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Overlapping additive and multiplicative Schwarz iterations for H -matrices

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Abstract

In recent years, an algebraic framework was introduced for the analysis of convergence of Schwarz methods for the solution of linear systems of the form $Ax = b$. Within this framework, additive and multiplicative Schwarz were shown to converge when the coefficient matrix A is a nonsingular M -matrix, or a symmetric positive definite matrix. In this paper, many of these results are extended to the case of A being an H -matrix. The case of inexact local solves is also considered. In addition, the two-level scheme is studied, i.e., when a coarse grid correction is used in conjunction with the additive or the multiplicative Schwarz iterations.

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1. Introduction

We consider linear systems of equations of the form

$$Ax = b, \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$ is an H -matrix (and thus nonsingular; see, e.g., [5] and the references therein) and x and b are vectors of $V = \mathbb{R}^n$; we review some definitions later in this section. We study the solution of (1) by Schwarz iterations with p overlapping blocks. These are iterative methods originally developed for operators A arising from discretizations of partial differential equations (p.d.e.). In these cases, Schwarz iterations correspond to domain decomposition methods; see, e.g., [12,13]. Schwarz methods are most often used as preconditioners, but in some instances they are used as stationary iterative methods of the form

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \dots, \quad (2)$$

where x^0 is an initial guess, c is an appropriate vector, and T is the iteration matrix; see, e.g., [3,8], and the references given therein for such use. In our context, the convergence of the iteration (2), which holds for any initial vector x^0 if and only if $\rho = \rho(T) < 1$ ($\rho(T)$ denoting the spectral radius; see, e.g., [2]), indicates that the spectrum of the preconditioned matrix $I - T$ ($= B^{-1}A$ for some nonsingular matrix B), i.e., its set of eigenvalues, is contained in a ball centered at one with radius ρ .

In the rest of this section we review the additive and multiplicative iterations, each corresponding to a different matrix T in (2). Our exposition is based on the algebraic formulation presented in [6] and [1]. In these references, convergence of the Schwarz iterations was studied when the matrix A in (1) is either a nonsingular M -matrix or symmetric positive definite. In this paper, in Sections 2 and 3 we study the convergence of these iterations when A is an H -matrix. The analysis for the case of inexact local solves is also included. In Section 4 we extend the convergence results to the two-level iterations, i.e., when a “coarse grid” correction is used.

A nonsingular matrix A having all nonpositive off-diagonal entries is called an M -matrix if the inverse is (entry-wise) nonnegative, i.e., $A^{-1} \geq O$; see, e.g., [2,11]. For any matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, its *comparison* matrix $\langle A \rangle = (\alpha_{ij})$ is defined by

$$\alpha_{ii} = |a_{ii}|, \quad \alpha_{ij} = -|a_{ij}|, \quad i \neq j.$$

A matrix A is said to be an H -matrix if $\langle A \rangle$ is an M -matrix. In particular, A is an H -matrix if and only if it is generalized diagonally dominant, i.e.,

$$|a_{ii}|u_i > \sum_{i \neq j} |a_{ij}|u_j, \quad i = 1, \dots, n \quad (3)$$

for some positive vector $u = (u_1, \dots, u_n)^T$. H -matrices were introduced in [11] as generalization of M -matrices. They appear in many applications, e.g., when discretizing certain nonlinear parabolic operators using high order finite elements and sufficiently small time steps [4]. The characterization (3) also indicates how general these matrices are; see further [15].

As we have mentioned, in this paper we study the solution of (1) when A is an H -matrix, using Schwarz iterative methods. These are block iterative methods in which

the blocks are overlapping, i.e., some variables are common to more than one block. When there is no overlap, additive Schwarz reduces to block Jacobi, while multiplicative Schwarz reduces to block Gauss–Seidel. To formally describe the overlapping blocks, let V_i be subspaces of $V = \mathbb{R}^n$ of dimension n_i , $i = 1, \dots, p$, ($p > 1$) such that the sum of these subspaces span the whole space. These subspaces are not pairwise disjoint; on the contrary, their intersection is precisely the overlap, and thus $\sum_{i=1}^p n_i > n$.

