On the subdirect sums of normal matrices

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Abstract: In this paper a characterization of the subdirect sum of normal matrices is addressed, i.e., given two normal matrices A and B we give necessary and sufficient conditions such that the k-subdirect sum $A \oplus_k B$ is also a normal matrix. We extend this result to the case of the subdirect sum of two overlapped normal matrices that are submatrices of a given matrix. In addition, new properties of the eigenvalues of the subdirect sums are presented for the particular case of 1-subdirect sums.

Key–Words: Normal matrix, subdirect sum, eigenvalues, domain decomposition, overlap, iterative methods, Schwarz preconditioning.

1 Introduction

The notion of k-subdirect sum of matrices, introduced in [5], appears in several contexts related to square matrices: e.g., matrix completion problems, analysis of matrix classes, overlapping subdomains in domain decomposition methods, global stiffness matrix in finite element method, etc; see [3], [2] and the references therein.

Some properties of the subdirect sums of nonsingular M-matrices and of their inverses have been studied in [2], while the case of diagonally dominant matrices is shown in [1]. In this paper we focus on the characterization of the subdirect sum of normal matrices including the extension of this result to the case of the sum of two overlapped normal matrices that are submatrices of a given matrix. We begin giving the definition of subdirect sum and presenting some new properties of the eigenvalues of 1-subdirect sums, which are relevant in some instances; see [5].

2 Subdirect sums

Let A and B be two square matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned into 2×2 blocks as follows,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{1}$$

where A_{22} and B_{11} are square matrices of order k. Following [5], we call the following square matrix of order $n = n_1 + n_2 - k$,

$$C = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \tag{2}$$

the k-subdirect sum of A and B and denote it by $C = A \oplus_k B$.

These sums may appear in practical problems. For example, when consider the assembly process of the finite element method [9]. The finite element mesh shown in figure 1 leads to an assembled matrix C that is of the form

$$C = \begin{bmatrix} \square & \square & \square & & \\ \square & \square & \square & & \\ \square & \square & \square & & \\ & & & & \\ & & & & \end{bmatrix} + \begin{bmatrix} & & & & \\ & \square & \square & \square & \\ & \square & \square & \square & \\ & \square & \square & \square & \\ & & & & \end{bmatrix} + \begin{bmatrix} & & & & \\ & & & & \\ & & \square & \square & \square \\ & & \square & \square & \square \\ & & \square & \square & \square \end{bmatrix} \tag{3}$$

and this expression resembles the concept of subdirect sum.

2.1 Eigenvalues of 1-subdirect sums

The particular case of $k = 1$, i.e., a 1-subdirect sum, is important in certain applications (see [5] and the refer-

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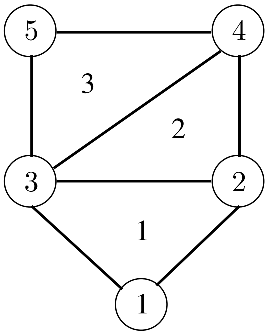

Figure 1: A simple finite element mesh with three triangular elements and five nodes.

ences therein). We denote as $\sigma(A)$ the spectrum (i.e., the set of eigenvalues) of a matrix A. Some properties of the eigenvalues of 1-subdirect sums can be summarized as follows.

Theorem 1 *Let A and B be matrices of order n_1 and n_2, respectively, partitioned as in* (1) *and let $k = 1$. Let $C = A \oplus_1 B$. Then any of the following statements*

i) $\lambda \in \sigma(A) \cap \sigma(A_{11})$

ii) $\lambda \in \sigma(A_{11}) \cap \sigma(B_{22})$

iii) $\lambda \in \sigma(B) \cap \sigma(B_{22})$

implies that $\lambda \in \sigma(C)$.

