & 5 & 0 & 0 \\ 3 & 2 & -1 & 1 & 4 & 0 & 0 \\ 2 & 1 & -1 & -1 & 3 & 0 & 0 \\ \hline 1 & 2 & 3 & 12 & 3 & 0 & 1 \\ 2 & -1 & 1 & 4 & 7 & 3 & 0 \\ \hline 0 & 0 & 0 & 1 & -2 & 5 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 4 \end{array} \right]$$

has the eigenvalues

$$\sigma(C) \approx \{14.9, -1.4, -0.3, 4.6 \pm 2.9i, 2.9, 4.7\},$$

and therefore we have $\lambda \in \sigma(A_{11}) \cap \sigma(A)$ but $\lambda \notin \sigma(A \oplus_2 B)$ and therefore part (i) of theorem 1 does not hold for $k > 1$.

Example 7 Given the matrices

$$A = \left[\begin{array}{ccc|cc} & & & 1 & -1 \\ & & & 3 & 2 \\ & & & 1 & 2 \\ \hline 1 & 3 & 1 & 2 & 3 \\ -1 & 2 & 2 & 1 & 1 \end{array} \right]$$

and

$$B = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & -3 & 2 \\ -1 & 1 & 1 & 1 & -1 \\ \hline 2 & 1 & & & \\ -3 & 1 & & & A_{11} \\ 2 & -1 & & & \end{array} \right],$$

with A_{11} given in example 6, we have $\sigma(A_{11}) \cap \sigma(B_{22}) = \sigma(A_{11})$, and a computation gives $\sigma(A_{11}) \cap \sigma(A \oplus_2 B) = \emptyset$. Then we conclude that statement (ii) of theorem 1 does not hold for $k > 1$.

Example 8 Let A be the matrix given in example 6

and let $B = \begin{bmatrix} A_{22}^T & A_{12}^T \\ A_{21}^T & A_{11}^T \end{bmatrix}$. A computation shows

that $\sigma(B_{22}) \cap \sigma(B) = \sigma(B_{22}) = \sigma(A_{11}^T)$, and $\sigma(A_{11}) \cap \sigma(A \oplus_2 B) = \emptyset$. Then we conclude that statement (iii) of theorem 1 does not hold for $k > 1$.

We have seen that theorem 1 allows to obtain some of the eigenvalues of the 1-subdirect sum when certain conditions of the eigenvalues of A , B and some submatrices are satisfied. The following result, which relates the eigenvalues of a matrix and a submatrix shall be useful to further explore theorem 1.

Theorem 9 Let A be an square matrix of order n and let $\lambda \in \sigma(A)$. Let A_m be a principal submatrix of A of order m . Let $g \geq 1$ be a positive integer. If the geometric multiplicity of λ is at least g and $m > n - g$ then it holds that $\lambda \in \sigma(A_m)$.

Combining theorems 1 and 9 we can state the following interesting result.

Theorem 10 *Let A and B be matrices of order n_1 and n_2 , respectively, partitioned as in (1) and let $k = 1$. If $\lambda \in \sigma(A) \cup \sigma(B)$ and the geometric multiplicity of λ is at least 2 (with reference to any of both matrices) then λ is an eigenvalue of the 1-subdirect sum $C = A \oplus_1 B$.*

Proof. From theorem 1 and theorem 9. In detail: If $\lambda \in \sigma(A)$ and its geometric multiplicity is greater or equal to 2 then, by theorem 9 applied to principal submatrix A_{11} of order $m = n_1 - k = n_1 - 1$ we have that $m > n_1 - g$ since $m = n_1 - 1 > n_1 - 2$. Therefore we conclude that λ is also an eigenvalue of A_{11} . From theorem 1 we conclude that λ is also an eigenvalue of the 1-subdirect sum C . If $\lambda \in \sigma(B)$, from theorem 9 we conclude that λ is also an eigenvalue of B_{22} and from theorem 1 we conclude that λ is also an eigenvalue of the 1-subdirect sum C . \square

