



SUBDIRECT SUMS OF NONSINGULAR M -MATRICES AND OF THEIR INVERSES*

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Abstract. The question of when the subdirect sum of two nonsingular M -matrices is a nonsingular M -matrix is studied. Sufficient conditions are given. The case of inverses of M -matrices is also studied. In particular, it is shown that the subdirect sum of overlapping principal submatrices of a nonsingular M -matrix is a nonsingular M -matrix. Some examples illustrating the conditions presented are also given.

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1. Introduction. Subdirect sum of matrices are generalizations of the usual sum of matrices (a k -subdirect sum is formally defined below in Section 2). They were introduced by Fallat and Johnson in [3], where many of their properties were analyzed. For example, they showed that the subdirect sum of positive definite matrices, or of symmetric M -matrices, are positive definite or symmetric M -matrices, respectively. They also showed that this is not the case for M -matrices: the sum of two M -matrices may not be an M -matrix. One goal of the present paper is to give sufficient conditions so that the subdirect sum of nonsingular M -matrices is a nonsingular M -matrix. We also treat the case of the subdirect sum of inverses of M -matrices.

Subdirect sums of two overlapping principal submatrices of a nonsingular M -matrix appear naturally when analyzing additive Schwarz methods for Markov chains or other matrices [2], [4]. In this paper we show that the subdirect sum of two overlapping principal submatrices of a nonsingular M -matrix is a nonsingular M -matrix.

The paper is structured as follows. In Section 2 we focus on the nonsingularity of the subdirect sum of any pair of nonsingular matrices, giving an explicit expression for the inverse. In Section 2.1 we study the k -subdirect sum of two nonsingular M -matrices and in particular, the case of subdirect sums of overlapping blocks of nonsingular M -matrices. In Section 2.3 we extend some results to the subdirect sum of more than two nonsingular M -matrices. In Section 3 we analyze the subdirect sum of two inverses. Finally, in Section 4 we mention some open questions on subdirect sums of P -matrices. Throughout the paper we give examples which help illustrate the theoretical results.

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2. Subdirect sums of nonsingular matrices. Let A and B be two square matrices of order n_1 and n_2 , respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned into 2×2 blocks as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad (2.1)$$

where A_{22} and B_{11} are square matrices of order k . Following [3], we call the following square matrix of order $n = n_1 + n_2 - k$,

$$C = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix} \quad (2.2)$$

the k -subdirect sum of A and B and denote it by $C = A \oplus_k B$.

We are interested in the case when A and B are nonsingular matrices. We partition the inverses of A and B conformably to (2.1) and denote its blocks as follows:

$$A^{-1} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix}, \quad (2.3)$$

where, as before, \hat{A}_{22} and \hat{B}_{11} are square of order k .

In the following result we show that nonsingularity of matrix $\hat{A}_{22} + \hat{B}_{11}$ is a necessary and sufficient condition for the k -subdirect sum C to be nonsingular. The proof is based on the use of the relation $n = n_1 + n_2 - k$ to properly partition the indicated matrices.

THEOREM 2.1. *Let A and B be nonsingular matrices of order n_1 and n_2 , respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in (2.1) and their inverses be partitioned as in (2.3). Let $C = A \oplus_k B$. Then C is nonsingular if and only if $\hat{H} = \hat{A}_{22} + \hat{B}_{11}$ is nonsingular.*

Proof. Let I_m be the identity matrix of order m . The theorem follows from the following relation:

$$\begin{bmatrix} A^{-1} & O \\ O & I_{n-n_1} \end{bmatrix} C \begin{bmatrix} I_{n-n_2} & O \\ O & B^{-1} \end{bmatrix} = \begin{bmatrix} I_{n-n_2} & \hat{A}_{12} & O \\ O & \hat{H} & \hat{B}_{12} \\ O & O & I_{n-n_1} \end{bmatrix}. \quad \square \quad (2.4)$$

2.1. Nonsingular M -matrices. Given $A = \{a_{ij}\} \in \mathbb{R}^{m \times n}$, we write $A > O$ ($A \geq O$), to indicate $a_{ij} > 0$ ($a_{ij} \geq 0$), for $i = 1, \dots, m$, $j = 1, \dots, n$, and such matrices are called positive (nonnegative). Similarly, $A \geq B$ when $A - B \geq O$. Square matrices which have nonpositive off-diagonal entries are called Z -matrices. We call a Z -matrix M a nonsingular M -matrix if $M^{-1} \geq O$. We recall some properties of these matrices; see [1], [8]:

- (i) The diagonal of a nonsingular M -matrix is positive.
- (ii) If B is a Z -matrix and M is a nonsingular M -matrix, and $M \leq B$, then B is also a nonsingular M -matrix. In particular, any matrix obtained from a nonsingular M -matrix by setting certain off-diagonal entries to zero is also a nonsingular M -matrix.

(iii) A matrix M is a nonsingular M -matrix if and only if each principal submatrix of M is a nonsingular M -matrix.

(iv) A Z -matrix M is a nonsingular M -matrix if and only if there exists a positive vector $x > 0$ such that $Mx > 0$.