Proof. Let us denote $a_{22} = A_{22}$ and $b_{11} = B_{11}$ to display that this quantities are matrices of order 1×1. A direct computation shows that

$$det(C \quad -\lambda I) = \\ = \begin{bmatrix} A_{11} - \lambda I & A_{12} & O \\ A_{21} & a_{22} + b_{11} - \lambda & B_{12} \\ O & B_{21} & B_{22} - \lambda I \end{bmatrix} \tag{4}$$

and expanding the n_1th-row as a sum of the rows $(A_{21}, a_{22} - \lambda, 0)$ and $(0, b_{11}, B_{12})$ we have

$$det(C - \lambda I) =$$

$$det \begin{bmatrix} A_{11} - \lambda I & A_{12} & O \\ A_{21} & a_{22} - \lambda & 0 \\ O & B_{21} & B_{22} - \lambda I \end{bmatrix} \quad +$$

$$det \begin{bmatrix} A_{11} - \lambda I & A_{12} & O \\ 0 & b_{11} & B_{12} \\ O & B_{21} & B_{22} - \lambda I \end{bmatrix},$$

which leads to

$$det(C - \lambda I) =$$

$$det(A - \lambda I) det(B_{22} - \lambda I) +$$

$$det(A_{11} - \lambda I) det \begin{bmatrix} b_{11} & B_{12} \\ B_{21} & B_{22} - \lambda I \end{bmatrix}, \tag{5}$$

and from this expression we have that each of the statements (i) and (ii) implies $\lambda \in \sigma(C)$. To prove (iii) the technique is the same but now we expand the n_1th-row of equation (4) as a sum of the rows $(A_{21}, a_{22}, 0)$ and $(0, b_{11} - \lambda, B_{12})$ to finally get

$$det(C - \lambda I) =$$

$$det \begin{bmatrix} A_{11} - \lambda I & A_{12} \\ A_{21} & a_{22} \end{bmatrix} det(B_{22} - \lambda I) +$$

$$det(A_{11} - \lambda I) det(B - \lambda I), \tag{6}$$

from which we conclude that statement (iii) implies $\lambda \in \sigma(C)$. □

Example 2 *Given the matrices*

$$A_{11} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}, \; A = \left[\begin{array}{ccc|c} & & & 5 \\ & A_{11} & & 4 \\ & & & 3 \\ \hline 1 & 0 & 1 & 5 \end{array}\right]$$

$$B = \left[\begin{array}{c|ccc} 3 & 1 & 0 & 1 \\ \hline 2 & 6 & 3 & 0 \\ 1 & -2 & 5 & -1 \\ 1 & -1 & 1 & 4 \end{array}\right]$$

the spectra of A and A_{11} are

$$\sigma(A) = \{-2, 1, 1, 7\}, \quad \sigma(A_{11}) = \{-2, 1, 3\}$$

and we obtain that the 1-subdirect sum

$$C = A \oplus_1 B = \left[\begin{array}{ccc|c|ccc} 1 & -1 & 4 & 5 & 0 & 0 & 0 \\ 3 & 2 & -1 & 4 & 0 & 0 & 0 \\ 2 & 1 & -1 & 3 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 8 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 6 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 5 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 4 \end{array}\right]$$

has the eigenvalues

$$\sigma(C) \approx \{9.8,\ -2,\ 1,\ 1.7, 4.9 \pm 2.4i, 4.7\}$$

and according to theorem 1, since the eigenvalues -2 and 1 are common to A and A_{11} they are also eigenvalues of C.

Example 3 *Given the matrix* A *of example 2 and matrix*

$$B = \left[\begin{array}{c|ccc} 4 & 1 & 2 & -1 \\ \hline 2 & & & \\ -2 & & B_{22} & \\ 1 & & & \end{array}\right]$$

with $B_{22} = \left[\begin{array}{ccc} 3 & 5 & -5 \\ -1 & -3 & 7 \\ -1 & -1 & 5 \end{array}\right]$*, the spectra of* B *and* B_{22} *are* $\sigma(B) \approx \{-1.5, 4.8, 2.8 \pm 1.0i\}$*,* $\sigma(B_{22}) = \{-2, 3, 4\}$*, and we obtain that the* 1*-subdirect sum*