The restriction and prolongation operators map vectors from V to V_i and vice-versa. The restriction operators used here are of the form

$$R_i = [I_i | O]\pi_i, \quad i = 1, \dots, p, \quad (4)$$

where I_i is the identity on \mathbb{R}^{n_i} , and π_i is a permutation matrix on \mathbb{R}^n . The prolongation operator considered here is R_i^T . We now define the following matrices

$$E_i = R_i^T R_i, \quad i = 1, \dots, p. \quad (5)$$

Note that the diagonal matrices E_i given by (5) have nonzero diagonal elements (with value one) only in the columns which have a nonzero element the matrix R_i . We denote by q the *measure of overlap*, i.e., the maximum over all possible rows, of the number of matrices E_i with a nonzero in the row. Thus

$$\sum_{i=1}^p E_i \leq qI; \quad (6)$$

and usually $q \ll p$.

The restriction of the matrix A to the subspace V_i is

$$A_i = R_i A R_i^T, \quad (7)$$

which is a symmetric permutation of an $n_i \times n_i$ principal submatrix of A , $i = 1, \dots, p$. These are precisely the p overlapping blocks; see [6] for more details.

We are ready to describe the Schwarz iterations for the solution of (1). The damped additive Schwarz iteration has the following form

$$x^{k+1} = x^k + \theta \sum_{i=1}^p R_i^T A_i^{-1} R_i (b - Ax^k), \quad (8)$$

where $0 < \theta \leq 1$ is the *damping factor*.

In practical implementations of the additive Schwarz iterations (8), for each iteration, the residual vector $r^k = b - Ax^k$ is restricted to each subspace V_i using the operator R_i . Then, the local problem

$$A_i e_i = r_i^k = R_i r^k \quad (9)$$

is solved, the obtained local errors, $e_i^k = e_i$, are prolonged, summing them (and damping its value) and finally that correction is added to x^k yielding the new approximation vector x^{k+1} .

It follows that the process (8) is (2) with the iteration matrix

$$T_\theta = I - \theta \sum_{i=1}^p R_i^T A_i^{-1} R_i A = I - \theta \sum_{i=1}^p P_i, \quad (10)$$

where $P_i = R_i^T A_i^{-1} R_i A$ is a projection onto V_i . In the case of A symmetric and positive definite, this projection is orthogonal with respect to the A -inner product; see, e.g., [13].

The iteration matrix for the multiplicative Schwarz iterations is

$$T_\mu = (I - P_p)(I - P_{p-1}) \cdots (I - P_1) = \prod_{i=p}^1 (I - P_i). \quad (11)$$

In contrast to the additive Schwarz iteration (8), here, the correction in each subspace is followed by another correction, until all corrections have been made.

When the local problem (9) is not solved exactly, but only approximately, its solution can be considered the exact solution of another approximate local problem, namely $\tilde{A}_i e_i = r_i^k$. The matrix \tilde{A}_i could be, for example, an incomplete factorization of A_i ; see, e.g., [13]. In this case, the iteration matrix for the damped additive Schwarz iteration with inexact local solves is

$$\tilde{T}_\theta = I - \theta \sum_{i=1}^p R_i^T \tilde{A}_i^{-1} R_i A = I - \theta \sum_{i=1}^p \tilde{P}_i, \quad (12)$$

where $\tilde{P}_i = R_i^T \tilde{A}_i^{-1} R_i A$.

Similarly, the iteration matrix for the multiplicative Schwarz iterations with inexact local solves is

$$\tilde{T}_\mu = (I - \tilde{P}_p)(I - \tilde{P}_{p-1}) \cdots (I - \tilde{P}_1) = \prod_{i=p}^1 (I - \tilde{P}_i). \quad (13)$$

We proceed in the next sections to study the convergence of the iterations (10)–(13).