Example 11 *Given the matrices*

$$A = \left[\begin{array}{cc|c} 2 & -1 & 3 \\ -2 & 3 & 3 \\ \hline 2 & 1 & 1 \end{array} \right], \quad B = \left[\begin{array}{c|cc} 1 & 2 & 1 \\ -1 & -3 & 1 \\ \hline 1 & 3 & 3 \end{array} \right]$$

their spectra are given by $\sigma(A) = \{-2, 4, 4\}$ and

$$\sigma(B) \approx \{-3.0, 0.4, 3.7\}$$

and the eigenvalue $\lambda = 4$ of A has geometric multiplicity equal to 2 with the associated eigenspace spanned by eigenvectors $[-1, 2, 0]^T$ and $[3, 0, 2]^T$. Therefore, according to theorem 10, the 1-subdirect sum

$$C = A \oplus_1 B = \left[\begin{array}{cc|c|cc} 2 & -1 & 3 & 0 & 0 \\ -2 & 3 & 3 & 0 & 0 \\ \hline 2 & 1 & 2 & 2 & 1 \\ \hline 0 & 0 & -1 & -3 & 1 \\ 0 & 0 & 1 & 3 & 3 \end{array} \right]$$

has $\lambda = 4$ as an eigenvalue. In fact we obtain:

$$\sigma(C) \approx \{-2.7, -2.2, 4.7, 4, 3.2\}$$

In certain cases, imposing more conditions to the eigenvalues and eigenvectors we can obtain more results concerning the eigenvalues and eigenvectors of the k -subdirect sums for an arbitrary k . We do this in the next section.

3 Eigenvectors of subdirect sums

In this section we analyze some properties of the eigenvalues and eigenvectors of the k -subdirect sums. We first consider the case when $\lambda = 0$ is an eigenvalue of A or B and then we impose certain conditions to their eigenvectors. After that we extend to the case of general λ .

The following result shows how to obtain an eigenvector of the k -sub-direct sum assuming certain conditions for two blocks of matrix A .

Lemma 12 *Let A and B be matrices of order n_1 and n_2 , respectively, partitioned as in (1), with $1 \leq k \leq \min(n_1, n_2)$. Let $x_1 \in \mathbf{C}^{(n_1-k) \times 1}$ be a nonzero vector such that $x_1 \in \ker A_{11}$ and $x_1 \in \ker A_{21}$. Then it holds that vector $x = \begin{bmatrix} x_1 \\ O_{n_2 \times 1} \end{bmatrix}$ is an eigenvector of the k -subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue $\lambda = 0$.*

Proof. We only have to show that $Cx = 0$. A direct computation gives

$$Cx = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ O_{k \times 1} \\ O_{(n_2-k) \times 1} \end{bmatrix} = \begin{bmatrix} A_{11}x_1 \\ A_{21}x_1 \\ O_{(n_2-k) \times 1} \end{bmatrix},$$

and since $x_1 \in \ker A_{11}$ and $x_1 \in \ker A_{21}$ we get

$$Cx = \begin{bmatrix} O_{(n_1-k) \times 1} \\ O_{n_2 \times 1} \end{bmatrix} = 0x. \quad \square$$

We have an analogous result when focusing on matrix B .

Lemma 13 *Let A and B be a matrices of order n_1 and n_2 , respectively, partitioned as in (1), with $1 \leq k \leq \min(n_1, n_2)$. Let $x_2 \in \mathbf{C}^{(n_2-k) \times 1}$ be a nonzero vector such that $x_2 \in \ker B_{11}$ and $x_2 \in \ker B_{12}$.*

Then it holds that vector $x = \begin{bmatrix} O_{n_1 \times 1} \\ x_2 \end{bmatrix}$ is an eigenvector the k -subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue $\lambda = 0$.

Proof. A direct computation gives $Cx = 0$. \square

Remark. Note that conditions $x_1 \in \ker A_{11}$ and $x_1 \in \ker A_{21}$ imply that the first $n_1 - k$ columns of A are a linearly dependent set and therefore the same columns of C are also a linearly dependent set and

then the k -subdirect sum C is a singular matrix. What lemma 12 offers is an explicit eigenvector associated with the eigenvalue $\lambda = 0$ of C . The same argument can be made for conditions $x_2 \in \ker B_{11}$ and $x_2 \in \ker B_{12}$ in lemma 13.