$$C = A \oplus_1 B = \left[\begin{array}{ccc|c|ccc} 1 & -1 & 4 & 5 & 0 & 0 & 0 \\ 3 & 2 & -1 & 4 & 0 & 0 & 0 \\ 2 & 1 & -1 & 3 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 9 & 1 & 2 & -1 \\ \hline 0 & 0 & 0 & 2 & 3 & 5 & -5 \\ 0 & 0 & 0 & -2 & -1 & -3 & 7 \\ 0 & 0 & 0 & 1 & -1 & -1 & 5 \end{array}\right]$$

has the eigenvalues

$$\sigma(C) \approx \{9.9, -2, -1.7, 3.8, 1, 1, 3\},$$

and according to theorem 1, since the eigenvalues -2 *and* 3 *are common to* A_{11} *and* B_{22} *they are also eigenvalues of* C*.*

It is easy to find examples such that $\lambda \in \sigma(A) \cap \sigma(B)$ but $\lambda \notin \sigma(A \oplus_1 B)$. Indeed, although $\sigma(A) = \sigma(B)$ we can not ensure that $A \oplus_1 B$ shares eigenvalues with A and B, as we show in the next example.

Example 4 *Given the matrices*

$$A = \left[\begin{array}{cc|c} 2 & 1 & 1 \\ -8 & 3 & 9 \\ \hline 8 & 1 & -5 \end{array}\right], \quad B = \left[\begin{array}{c|cc} -3 & -13 & 10 \\ \hline -5 & -11 & 10 \\ -6 & -14 & 14 \end{array}\right]$$

with the same spectrum, $\sigma(A) = \sigma(B) = \{2, 4, -6\}$*, we have, notwithstanding, that the* 1*-subdirect sum*

$$C = A \oplus_1 B = \left[\begin{array}{cc|c|cc} 2 & 1 & 1 & 0 & 0 \\ -8 & 3 & 9 & 0 & 0 \\ \hline 8 & 1 & -8 & -13 & 10 \\ \hline 0 & 0 & -5 & -11 & 10 \\ 0 & 0 & -6 & -14 & 14 \end{array}\right]$$

has not even one eigenvalue in common with A *and* B*, since* $\sigma(C) \approx \{-10.1, 0, 0.8, 4.6 \pm 1.3i\}$*.*

There is a particular case in which $\lambda \in \sigma(A) \cap \sigma(B)$ implies $\lambda \in \sigma(A \oplus_1 B)$. This happens when $\lambda = 0$ as we state in the following result.

Corollary 5 *Let* A *and* B *be matrices of order* n_1 *and* n_2*, respectively, partitioned as in* (1) *and let* $k = 1$*. If* A *and* B *are singular matrices then* $C = A \oplus_1 B$ *is also a singular matrix.*

Proof. From equation (5), or (6), making $\lambda = 0$ we have

$$det(C) = det(A_{11})det(B) + det(A)det(B_{22}) \quad (7)$$

and the proof follows. □

Remark. Expression (7) was already known in [5] but we have just obtained it as a particular case of the more general expressions (5) or (6).

The following examples show that theorem 1 does not hold when $k > 1$.

Example 6 *Given the matrices*

$$A = \left[\begin{array}{ccc|cc} & & & 1 & 5 \\ & A_{11} & & 1 & 4 \\ & & & -1 & 3 \\ \hline 1 & 2 & 3 & 9 & 2 \\ 2 & -1 & 1 & 2 & 1 \end{array}\right]$$

with $A_{11} = \left[\begin{array}{ccc} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{array}\right]$*, and*

$$B = \left[\begin{array}{cc|cc} 3 & 1 & 0 & 1 \\ 2 & 6 & 3 & 0 \\ \hline 1 & -2 & 5 & -1 \\ 1 & -1 & 1 & 4 \end{array}\right],$$