2. Convergence of additive Schwarz iterations

We begin by establishing a different algebraic representation of the iteration matrix (10). Given a matrix $A = (a_{ij})$, we define the matrix $|A| = (|a_{ij}|)$. It follows that $|A| \geq O$ and that $|AB| \leq |A||B|$ for any two matrices A and B of compatible size. Let

$$A_{-i} = [O | I_{-i}] \pi_i A \pi_i^T [O | I_{-i}]^T, \quad (14)$$

where I_{-i} is the $(n - n_i) \times (n - n_i)$ identity matrix, and let

$$M_i = \pi_i^T \begin{bmatrix} A_i & O \\ O & H_{-i} \end{bmatrix} \pi_i, \quad (15)$$

where H_{-i} is some $(n - n_i) \times (n - n_i)$ nonsingular matrix such that

$$|H_{-i} - A_{-i}| = \langle H_{-i} \rangle - \langle A_{-i} \rangle. \quad (16)$$

In fact, this condition gives us a lot of freedom in choosing H_{-i} . In [1,6], different choices were $H_{-i} = A_{-i}$ or $H_{-i} = D_{-i} = \text{diag } A_{-i}$. These choices clearly satisfy our condition (16).

It follows then from the form of the matrices (5) and (15) that

$$E_i M_i^{-1} = R_i^T A_i^{-1} R_i. \quad (17)$$

Using this equality, the iteration matrix T_θ can be expressed as

$$T_\theta = I - \theta \sum_{i=1}^p E_i M_i^{-1} A. \quad (18)$$

Our proof of convergence consists of showing that if $\theta \leq 1/q$, then $\rho(T_\theta) < 1$; cf. [6]. Our strategy is to show that $|T_\theta| \leq \widehat{T}_\theta$, for the matrix

$$\widehat{T}_\theta = I - \theta \sum_{i=1}^p R_i^T \langle A \rangle_i^{-1} R_i \langle A \rangle, \quad (19)$$

which we show in the next subsection to be nonnegative and to have spectral radius less than one.

2.1. Properties of \widehat{T}_θ

Let us consider the following linear system associated with the original problem (1),

$$\langle A \rangle x = b, \quad (20)$$

and apply the additive Schwarz iterations with the *same* p overlapping blocks that we considered in Section 1. Then, given an initial approximation x^0 for the solution of (1), the damped additive Schwarz iteration applied to (20) reads, for $k = 0, 1, \dots$,

$$x^{k+1} = x^k + \theta \sum_{i=1}^p R_i^T \langle A \rangle_i^{-1} R_i (b - \langle A \rangle x^k), \quad (21)$$

where the R_i are given by (4), and in a way similar to (7),

$$\langle A \rangle_i = R_i \langle A \rangle R_i^T,$$

and the iteration matrix for this scheme is precisely (19).

It is not hard to see that $\langle A \rangle_i = \langle A_i \rangle$, and since any principal submatrix of an M -matrix is also an M -matrix [2,11], we have the following useful result.

Lemma 1. *If $A \in \mathbb{R}^{n \times n}$ is an H -matrix, then any principal submatrix of A , and any symmetric permutation of it is an H -matrix. In particular the matrix A_i given by (7) is an H -matrix.*

We note that the coefficient matrix of (20) is an M -matrix, and therefore, we can use the results in [6]. In particular, we have that

$$\widehat{B} = \sum_{i=1}^p R_i^T \langle A_i \rangle^{-1} R_i = \sum_{i=1}^p E_i \langle M_i \rangle^{-1}$$

is nonnegative and nonsingular, where E_i is given by (5). We also have that the iteration matrix (19) can be written as

$$\widehat{T}_\theta = I - \theta \sum_{i=1}^p E_i \langle M_i \rangle^{-1} \langle A \rangle \quad (22)$$

and that if $\theta \leq 1/q$, $\widehat{T}_\theta \geq O$, and the damped additive Schwarz iteration (21) converges to the solution of (20) for any initial vector x^0 . Therefore we have

$$\rho(\widehat{T}_\theta) < 1. \quad (23)$$

2.2. Convergence for H -matrices

Before proceeding with the convergence analysis of (21) we prove an important result concerning the matrices M_i defined in (15). A splitting $A = M_i - N_i$ is called regular if $M_i^{-1} \geq O$ and $N_i \geq O$ [14]; it is called H -compatible if $\langle A \rangle = \langle M_i \rangle - |N_i|$; see [5].