We first consider the k -subdirect sum of nonsingular Z -matrices. From (2.4) we can explicitly write

$$C^{-1} = \begin{bmatrix} I_{n-n_2} & O \\ O & B^{-1} \end{bmatrix} \begin{bmatrix} I_{n-n_2} & -\hat{A}_{12}\hat{H}^{-1} & \hat{A}_{12}\hat{H}^{-1}\hat{B}_{12} \\ O & \hat{H}^{-1} & -\hat{H}^{-1}\hat{B}_{12} \\ O & O & I_{n-n_1} \end{bmatrix} \begin{bmatrix} A^{-1} & O \\ O & I_{n-n_1} \end{bmatrix}$$

from which we obtain

$$C^{-1} = \begin{bmatrix} \hat{A}_{11} - \hat{A}_{12}\hat{H}^{-1}\hat{A}_{21} & \hat{A}_{12} - \hat{A}_{12}\hat{H}^{-1}\hat{A}_{22} & \hat{A}_{12}\hat{H}^{-1}\hat{B}_{12} \\ \hat{B}_{11}\hat{H}^{-1}\hat{A}_{21} & \hat{B}_{11}\hat{H}^{-1}\hat{A}_{22} & -\hat{B}_{11}\hat{H}^{-1}\hat{B}_{12} + \hat{B}_{12} \\ \hat{B}_{21}\hat{H}^{-1}\hat{A}_{21} & \hat{B}_{21}\hat{H}^{-1}\hat{A}_{22} & -\hat{B}_{21}\hat{H}^{-1}\hat{B}_{12} + \hat{B}_{22} \end{bmatrix} \quad (2.5)$$

and therefore we can state the following immediate result.

THEOREM 2.2. *Let A and B be nonsingular Z -matrices of order n_1 and n_2 , respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in (2.1) and their inverses be partitioned as in (2.3). Let $C = A \oplus_k B$. Let $\hat{H} = \hat{A}_{22} + \hat{B}_{11}$ be nonsingular. Then C is a nonsingular M -matrix if and only if each of the nine blocks of C^{-1} in (2.5) is nonnegative.*

We consider now the case where A and B are nonsingular M -matrices. It was shown in [3] that even if $H = A_{22} + B_{11}$ is a nonsingular M -matrix, this does not guarantee that $C = A \oplus_k B$ is a nonsingular M -matrix. We point out that this matrix H is not the matrix \hat{H} obtained from A^{-1} and B^{-1} and used in Theorem 2.1. The fact that H is a nonsingular M -matrix is a necessary but not a sufficient condition for C to be a nonsingular M -matrix. Sufficient conditions are presented in the following result.

THEOREM 2.3. *Let A and B be nonsingular M -matrices partitioned as in (2.1). Let $x_1 > 0 \in \mathbb{R}^{(n_1-k) \times 1}$, $y_1 > 0 \in \mathbb{R}^{k \times 1}$, $x_2 > 0 \in \mathbb{R}^{k \times 1}$ and $y_2 > 0 \in \mathbb{R}^{(n_2-k) \times 1}$ be such that*

$$A \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} > 0, \quad B \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} > 0. \quad (2.6)$$

Let $H = A_{22} + B_{11}$ be a nonsingular M -matrix and let

$$y = H^{-1}(A_{22}y_1 + B_{11}x_2). \quad (2.7)$$

Then if $y \leq y_1$ and $y \leq x_2$ the k -subdirect sum $C = A \oplus_k B$ is a nonsingular M -matrix.

Proof. We will show that there exists $u > 0$ such that $Cu > 0$. We first note that from (2.6) we get

$$\left. \begin{array}{l} A_{11}x_1 + A_{12}y_1 > 0 \\ A_{21}x_1 + A_{22}y_1 > 0 \end{array} \right\}, \quad \left. \begin{array}{l} B_{11}x_2 + B_{12}y_2 > 0 \\ B_{21}x_2 + B_{22}y_2 > 0 \end{array} \right\}. \quad (2.8)$$

Taking $u = \begin{bmatrix} x_1 \\ y \\ y_2 \end{bmatrix}$ and partitioning C as in (2.2) we obtain

$$Cu = \begin{bmatrix} A_{11}x_1 + A_{12}y \\ A_{21}x_1 + (A_{22} + B_{11})y + B_{12}y_2 \\ B_{21}y + B_{22}y_2 \end{bmatrix}. \quad (2.9)$$

Since $A_{21} \leq O$ and $B_{12} \leq O$, from (2.8) it follows that $A_{22}y_1 > 0$ and $B_{11}x_2 > 0$. Since $H^{-1} \geq O$, from (2.7) we have that y is positive, and consequently, so is u , i.e., $u > 0$. We will show that $Cu > 0$ one block of rows in (2.9) at a time. If $y \leq y_1$, as $A_{12} \leq 0$, we have that $A_{12}y \geq A_{12}y_1$ and again using (2.8) we obtain that the first block of rows of Cu is positive. In a similar way, the condition $y \leq x_2$ together with the last equation of (2.8) allows to conclude that the third block of rows of Cu is positive. Finally, substituting y given by (2.7) in the second row of Cu and considering (2.8) we conclude that the second block of rows of Cu is also positive. \square

Note that A and B are nonsingular M -matrices and therefore the positive vectors (x_1, y_1) and (x_2, y_2) of (2.6) always exist. This theorem gives sufficient but not necessary conditions for $C = A \oplus_k B$ to be a nonsingular M -matrix, as illustrated in Example 2.5 further below.