we have $\sigma(A) \approx \{10.4, 3, -2, -0.4, 1\}$ *and* $\sigma(A_{11}) = \{-2, 1, 3\}$*, and therefore*

$$\sigma(A_{11}) \cap \sigma(A) = \sigma(A_{11}) = \{1, -2, 3\},$$

but we obtain that the 2*-subdirect sum*

$$C = A \oplus_2 B = \left[\begin{array}{ccc|cc|cc} 1 & -1 & 4 & 1 & 5 & 0 & 0 \\ 3 & 2 & -1 & 1 & 4 & 0 & 0 \\ 2 & 1 & -1 & -1 & 3 & 0 & 0 \\ \hline 1 & 2 & 3 & 12 & 3 & 0 & 1 \\ 2 & -1 & 1 & 4 & 7 & 3 & 0 \\ \hline 0 & 0 & 0 & 1 & -2 & 5 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 4 \end{array}\right]$$

has the eigenvalues

$$\sigma(C) \approx \{14.9,\ -1.4,\ -0.3,\ 4.6 \pm 2.9i,\ 2.9,\ 4.7\},$$

and therefore we have $\lambda \in \sigma(A_{11}) \cap \sigma(A)$ *but* $\lambda \notin \sigma(A \oplus_2 B)$ *and therefore part* (i) *of theorem 1 does not hold for* $k > 1$*.*

Example 7 *Given the matrices*

$$A = \left[\begin{array}{ccc|cc} & & & 1 & -1 \\ & A_{11} & & 3 & 2 \\ & & & 1 & 2 \\ \hline 1 & 3 & 1 & 2 & 3 \\ -1 & 2 & 2 & 1 & 1 \end{array}\right]$$

and

$$B = \left[\begin{array}{cc|ccc} 1 & -1 & 2 & -3 & 2 \\ -1 & 1 & 1 & 1 & -1 \\ \hline 2 & 1 & & & \\ -3 & 1 & & A_{11} & \\ 2 & -1 & & & \end{array}\right],$$

with A_{11} given in example 6, we have $\sigma(A_{11}) \cap \sigma(B_{22}) = \sigma(A_{11})$, and a computation gives $\sigma(A_{11}) \cap \sigma(A \oplus_2 B) = \emptyset$. Then we conclude that statement (ii) of theorem 1 does not hold for $k > 1$.

Example 8 *Let A be the matrix given in example 6 and let $B = \left[\begin{array}{cc} A_{22}^T & A_{12}^T \\ A_{21}^T & A_{11}^T \end{array}\right]$. A computation shows that $\sigma(B_{22}) \cap \sigma(B) = \sigma(B_{22}) = \sigma(A_{11}^T)$, and $\sigma(A_{11}) \cap \sigma(A \oplus_2 B) = \emptyset$. Then we conclude that statement (iii) of theorem 1 does not hold for $k > 1$.*

We have seen that theorem 1 allows to obtain some of the eigenvalues of the 1-subdirect sum when certain conditions of the eigenvalues of A, B and some submatrices are satisfied. The following result, which relates the eigenvalues of a matrix and a submatrix shall be useful to further explode theorem 1.

Theorem 9 *Let A be an square matrix of order n and let $\lambda \in \sigma(A)$. Let A_m be a principal submatrix of A of order m. Let $g \geq 1$ be a positive integer. If the geometric multiplicity of λ is at least g and $m > n - g$ then it holds that $\lambda \in \sigma(A_m)$.*

Proof. See [8], p. 60.

Combining theorems 1 and 9 we can state the following interesting result.