As a direct consequence of lemmas 12 and 13 we have the following.

Theorem 14 *Let A and B be matrices of order n_1 and n_2 , respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in (1). Let x_1 and x_2 be nonzero vectors such that $x_1 \in \ker A_{11}$, $x_1 \in \ker A_{21}$, $x_2 \in \ker B_{11}$ and $x_2 \in \ker B_{12}$. Then it holds that vector $x = \begin{bmatrix} x_1 \\ O_{k \times 1} \\ x_2 \end{bmatrix}$ is an eigenvector of the k -subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue $\lambda = 0$.*

Proof. It is easy to show that $Cx = O$. \square

Now we can make a step forward leaving the condition of null eigenvalue.

Lemma 15 *Let A and B be matrices of order n_1 and n_2 , respectively, partitioned as in (1), with $1 \leq k \leq \min(n_1, n_2)$. Let $x_1 \in \mathbf{C}^{(n_1-k) \times 1}$ be a nonzero vector such that x_1 is an eigenvector of A_{11} associated with the eigenvalue λ and such that $x_1 \in \ker A_{21}$. Then it holds that vector $x = \begin{bmatrix} x_1 \\ O_{n_2 \times 1} \end{bmatrix}$ is an eigenvector of the k -subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue λ .*

Proof. As in the proof of lemma 12 it is easy to show that $Cx = \lambda x$. \square

The counterpart, focusing on matrix B , is the following.

Lemma 16 *Let A and B be a matrices of order n_1 and n_2 , respectively, partitioned as in (1), with $1 \leq k \leq \min(n_1, n_2)$. Let $x_2 \in \mathbf{C}^{(n_2-k) \times 1}$ be a nonzero vector such that x_2 is an eigenvector of B_{22} associated with the eigenvalue λ and such that $x_2 \in \ker B_{12}$. Then it holds that vector $x = \begin{bmatrix} O_{n_1 \times 1} \\ x_2 \end{bmatrix}$ is an eigenvector of the k -subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue λ .*

Proof. A direct calculation gives $Cx = \lambda x$. \square

It is easy to see that if $\begin{bmatrix} x_1 \\ O_{k \times 1} \end{bmatrix} \in \mathbf{C}^{n_1 \times 1}$ is an eigenvector of A corresponding to the eigenvalue λ then it implies that x_1 is an eigenvector of A_{11} associated with λ and also that $x_1 \in \ker A_{21}$. Therefore as a consequence of lemmas 15 and 16 we have the following.

Theorem 17 *Let A and B be matrices of order n_1 and n_2 , respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in (1). Let $C = A \oplus_k B$. Let us assume that A and B have the same eigenvalue λ , let*

$$\begin{bmatrix} x_1 \\ O_{k \times 1} \end{bmatrix} \in \mathbf{C}^{n_1 \times 1}$$

be an eigenvector of A corresponding to the eigenvalue λ and let

$$\begin{bmatrix} O_{k \times 1} \\ x_2 \end{bmatrix} \in \mathbf{C}^{n_2 \times 1}$$

be an eigenvector of B corresponding to the eigenvalue λ . Then it holds that

$$x = \begin{bmatrix} x_1 \\ O_{k \times 1} \\ x_2 \end{bmatrix}$$

is an eigenvector of C corresponding to the eigenvalue λ .

Proof. It is easy to show that $Cx = \lambda x$. \square

Example 18 *Given the matrices*

$$A = \left[\begin{array}{ccc|cc} & & & 3 & 2 \\ & & & 5 & -1 \\ & & & 1 & 3 \\ \hline 4 & 5 & 1 & 2 & 1 \\ -2 & 0 & 2 & 1 & 4 \end{array} \right]$$

with $A_{11} = \begin{bmatrix} 2 & 1 & 1 \\ -8 & 3 & 9 \\ 8 & 1 & -5 \end{bmatrix}$, we have that $x_1^T =$