EXAMPLE 2.4. The matrices

$$A = \left[\begin{array}{c|cc} 3 & -2 & -1 \\ \hline -1/2 & 2 & -3 \\ -1 & -1 & 4 \end{array} \right] \quad \text{and} \quad B = \left[\begin{array}{cc|c} 1 & -2 & -1/3 \\ \hline -3 & 9 & 0 \\ -2 & -1/2 & 6 \end{array} \right],$$

and the vectors

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1 \end{bmatrix}$$

satisfy the inequalities (2.6), and computing the vector y from (2.7) we get $y \approx (1.95, 0.87)^T$, which satisfy $y \leq y_1$ and $y \leq x_2$. Therefore the 2-subdirect sum

$$C = \left[\begin{array}{ccc|c} 3 & -2 & -1 & 0 \\ \hline -1/2 & 3 & -5 & -1/3 \\ -1 & -4 & 13 & 0 \\ \hline 0 & -2 & -1/2 & 6 \end{array} \right]$$

is a nonsingular M -matrix in accordance with Theorem 2.3.

EXAMPLE 2.5. The matrices

$$A = \left[\begin{array}{c|cc} 5 & -1/2 & -1/3 \\ \hline -1 & 4 & -2 \\ -1 & -6 & 10 \end{array} \right] \quad \text{and} \quad B = \left[\begin{array}{cc|c} 1 & -2 & -1/3 \\ \hline -3 & 9 & 0 \\ -2 & -1/2 & 6 \end{array} \right]$$

and the vectors

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1 \end{bmatrix}$$

satisfy the inequalities (2.6), but computing vector y from (2.7) we obtain

$y \approx (1.18, 0.85)^T$, which does not satisfy the conditions of Theorem 2.3. Nevertheless the 2-subdirect sum

$$C = A \oplus_2 B = \left[\begin{array}{c|cc|c} 5 & -1/2 & -1/3 & 0 \\ \hline -1 & 5 & -4 & -1/3 \\ -1 & -9 & 19 & 0 \\ \hline 0 & -2 & -1/2 & 6 \end{array} \right]$$

is a nonsingular M -matrix.

In the special case of A and B block lower and upper triangular nonsingular M -matrices, respectively, the results of Theorems 2.2 and 2.3 are easy to establish. Let

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}, \quad (2.10)$$

with A_{22} and B_{11} square matrices of order k .

THEOREM 2.6. *Let A and B be nonsingular lower and upper block triangular nonsingular M -matrices, respectively, partitioned as in (2.10). Then $C = A \oplus_k B$ is a nonsingular M -matrix.*

Proof. We can repeat the same argument as in the proof of Theorem 2.3 with the advantage of having $A_{12} = O$ and $B_{21} = O$. Note that conditions $y \leq y_1$ and $y \leq x_2$ are not necessary here because the first and last block of rows of Cu in (2.9) are automatically positive in this case. \square

REMARK 2.7. The expression of C^{-1} is given by (2.5). In this particular case of block triangular matrices we have $\hat{A}_{12} = O$, $\hat{B}_{21} = O$, $\hat{A}_{22} = A_{22}^{-1}$, $\hat{B}_{11} = B_{11}^{-1}$, from which $\hat{H} = A_{22}^{-1} + B_{11}^{-1}$. If, in addition, $A_{22} = B_{11}$, then we obtain

$$C^{-1} = \left[\begin{array}{cc|cc} A_{11}^{-1} & O & O & O \\ \hline -\frac{1}{2}A_{22}^{-1}A_{21}A_{11}^{-1} & \frac{1}{2}A_{22}^{-1} & -\frac{1}{2}A_{22}^{-1}B_{12}B_{22}^{-1} & \\ \hline O & O & B_{22}^{-1} & \end{array} \right] \geq O.$$

EXAMPLE 2.8. The matrices

$$A = \left[\begin{array}{c|cc} 3 & 0 & 0 \\ \hline -1 & 5 & -1 \\ -1 & -9 & 5 \end{array} \right] \quad \text{and} \quad B = \left[\begin{array}{cc|c} 6 & -2 & -1 \\ \hline -4 & 3 & -3 \\ 0 & 0 & 2 \end{array} \right]$$

satisfy the hypotheses of Theorem 2.6. The matrices $C = A \oplus_2 B$ and C^{-1} are

$$C = \left[\begin{array}{c|cc|c} 3 & 0 & 0 & 0 \\ \hline -1 & 11 & -3 & -1 \\ -1 & -13 & 8 & -3 \\ \hline 0 & 0 & 0 & 2 \end{array} \right], \quad C^{-1} = \left[\begin{array}{c|cc|c} 1/3 & 0 & 0 & 0 \\ \hline 11/147 & 8/49 & 3/49 & 17/98 \\ 8/49 & 13/49 & 11/49 & 23/49 \\ \hline 0 & 0 & 0 & 1/2 \end{array} \right]$$

and therefore C is a nonsingular M -matrix as expected.