Theorem 1. *Let $A \in \mathbb{R}^{n \times n}$ be an H -matrix and let the matrices M_i be of the form (15), satisfying (16). Then, $A = M_i - N_i$, $i = 1, \dots, p$, are H -compatible splittings.*

Proof. First, from the definition of $\langle A \rangle$ notice that $\langle \pi_i^T A \pi_i \rangle = \pi_i^T \langle A \rangle \pi_i$ since the diagonal of the matrix $\pi_i^T A \pi_i$ is a permutation of the diagonal of A . Let A be written by blocks as

$$A = \pi_i^T \begin{bmatrix} A_i & U \\ V & A_{-i} \end{bmatrix} \pi_i,$$

where U and V are matrices of appropriate size.

Then, we have that

$$\langle A \rangle = \pi_i^T \begin{bmatrix} \langle A_i \rangle & -|U| \\ -|V| & \langle A_{-i} \rangle \end{bmatrix} \pi_i \quad (24)$$

and

$$\langle M_i \rangle = \pi_i^T \begin{bmatrix} \langle A_i \rangle & O \\ O & \langle H_{-i} \rangle \end{bmatrix} \pi_i.$$

From these expressions we may write, using (16)

$$\langle M_i \rangle - \langle A \rangle = \pi_i^T \begin{bmatrix} \langle A_i \rangle & O \\ O & \langle H_{-i} \rangle \end{bmatrix} \pi_i - \pi_i^T \begin{bmatrix} \langle A_i \rangle & -|U| \\ -|V| & \langle A_{-i} \rangle \end{bmatrix} \pi_i$$

$$= \pi_i^T \begin{bmatrix} O & |U| \\ |V| & |H_{-i} - A_{-i}| \end{bmatrix} \pi_i.$$

Consider now the splittings $A = M_i - N_i, i = 1, \dots, p$. Then,

$$\begin{aligned} N_i = M_i - A &= \pi_i^T \begin{bmatrix} A_i & O \\ O & H_{-i} \end{bmatrix} \pi_i - \pi_i^T \begin{bmatrix} A_i & U \\ V & A_{-i} \end{bmatrix} \pi_i \\ &= \pi_i^T \begin{bmatrix} O & -U \\ -V & H_{-i} - A_{-i} \end{bmatrix} \pi_i. \end{aligned}$$

Hence, we have that $|N_i| = \langle M_i \rangle - \langle A \rangle$ and the proof is complete. \square

We are ready now to prove the following convergence result.

Theorem 2. *Let $A \in \mathbb{R}^{n \times n}$ be an H -matrix. Let the matrices R_i be of the form (4). Then, if $\theta \leq 1/q$, the damped additive Schwarz iteration (8) converges to the solution of (1) for any initial vector x^0 .*

Proof. We first show that

$$|T_\theta| \leq \widehat{T}_\theta. \tag{25}$$

From the expressions (22) and (18) we have

$$\widehat{T}_\theta - T_\theta = \theta \sum_{i=1}^p E_i \left[M_i^{-1} A - \langle M_i \rangle^{-1} \langle A \rangle \right]$$

and applying Theorem 1, we have

$$\widehat{T}_\theta - T_\theta = \theta \sum_{i=1}^p E_i \left[\langle M_i \rangle^{-1} |N_i| - M_i^{-1} N_i \right]. \tag{26}$$