Theorem 10 *Let A and B be matrices of order n_1 and n_2, respectively, partitioned as in* (1) *and let $k = 1$. If $\lambda \in \sigma(A) \cup \sigma(B)$ and the geometric multiplicity of λ is at least* 2 *(with reference to any of both matrices) then λ is an eigenvalue of the* 1*-subdirect sum $C = A \oplus_1 B$.*

Proof. From theorem 1 and theorem 9. In detail: If $\lambda \in \sigma(A)$ and its geometric multiplicity is greater or equal to 2 then, by theorem 9 applied to principal submatrix A_{11} of order $m = n_1 - k = n_1 - 1$ we have that $m > n_1 - g$ since $m = n_1 - 1 > n_1 - 2$. Therefore we conclude that λ is also an eigenvalue of A_{11}. From theorem 1 we conclude that λ is also an eigenvalue of the 1-subdirect sum C. If $\lambda \in \sigma(B)$, from theorem 9 we conclude that λ is also an eigenvalue of B_{22} and from theorem 1 we conclude that λ is also an eigenvalue of the 1-subdirect sum C. □

Example 11 *Given the matrices*

$$A = \left[\begin{array}{cc|c} 2 & -1 & 3 \\ -2 & 3 & 3 \\ \hline 2 & 1 & 1 \end{array}\right], \quad B = \left[\begin{array}{c|cc} 1 & 2 & 1 \\ \hline -1 & -3 & 1 \\ 1 & 3 & 3 \end{array}\right]$$

their spectra are given by $\sigma(A) = \{-2, 4, 4\}$ and $\sigma(B) \approx \{-3.0,\ 0.4,\ 3.7\}$ and the eigenvalue $\lambda = 4$ of A has geometric multiplicity equal to 2 *with the associated eigenspace spanned by eigenvectors $[-1, 2, 0]^T$ and $[3, 0, 2]^T$. Therefore, according to theorem 10, the* 1*-subdirect sum*

$$C = A \oplus_1 B = \left[\begin{array}{cc|c|cc} 2 & -1 & 3 & 0 & 0 \\ -2 & 3 & 3 & 0 & 0 \\ \hline 2 & 1 & 2 & 2 & 1 \\ \hline 0 & 0 & -1 & -3 & 1 \\ 0 & 0 & 1 & 3 & 3 \end{array}\right]$$

has $\lambda = 4$ as an eigenvalue. In fact we obtain: $\sigma(C) \approx \{-2.7,\ -2.2,\ 4.7,\ 4,\ 3.2\}$.

3 Subdirect sums of normal matrices

An square matrix A is said to be normal if $AA^* = A^*A$, where A^* denotes the conjugate transpose of A; see [7], [4] for more characterizations. In this class of matrices lay the subclasses of unitary ($UU^* = I = U^*U$), Hermitian ($A^* = A$), and skew-Hermitian ($A^* = -A$) matrices. Some properties of the subdirect sums for certain subclasses of Hermitian matrices were analyzed in [5]. In this section we establish necessary and sufficient conditions such that the k-subdirect sum of normal matrices is also a normal matrix.

We make two observations. First, we note that in general the k-subdirect sum of normal matrices is not a normal matrix. For example, given $A = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]$ and $B = \left[\begin{array}{cc} 1 & 3 \\ 3 & 2 \end{array}\right]$, the 1-subdirect sum $C = A \oplus_1 B$ is not a normal matrix. Then it is a natural question to seek for conditions such that the subdirect sum lays on the class. On the other hand, we also observe that given a normal matrix their submatrices need not to be normal. For example, $\begin{bmatrix} A & A^* \\ A^* & A \end{bmatrix}$ is a normal matrix for any square matrix A.

To set the notation of our main result let us denote by A a normal matrix partitioned as in (1). Then one obtains

$$AA^* = \begin{bmatrix} A_{11}A_{11}^* + A_{12}A_{12}^* & A_{11}A_{21}^* + A_{12}A_{22}^* \\ A_{21}A_{11}^* + A_{22}A_{12}^* & A_{21}A_{21}^* + A_{22}A_{22}^* \end{bmatrix} \tag{8}$$

and

$$A^*A = \begin{bmatrix} A_{11}^*A_{11} + A_{21}^*A_{21} & A_{11}^*A_{12} + A_{21}^*A_{22} \\ A_{12}^*A_{11} + A_{22}^*A_{21} & A_{12}^*A_{12} + A_{22}^*A_{22} \end{bmatrix} \tag{9}$$