In some applications, such as in domain decomposition [6], [7], matrices A and B partitioned as in (2.1) arise with a common block, i.e., $A_{22} = B_{11}$. In the next example we show that even if A and B are nonsingular M -matrices, and so is the common block, we can not ensure that $C = A \oplus_k B$ is a nonsingular M -matrix.

EXAMPLE 2.9. The matrices

$$A = \left[\begin{array}{c|cc} 370 & -342 & -318 \\ \hline -448 & 737 & -107 \\ -46 & -190 & 444 \end{array} \right], \quad B = \left[\begin{array}{cc|c} 737 & -107 & -134 \\ \hline -190 & 444 & -440 \\ -885 & -182 & 603 \end{array} \right]$$

are nonsingular M -matrices with $A_{22} = B_{11}$ an M -matrix, but $C = A \oplus_2 B$ is not an M -matrix, since we have

$$C = \left[\begin{array}{c|cc|c} 370 & -342 & -318 & 0 \\ \hline -448 & 1474 & -214 & -134 \\ -46 & -380 & 888 & -440 \\ \hline 0 & -885 & -182 & 603 \end{array} \right]$$

$$\text{and } C^{-1} \approx \left[\begin{array}{c|cc|c} -0.0291 & -0.0242 & -0.0204 & -0.0203 \\ \hline -0.0145 & -0.0109 & -0.0098 & -0.0096 \\ -0.0214 & -0.0163 & -0.0132 & -0.0133 \\ \hline -0.0277 & -0.0210 & -0.0183 & -0.0164 \end{array} \right].$$

In the next section we shall see that when A and B share a block and they are submatrices of a given nonsingular M -matrix, the resulting k -subdirect sum is in fact a nonsingular M -matrix.

2.2. Overlapping M -matrices. In this section we restrict A and B to be principal submatrices of a given nonsingular M -matrix and such that they have a common block. Let

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \tag{2.11}$$

be a nonsingular M -matrix with M_{22} square matrix of order $k \geq 1$ and let

$$A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} \tag{2.12}$$

be of order n_1 and n_2 , respectively. The k -subdirect sum of A and B is thus given by

$$C = A \oplus_k B = \begin{bmatrix} M_{11} & M_{12} & O \\ M_{21} & 2M_{22} & M_{23} \\ O & M_{32} & M_{33} \end{bmatrix}. \tag{2.13}$$

In the following theorem we show that C is a nonsingular M -matrix.

THEOREM 2.10. *Let M be a nonsingular M -matrix partitioned as in (2.11), and let A and B be two overlapping principal submatrices given by (2.12). Then the k -subdirect sum $C = A \oplus_k B$ is a nonsingular M -matrix.*

Proof. Let us construct an $n \times n$ Z -matrix T as follows:

$$T = \begin{bmatrix} M_{11} & 2M_{12} & M_{13} \\ M_{21} & 2M_{22} & M_{23} \\ M_{31} & 2M_{32} & M_{33} \end{bmatrix}. \tag{2.14}$$

Then $T = M \operatorname{diag}(I, 2I, I)$ and we get $T^{-1} = \operatorname{diag}(I, (1/2)I, I)M^{-1} \geq O$. Then T is a nonsingular M -matrix. Finally since C is a Z -matrix and $C \geq T$ we conclude that C is a nonsingular M -matrix. \square

EXAMPLE 2.11. The following nonsingular M -matrix is partitioned as in (2.11):

$$M = \left[\begin{array}{cc|ccc|c} 13/14 & -4/23 & -3/20 & -1/42 & -19/186 & -3/46 \\ -3/7 & 21/23 & -1/5 & -1/21 & -1/93 & -6/23 \\ \hline -1/7 & -7/46 & 17/20 & -1/14 & -1/186 & -2/23 \\ -4/21 & -27/92 & -1/15 & 4/7 & -58/93 & -27/92 \\ -1/14 & -9/46 & -3/10 & -1/7 & 53/62 & -9/46 \\ \hline -2/21 & -9/92 & -2/15 & -2/7 & -7/62 & 83/92 \end{array} \right]. \tag{2.15}$$