$[1, -1, -1]$ is an eigenvector of A_{11} associated with the eigenvalue $\lambda = 2$, and $[x_1, 0, 0]^T$ is an eigenvector of A associated with the same eigenvalue. Given the matrix

$$B = \left[\begin{array}{cc|ccc} 1 & 2 & 1 & 1 & 0 \\ -1 & -3 & 1 & 0 & 1 \\ \hline 1 & 3 & & & \\ 2 & -1 & & & \\ -4 & 4 & & & \end{array} \right],$$

with $B_{22} = \begin{bmatrix} 3 & 6 & -5 \\ 1 & 7 & -4 \\ 1 & 6 & -3 \end{bmatrix}$, we have that $x_2^T =$

$[1, -1, 1]$ is an eigenvector of B_{22} associated with the eigenvalue $\lambda = 2$, and $[0, 0, x_2]^T$ is an eigenvector of B associated with the same eigenvalue. Therefore,

sum $C = A \oplus_2 B =$

$$= \left[\begin{array}{ccc|cc|ccc} 2 & 1 & 1 & 3 & 2 & 0 & 0 & 0 \\ -8 & 3 & 9 & 5 & -1 & 0 & 0 & 0 \\ 8 & 1 & -5 & 1 & 3 & 0 & 0 & 0 \\ \hline 4 & 5 & 1 & 3 & 3 & 1 & 1 & 0 \\ -2 & 0 & 2 & 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 3 & 3 & 6 & -5 \\ 0 & 0 & 0 & 2 & -1 & 1 & 7 & -4 \\ 0 & 0 & 0 & -4 & 4 & 1 & 6 & -3 \end{array} \right]$$

has the eigenvector $[1, -1, -1, 0, 0, 1, -1, 1]^T$ associated with the eigenvalue $\lambda = 2$.

In a similar fashion as in the preceding theorems we can state the following result.

Theorem 19 Let A and B be matrices of order n_1 and n_2 , respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in (1). Let $C = A \oplus_k B$. Let $w \in \mathbb{C}^{k \times 1} \in \ker A_{12} \cap \ker B_{21}$ such that w is a common eigenvector of A_{22} and B_{11} associated with the eigenvalue λ . Then it holds that

$$x = \begin{bmatrix} O_{(n_1-k) \times 1} \\ w \\ O_{(n_2-k) \times 1} \end{bmatrix}$$

is an eigenvector of C corresponding to the eigenvalue 2λ .

Proof. It is easy to show that $Cx = 2\lambda x$, and x is nonzero since w , being an eigenvector, is nonzero \square

The following example shows a couple of matrices that satisfy the above theorem.

Example 20 Given the matrices

$$A = \left[\begin{array}{cc|cc} 4 & -1 & -1 & \\ 3 & 26 & -4 & \\ 2 & -6 & 24 & \end{array} \right], B = \left[\begin{array}{cc|c} 32 & 2 & 5 \\ 15 & 45 & -4 \\ -1 & -1 & 3 \end{array} \right]$$

we have that $w^T = [1, -1]$ satisfies $A_{12}w = B_{21}w = [0, 0]^T$ and is an eigenvector of $A_{22} = \begin{bmatrix} 26 & -4 \\ -6 & 24 \end{bmatrix}$

and of $B_{11} = \begin{bmatrix} 32 & 2 \\ 15 & 45 \end{bmatrix}$ associated with the eigenvalue $\lambda = 30$, and according to the theorem 19 we obtain that $[0, 1, -1, 0]^T$ is an eigenvector of the sub-direct sum

$$C = A \oplus_2 B = \left[\begin{array}{ccc|cc} 4 & -1 & -1 & 0 \\ 3 & 58 & -2 & 5 \\ 2 & 9 & 69 & -4 \\ \hline 0 & -1 & -1 & 3 \end{array} \right]$$

associated with the eigenvalue $\lambda = 60$.

4 Conclusions and open issues

Some new properties of the eigenvalues of the sub-direct sums have been presented for the particular case of 1-subdirect sums and some results concerning the eigenvectors of the k -subdirect sums have also been given. Several numerical examples illustrate the results. The characterization of the eigenvalues of the k -subdirect sum for general k is still an open problem. Some recent results on eigenvalue inclusion sets (see [3], [6]) may help outline some properties of the eigenvalues of the k -subdirect sum $C = A \oplus_k B$ in terms of the spectra of matrices A and B .

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