From (8) and (9) and the fact that A is normal, we obtain

$$\left.\begin{array}{l} A_{11}A_{11}^* + A_{12}A_{12}^* = A_{11}^*A_{11} + A_{21}^*A_{21} \\ A_{11}A_{21}^* + A_{12}A_{22}^* = A_{11}^*A_{12} + A_{21}^*A_{22} \\ A_{21}A_{11}^* + A_{22}A_{12}^* = A_{12}^*A_{11} + A_{22}^*A_{21} \\ A_{21}A_{21}^* + A_{22}A_{22}^* = A_{12}^*A_{12} + A_{22}^*A_{22} \end{array}\right\} \tag{10}$$

These equations shall be useful in the proof of the following theorem.

Theorem 12 *Let A and B be normal matrices of order n_1 and n_2, respectively, partitioned as in* (1)*. Then the k subdirect sum $C = A \oplus_k B$ given by* (2) *is a normal matrix if and only if the following conditions hold*

i) $A_{12}B_{11}^* = A_{21}^*B_{11}$

ii) $A_{12}B_{21}^* = A_{21}^*B_{12}$

iii) $A_{22}B_{11}^* + B_{11}A_{22}^* = A_{22}^*B_{11} + B_{11}^*A_{22}$

iv) $A_{22}B_{21}^* = A_{22}^*B_{12}$

Proof. Let us denote by $(CC^*)_{ij}$ the ijth-block of the product CC^*. A direct computation gives

$$\left.\begin{array}{lcl} (CC^*)_{11} & = & A_{11}A_{11}^* + A_{12}A_{12}^* \\ (CC^*)_{12} & = & A_{11}A_{21}^* + A_{12}H^* \\ (CC^*)_{13} & = & A_{12}B_{21}^* \\ (CC^*)_{21} & = & A_{21}A_{11}^* + HA_{12}^* \\ (CC^*)_{22} & = & A_{21}A_{21}^* + HH^* + B_{12}B_{12}^* \\ (CC^*)_{23} & = & HB_{21}^* + B_{12}B_{22}^* \\ (CC^*)_{31} & = & B_{21}A_{12}^* \\ (CC^*)_{32} & = & B_{21}H^* + B_{22}B_{12}^* \\ (CC^*)_{33} & = & B_{21}B_{21}^* + B_{22}B_{22}^* \end{array}\right\} \tag{11}$$

where $H = A_{22} + B_{11}$. Since A and B are normal matrices they both satisfy equations (10). From (10) and (11) it is easy to get

$$\left.\begin{array}{lcl} (CC^*)_{11} & = & A_{11}^*A_{11} + A_{21}^*A_{21} \\ (CC^*)_{12} & = & A_{11}^*A_{12} + A_{21}^*A_{22} + A_{12}B_{11}^* \\ (CC^*)_{13} & = & A_{12}B_{21}^* \\ (CC^*)_{21} & = & A_{12}^*A_{11} + A_{22}^*A_{21} + B_{11}A_{12}^* \\ (CC^*)_{22} & = & A_{12}^*A_{12} + A_{22}^*A_{22} + \\ & & A_{22}B_{11}^* + B_{11}A_{22}^* + \\ & & B_{11}^*B_{11} + B_{21}^*B_{21} \\ (CC^*)_{23} & = & A_{22}B_{21}^* + B_{11}^*B_{12} + B_{21}^*B_{22} \\ (CC^*)_{31} & = & B_{21}A_{12}^* \\ (CC^*)_{32} & = & B_{21}A_{22}^* + B_{12}^*B_{11} + B_{22}^*B_{21} \\ (CC^*)_{33} & = & B_{12}^*B_{12} + B_{22}^*B_{22} \end{array}\right\} \tag{12}$$