Taking overlapping submatrices A and B as in (2.12) the 3-subdirect sum $C = A \oplus_3 B$ is given by

$$C = \left[\begin{array}{cc|ccc|c} 13/14 & -4/23 & -3/20 & -1/42 & -19/186 & 0 \\ -3/7 & 21/23 & -1/5 & -1/21 & -1/93 & 0 \\ \hline -1/7 & -7/46 & 17/10 & -1/7 & -1/93 & -2/23 \\ -4/21 & -27/92 & -2/15 & 8/7 & -116/93 & -27/92 \\ -1/14 & -9/46 & -3/5 & -2/7 & 53/31 & -9/46 \\ \hline 0 & 0 & -2/15 & -2/7 & -7/62 & 83/92 \end{array} \right]$$

and it is a nonsingular M -matrix according to Theorem 2.10. In fact, we have that

$$C^{-1} \approx \left[\begin{array}{cc|ccc|c} 1.3500 & 0.3977 & 0.2624 & 0.1609 & 0.2103 & 0.1232 \\ 0.7628 & 1.4108 & 0.3383 & 0.2085 & 0.2185 & 0.1478 \\ \hline 0.3007 & 0.2845 & 0.7422 & 0.2006 & 0.1824 & 0.1763 \\ 1.1024 & 1.1571 & 0.8927 & 1.6092 & 1.3118 & 0.8940 \\ 0.4854 & 0.5256 & 0.5116 & 0.4379 & 0.9664 & 0.4013 \\ \hline 0.4543 & 0.4743 & 0.4564 & 0.5941 & 0.5634 & 1.4679 \end{array} \right].$$

2.3. k -subdirect sum of p M -matrices. In this section we extend Theorems 2.3 and 2.10 to the subdirect sum of several nonsingular M -matrices. Example 2.14 later in the section illustrates the notation used in the proofs.

THEOREM 2.12. *Let $A_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \dots, p$, be nonsingular M -matrices partitioned as*

$$A_i = \begin{bmatrix} A_{i,11} & A_{i,12} \\ A_{i,21} & A_{i,22} \end{bmatrix} \tag{2.16}$$

with $A_{i,11}$ a square matrix of order $k_{i-1} \geq 1$ and $A_{i,22}$ a square matrix of order $k_i \geq 1$, i.e., $n_i = k_{i-1} + k_i$. Since A_i are nonsingular M -matrices we have that there exist $x_i > 0 \in \mathbb{R}^{(n_i - k_i) \times 1}$ and $y_i > 0 \in \mathbb{R}^{k_i \times 1}$ such that

$$A_i \begin{bmatrix} x_i \\ y_i \end{bmatrix} > 0, \quad i = 1, \dots, p. \quad (2.17)$$

Let $C_0 = A_1$ and define the following $p - 1$ k_i -subdirect sums:

$$C_i = C_{i-1} \oplus_{k_i} A_{i+1}, \quad i = 1, \dots, p - 1, \quad (2.18)$$

i.e.,

$$\begin{aligned} C_1 &= A_1 \oplus_{k_1} A_2, \\ C_2 &= (A_1 \oplus_{k_1} A_2) \oplus_{k_2} A_3 = C_1 \oplus_{k_2} A_3, \\ &\vdots \\ C_{p-1} &= (A_1 \oplus_{k_1} A_2 \oplus_{k_2} \dots \oplus_{k_{p-2}} A_{p-1}) \oplus_{k_{p-1}} A_p = C_{p-2} \oplus_{k_{p-1}} A_p. \end{aligned}$$

Each subdirect sum C_i is of order m_i , such that $m_0 = n_1$ and

$$m_i = m_{i-1} + n_{i+1} - k_i = m_{i-1} + k_{i+1}, \quad i = 1, \dots, p - 1.$$

Let us partition C_i in the form

$$C_i = \begin{bmatrix} C_{i,11} & C_{i,12} \\ C_{i,21} & C_{i,22} \end{bmatrix}, \quad i = 1, \dots, p - 1, \quad (2.19)$$

with $C_{i,22}$ a square matrix of order k_{i+1} . Let

$$H_i = C_{i-1,22} + A_{i+1,11}, \quad i = 1, \dots, p - 1,$$

be nonsingular M -matrices and let

$$z_i = H_i^{-1}(C_{i-1,22}y_i + A_{i+1,11}x_{i+1}), \quad i = 1, \dots, p - 1.$$

Then, if $z_i \leq y_i$ and $z_i \leq x_{i+1}$, the subdirect sums C_i given by (2.18) are nonsingular M -matrices for $i = 1, \dots, p - 1$.

Proof. It is easy to see that applying Theorem 2.3 to each consecutive pair of matrices C_i we have that C_1, C_2, \dots, C_{p-1} are nonsingular M -matrices. This can be shown by induction. \square

We now extend Theorem 2.10 to the sub-direct sum of p submatrices of a given nonsingular M -matrix M . To that end, we first define $M(S)$ a principal submatrix of M with rows and columns with indices in the set of indices $S = \{i, i + 1, i + 2, \dots, j\}$. In [2] we call these *consecutive principal submatrices*. For example, matrices A and B given by (2.12) can be expressed as a submatrices of M given by (2.11) as $A = M(S_1)$, $B = M(S_2)$ with $S_1 = \{1, 2\}$ and $S_2 = \{2, 3\}$.

THEOREM 2.13. *Let M be a nonsingular M -matrix. Let $A_i = M(S_i)$, $i = 1, \dots, p$, be principal consecutive submatrices of M and consider the $p - 1$ k_i -subdirect sums given by*

$$C_i = C_{i-1} \oplus_{k_i} A_{i+1}, \quad i = 1, \dots, p - 1, \tag{2.20}$$

in which $C_0 = A_1$. Then each of the k_i -subdirect sums C_i is a nonsingular M -matrix.

Proof. It is easy to relate the structure of each C_i to that of the submatrices A_i involved. We consider that A_i are overlapping principal submatrices of the form (2.12) but allowing that each A_i has different number of blocks. Let M be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots & M_{1n} \\ M_{21} & M_{22} & M_{23} & \cdots & M_{2n} \\ M_{31} & M_{32} & M_{33} & \cdots & M_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & M_{n3} & \cdots & M_{nn} \end{bmatrix} \tag{2.21}$$

according with the size of the principal submatrices A_i . Each block M_{ij} may be a submatrix of more than one A_m , $m = 1, \dots, p$. Let $b_{ij}^{(l)} \geq 0$ be the number of matrices A_m such that M_{ij} is a submatrix of A_m , for $m = 1, \dots, l + 1$. Of course we can have $b_{ij}^{(l)} = 0$. Let us consider the l th subdirect sum C_l , $1 \leq l \leq p - 1$, which is of the form

$$C_l = \begin{bmatrix} b_{11}^{(l)} M_{11} & b_{12}^{(l)} M_{12} & b_{13}^{(l)} M_{13} & \cdots & b_{1l}^{(l)} M_{1l} \\ b_{21}^{(l)} M_{21} & b_{22}^{(l)} M_{22} & b_{23}^{(l)} M_{23} & \cdots & b_{2l}^{(l)} M_{2l} \\ b_{31}^{(l)} M_{31} & b_{32}^{(l)} M_{32} & b_{33}^{(l)} M_{33} & \cdots & b_{3l}^{(l)} M_{3l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{l1}^{(l)} M_{l1} & b_{l2}^{(l)} M_{l2} & b_{l3}^{(l)} M_{l3} & \cdots & b_{ll}^{(l)} M_{ll} \end{bmatrix}. \tag{2.22}$$

Observe that C_l is a Z -matrix and that $b_{ii}^{(l)} > 0$. Furthermore, for each column it holds that $b_{ii}^{(l)} \geq b_{ji}^{(l)}$, $j = 1, \dots, l$.

The proof proceeds in a manner similar to that of Theorem 2.10. Consider the Z -matrix (partitioned in the same manner as M)

$$T_l = M_l \text{diag}(b_{11}^{(l)} I, b_{22}^{(l)} I, b_{33}^{(l)} I, \dots, b_{ll}^{(l)} I),$$

where M_l is the principal submatrix of (2.21) with row and column blocks from 1 to l . It follows that $T_l^{-1} \geq O$ and therefore T_l is a nonsingular M -matrix. Finally, since $C_l \geq T_l$, we conclude that C_l is a nonsingular M -matrix, $l = 1, \dots, p$. \square

EXAMPLE 2.14. Given the nonsingular M -matrix M of Example 2.11, let us consider the following overlapping blocks

$$A_1 = M(\{1, 2, 3\}) = \begin{bmatrix} 13/14 & -4/23 & -3/20 \\ -3/7 & 21/23 & -1/5 \\ -1/7 & -7/46 & 17/20 \end{bmatrix},$$

$$A_2 = M(\{2, 3, 4, 5\}) = \begin{bmatrix} 21/23 & -1/5 & -1/21 & -1/93 \\ -7/46 & 17/20 & -1/14 & -1/186 \\ -27/92 & -1/15 & 4/7 & -58/93 \\ -9/46 & -3/10 & -1/7 & 53/62 \end{bmatrix},$$

$$A_3 = M(\{3, 4, 5, 6\}) = \begin{bmatrix} 17/20 & -1/14 & -1/186 & -2/23 \\ -1/15 & 4/7 & -58/93 & -27/92 \\ -3/10 & -1/7 & 53/62 & -9/46 \\ -2/15 & -2/7 & -7/62 & 83/92 \end{bmatrix}.$$

Then we have the 2-subdirect sum

$$C_1 = A_1 \oplus_2 A_2 = \begin{bmatrix} 13/14 & -4/23 & -3/20 & 0 & 0 \\ -3/7 & 42/23 & -2/5 & -1/21 & -1/93 \\ -1/7 & -7/23 & 17/10 & -1/14 & -1/186 \\ 0 & -27/92 & -1/15 & 4/7 & -58/93 \\ 0 & -9/46 & -3/10 & -1/7 & 53/62 \end{bmatrix},$$

which is a nonsingular M -matrix, and the 3-subdirect sum

$$C_2 = C_1 \oplus_3 A_3 = \begin{bmatrix} 13/14 & -4/23 & -3/20 & 0 & 0 & 0 \\ -3/7 & 42/23 & -2/5 & -1/21 & -1/93 & 0 \\ -1/7 & -7/23 & 51/20 & -1/7 & -1/93 & -2/23 \\ 0 & -27/92 & -2/15 & 8/7 & -116/93 & -27/92 \\ 0 & -9/46 & -3/5 & -2/7 & 53/31 & -9/46 \\ 0 & 0 & -2/15 & -2/7 & -7/62 & 83/92 \end{bmatrix},$$

which is also a nonsingular M -matrix in accordance with Theorem 2.13. Observe that in this example we have $k_1 = 2$ and $k_2 = 3$. Note also that, for example, we have $b_{22}^{(1)} = 2$, $b_{33}^{(1)} = 2$, $b_{14}^{(1)} = 0$, $b_{22}^{(2)} = 2$, $b_{22}^{(3)} = 2$, $b_{33}^{(2)} = 3$, $b_{14}^{(2)} = 0$.