Writing now the expression of C^*C we have

$$\left.\begin{array}{lcl} (C^*C)_{11} & = & A_{11}^*A_{11} + A_{21}^*A_{21} \\ (C^*C)_{12} & = & A_{11}^*A_{12} + A_{21}^*H \\ (C^*C)_{13} & = & A_{21}^*B_{12} \\ (C^*C)_{21} & = & A_{12}^*A_{11} + H^*A_{21} \\ (C^*C)_{22} & = & A_{12}^*A_{12} + H^*H + B_{21}^*B_{21} \\ (C^*C)_{23} & = & H^*B_{12} + B_{21}^*B_{22} \\ (C^*C)_{31} & = & B_{12}^*A_{21} \\ (C^*C)_{32} & = & B_{12}^*H + B_{22}^*B_{21} \\ (C^*C)_{33} & = & B_{12}^*B_{12} + B_{22}^*B_{22} \end{array}\right\} \tag{13}$$

and from (12) we obtain that $CC^* = C^*C$ if and only if the following conditions hold

1) $A_{12}B_{11}^* = A_{21}^*B_{11}$

2) $A_{12}B_{21}^* = A_{21}^*B_{12}$

3) $B_{11}A_{12}^* = B_{11}^*A_{21}$

4) $A_{22}B_{11}^* + B_{11}A_{22}^* = A_{22}^*B_{11} + B_{11}^*A_{22}$

5) $A_{22}B_{21}^* = A_{22}^*B_{12}$

6) $B_{21}A_{12}^* = B_{12}^*A_{21}$

7) $B_{21}A_{22}^* = B_{12}^*A_{22}$

Observing now that conditions (3), (6) and (7) are equivalent to conditions (1), (2) and (5), respectively, the proof is completed. □.

Some particular cases of normal matrices that verify sufficient conditions of this theorem are shown in the following corollaries.

Corollary 13 *Let A and B be normal matrices of order n_1 and n_2, respectively, partitioned as in* (1)*, and such that*

i) $A_{12} = A_{21}^*$

ii) $A_{22} = A_{22}^*$

iii) $B_{11} = B_{11}^*$

iv) $B_{12} = B_{21}^*$

Then the k-subdirect sum $C = A \oplus_k B$ given by (2) *is also a normal matrix.*

Note that the subclasses of real symmetric matrices and hermitian complex matrices verify the conditions of this corollary.

Corollary 14 *Let A and B be normal matrices of order n_1 and n_2, respectively, partitioned as in* (1)*, and such that*

i) $A_{12} = -A_{21}^*$

ii) $A_{22} = -A_{22}^*$

iii) $B_{11} = -B_{11}^*$

iv) $B_{12} = -B_{21}^*$

Then the k-subdirect sum $C = A \oplus_k B$ given by (2) *is also a normal matrix.*

Notice that the subclass of skew-Hermitian matrices verifies the conditions of this corollary.

4 Overlapping normal matrices

In this section we focus on the subdirect sum of normal matrices which are principal submatrices of a given matrix, and such that they have a common block. This situation may appear in some applications of the iterative method known as additive Schwarz; see, e.g., [3], [6], [10]. In fact, this problem was the base of our interest in subdirect sums.

Specifically, let

$$A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad B = \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} \tag{14}$$

normal matrices of order n_1 and n_2, respectively, and with M_{22} an square matrix of order k. Let

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \tag{15}$$

a matrix of order n, with $n = n_1 + n_2 - k$. Then it is easy to prove the following result.

Theorem 15 *Let A and B be normal matrices given by* (14) *and such that the following conditions hold*

i) $M_{12} = M_{21}^*$

ii) $M_{22} = M_{22}^*$

iii) $M_{23} = M_{32}^*$

Then the k-subdirect sum $C = A \oplus_k B$ given by

$$C = A \oplus_k B = \begin{bmatrix} M_{11} & M_{12} & O \\ M_{21} & 2M_{22} & M_{23} \\ O & M_{32} & M_{33} \end{bmatrix} \tag{16}$$

is also a normal matrix.

Proof. This result is a direct consequence of corollary 13. □.

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