3. Subdirect sums of inverses. Let A and B be nonsingular matrices partitioned as in (2.1). In this section we consider the k -subdirect sum of their inverses. We will establish counterparts to some of results in the previous sections. Let us denote by $G = A^{-1} \oplus_k B^{-1}$, with A^{-1} and B^{-1} partitioned as in (2.3), i.e.,

$$G = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & 0 \\ \hat{A}_{21} & \hat{A}_{22} + \hat{B}_{11} & \hat{B}_{12} \\ 0 & \hat{B}_{21} & \hat{B}_{22} \end{bmatrix}. \quad (3.1)$$

As a corollary to, and in analogy to Theorem 2.1, the next statement indicates that the nonsingularity of $A_{22} + B_{11}$ is a necessary condition to obtain G nonsingular.

THEOREM 3.1. *Let A and B be nonsingular matrices partitioned as in (2.1) and let their inverses be partitioned as in (2.3). Let $G = A^{-1} \oplus_k B^{-1}$ partitioned as in (3.1) with $k \geq 1$. Then G is nonsingular if and only if $H = A_{22} + B_{11}$ is nonsingular.*

We remark that in analogy to the expression (2.5) of C^{-1} , the explicit form of G^{-1} is

$$G^{-1} = \begin{bmatrix} A_{11} - A_{12}H^{-1}A_{21} & A_{12} - A_{12}H^{-1}A_{22} & A_{12}H^{-1}B_{12} \\ B_{11}H^{-1}A_{21} & B_{11}H^{-1}A_{22} & -B_{11}H^{-1}B_{12} + B_{12} \\ B_{21}H^{-1}A_{21} & B_{21}H^{-1}A_{22} & -B_{21}H^{-1}B_{12} + B_{22} \end{bmatrix}. \quad (3.2)$$

COROLLARY 3.2. *When A and B are nonsingular M -matrices with the common block $A_{22} = B_{11}$ a square matrix of order k , i.e., of the form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} A_{22} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad (3.3)$$

then $H = 2A_{22}$ is nonsingular and therefore $G = A^{-1} \oplus_k B^{-1}$ is nonsingular.

We note that this is the case when A and B are overlapping submatrices of an M -matrix, i.e., of the form (2.12) and (2.11) considered in Section 2.2, where we were interested in the subdirect sum of A and B . Here we conclude that the subdirect sum of their inverses is always nonsingular.

EXAMPLE 3.3. Let A and B be the matrices of Example 2.11, then according to Corollary 3.2, the 3-subdirect sum of the inverses

$$G = A^{-1} \oplus_3 B^{-1} \approx \begin{bmatrix} 1.5033 & 0.5513 & 0.5547 & 0.2757 & 0.3912 & 0 \\ 0.9540 & 1.5996 & 0.7158 & 0.3635 & 0.4038 & 0 \\ 0.6004 & 0.5636 & 2.9750 & 0.8144 & 0.7407 & 0.3708 \\ 2.0383 & 2.1242 & 3.5729 & 6.5498 & 5.3372 & 2.0139 \\ 0.8953 & 0.9650 & 2.0470 & 1.8025 & 3.9062 & 0.9048 \\ 0 & 0 & 0.8551 & 1.3803 & 1.2652 & 1.9143 \end{bmatrix}$$

is a nonsingular matrix.

In the above example a direct computation shows that G^{-1} is not an M -matrix:

$$G^{-1} \approx \begin{bmatrix} 0.8900 & -0.2337 & -0.0750 & -0.0119 & -0.0511 & 0.0512 \\ -0.4682 & 0.8566 & -0.1000 & -0.0238 & -0.0054 & 0.0470 \\ -0.0714 & -0.0761 & 0.4250 & -0.0357 & -0.0027 & -0.0435 \\ -0.0952 & -0.1467 & -0.0333 & 0.2857 & -0.3118 & -0.1467 \\ -0.0357 & -0.0978 & -0.1500 & -0.0714 & 0.4274 & -0.0978 \\ 0.1242 & 0.2045 & -0.0667 & -0.1429 & -0.0565 & 0.7123 \end{bmatrix}$$

which is not a Z -matrix. Note that when A and B are M -matrices we have from (3.1) that $G = A^{-1} \oplus B^{-1}$ is nonnegative. Therefore assuming that G^{-1} exists we have $(G^{-1})^{-1} \geq O$. Then it is a natural question to seek conditions so that G^{-1} is a nonsingular M -matrix. We study this question next.

The expressions (3.1) of G and (3.2) of G^{-1} , Theorem 3.1, and the observation that for nonsingular M -matrices we have $(G^{-1})^{-1} \geq O$, imply the following result.

THEOREM 3.4. *Let A and B be nonsingular M -matrices partitioned as in (2.1) and their inverses partitioned as in (2.3). Let $G = A^{-1} \oplus_k B^{-1}$ with $k \geq 1$, and let $H = A_{22} + B_{11}$ be nonsingular. Then G^{-1} is a nonsingular M -matrix if and only if G^{-1} is a Z -matrix.*

COROLLARY 3.5. *Let A and B be lower and upper block triangular nonsingular M -matrices, respectively, partitioned as in (2.10) with A_{22} and B_{11} square matrices of order k and $H = A_{22} + B_{11}$ nonsingular. Then $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ is a nonsingular M -matrix if and only if the following conditions hold:*

- i) $B_{11}H^{-1}A_{21} \leq O$.
- ii) $B_{11}H^{-1}A_{22}$ is a Z -matrix.
- iii) $-B_{11}H^{-1}B_{12} + B_{12} \leq O$.

Proof. >From (3.2) and (2.10) we have that

$$G^{-1} = \begin{bmatrix} A_{11} & 0 & 0 \\ B_{11}H^{-1}A_{21} & B_{11}H^{-1}A_{22} & -B_{11}H^{-1}B_{12} + B_{12} \\ 0 & 0 & B_{22} \end{bmatrix} \quad (3.4)$$

and therefore G^{-1} is a Z -matrix if and only if the conditions i), ii) and iii) hold. \square

Conditions i), ii) and iii) in the corollary are not as stringent as they may appear. For example, let A and B be block triangular nonsingular M -matrices partitioned as in (2.10) with a common block $A_{22} = B_{11}$, a square matrix of order k , i.e.,

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_{22} & B_{12} \\ 0 & B_{22} \end{bmatrix}. \quad (3.5)$$

Then $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ is a nonsingular M -matrix, since we have from (3.4) that

$$G^{-1} = \begin{bmatrix} A_{11} & O & O \\ \frac{1}{2}A_{21} & \frac{1}{2}A_{22} & \frac{1}{2}B_{12} \\ O & O & B_{22} \end{bmatrix},$$

and therefore G^{-1} is a Z -matrix. In fact, in this case, we have

$$G = \begin{bmatrix} A_{11}^{-1} & O & O \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & 2A_{22}^{-1} & -A_{22}^{-1}B_{12}B_{22}^{-1} \\ O & O & B_{22}^{-1} \end{bmatrix} \geq O.$$

The next example illustrates this situation.

EXAMPLE 3.6. Let A and B be the matrices of Example 2.8, then

$$G = A^{-1} \oplus_2 B^{-1} = \left[\begin{array}{c|cc|c} 1/3 & 0 & 0 & 0 \\ \hline 1/8 & 49/80 & 21/80 & 9/20 \\ 7/24 & 77/80 & 73/80 & 11/10 \\ \hline 0 & 0 & 0 & 1/2 \end{array} \right],$$

and

$$G^{-1} = \left[\begin{array}{c|cc|c} 3 & 0 & 0 & 0 \\ \hline -18/49 & 146/49 & -6/7 & -39/49 \\ -4/7 & -22/7 & 2 & -11/7 \\ \hline 0 & 0 & 0 & 2 \end{array} \right]$$

is a nonsingular M -matrix in accordance with Corollary 3.5.

Note that if the hypotheses of Corollary 3.5 are satisfied, and recalling Theorem 2.6, we have that each of the matrices $C = A \oplus_k B$ and $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ are both nonsingular M -matrices.

4. P -matrices. A square matrix is a P -matrix if all its principal minors are positive. As a consequence we have that all the diagonal entries of a P -matrix are positive. It also follows that a nonsingular M -matrix is a P -matrix. It can also be shown that if A is a nonsingular M -matrix, then A^{-1} is a P -matrix; see, e.g., [5].

In [3] it is shown that the k -subdirect sum (with $k > 1$) of two P -matrices is not necessarily a P -matrix. Our results in Sections 2.1 and 3 hold for nonsingular M -matrices and inverses of M -matrices, respectively. As these two classes of matrices are subsets of P -matrices, it is natural to ask if similar sufficient conditions can be found so that the k -subdirect sum of P -matrices is a P -matrix. The following example indicates that the answer may not be easy to obtain, since even in the simplest case of diagonal submatrices the k -subdirect sum may not be a P -matrix.

EXAMPLE 4.1. Given the P -matrices

$$A = \left[\begin{array}{c|cc} 543 & 388 & 322 \\ \hline 69 & 160 & 0 \\ 368 & 0 & 375 \end{array} \right], \quad B = \left[\begin{array}{c|cc} 136 & 0 & 219 \\ \hline 0 & 225 & 159 \\ 61 & 177 & 230 \end{array} \right]$$

we have that the 2-subdirect sum

$$C = A \oplus_2 B = \left[\begin{array}{c|ccc} 543 & 388 & 322 & 0 \\ \hline 69 & 296 & 0 & 219 \\ 368 & 0 & 600 & 159 \\ \hline 0 & 61 & 177 & 230 \end{array} \right]$$

is not a P -matrix, since $\det(C) < 0$.

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