Let us recall that if A is an H -matrix, then $|A^{-1}| \leq \langle A \rangle^{-1}$ [11]. Then, the right hand side of (26) is nonnegative, since

$$M_i^{-1} N_i \leq |M_i^{-1} N_i| \leq |M_i^{-1}| |N_i| \leq \langle M_i \rangle^{-1} |N_i| \tag{27}$$

and thus

$$T_\theta \leq \widehat{T}_\theta. \tag{28}$$

Now, from the expressions (22) and (18), we also have

$$\widehat{T}_\theta + T_\theta = 2I - \theta \sum_{i=1}^p E_i \left[\langle M_i \rangle^{-1} \langle A \rangle + M_i^{-1} A \right],$$

using again Theorem 1 and simplifying, we have

$$\widehat{T}_\theta + T_\theta = 2 \left[I - \theta \sum_{i=1}^p E_i \right] + \theta \sum_{i=1}^p E_i \left[\langle M_i \rangle^{-1} |N_i| + M_i^{-1} N_i \right]. \tag{29}$$

Now, from (6), if $\theta \leq 1/q$, we have

$$I \geq \theta q I \geq \theta \sum_{i=1}^p E_i,$$

and then the first term of the right hand side of (29) is nonnegative. Moreover, from (27), the second term of the right hand side of (29) is also nonnegative. Therefore, we have that if $\theta \leq 1/q$, then

$$-\widehat{T}_\theta \leq T_\theta. \quad (30)$$

Combining (28) and (30) we have the desired result (25).

To conclude the proof, we recall that if $A, B \in \mathbb{R}^{n \times n}$ and $|A| \leq B$ then $\rho(A) \leq \rho(B)$; see, e.g., [10, 2.4.9]. Applying this to (25) we have, using (23), that if $\theta \leq 1/q$, then $\rho(T_\theta) \leq \rho(|T_\theta|) \leq \rho(\widehat{T}_\theta) < 1$. \square

2.3. Inexact local solves

Given the matrices $\tilde{A}_i, i = 1, \dots, p$, representing the inexact local solves, for the convergence analysis one considers the matrices

$$\tilde{M}_i = \pi_i^T \begin{bmatrix} \tilde{A}_i & O \\ O & H_{-i} \end{bmatrix} \pi_i, \quad (31)$$

cf. (15). We assume as before that H_{-i} satisfies (16). As in (17) and (18), we have now $E_i \tilde{M}_i^{-1} = R_i^T \tilde{A}_i^{-1} R_i, i = 1, \dots, p$, and $\tilde{T}_\theta = I - \theta \sum_{i=1}^p E_i \tilde{M}_i^{-1} A$.

The conditions we impose on the local solves to guarantee convergence of the additive Schwarz methods are the following:

$$\langle \tilde{A}_i \rangle^{-1} \geq O \quad \text{and} \quad (32)$$

$$|\tilde{A}_i - A_i| = \langle \tilde{A}_i \rangle - \langle A_i \rangle, \quad i = 1, \dots, p. \quad (33)$$

We note that condition (32) is satisfied automatically if \tilde{A}_i is an H -matrix. This occurs, e.g., if \tilde{A}_i is an incomplete factorization of A_i [9,14]. Condition (33) is equivalent to having the splitting $A_i = \tilde{A}_i - (\tilde{A}_i - A_i)$ be H -compatible. Note also that under the conditions (32)–(33), since we have that $(\langle \tilde{A}_i \rangle - \langle A_i \rangle) \geq O$, we conclude that $\langle A_i \rangle = \langle \tilde{A}_i \rangle - (\langle \tilde{A}_i \rangle - \langle A_i \rangle)$ is a regular splitting. These conditions also provide us with the counterpart to Theorem 1. Its proof is analogous, and therefore it is omitted.

Theorem 3. *Let A be an H -matrix and let the matrices \tilde{M}_i be defined as in (31). Assume further that the inexact solves are such that the condition (33) holds. Then, the splittings $A = \tilde{M}_i - \tilde{N}_i, i = 1, \dots, p$, are H -compatible splittings.*

The following counterpart to the result (23) is obtained by applying [6, Theorem 3.5] to the M -matrix $\langle A \rangle$.

Theorem 4. Let A be an H -matrix and let the inexact solves be such that (32) holds and that $\langle \tilde{A}_i \rangle - \langle A_i \rangle \geq O$, $i = 1, \dots, p$, which hold if one imposes the condition (33) as well. Then, if $\theta \leq 1/q$, the damped additive Schwarz iteration with inexact local solves, defined by (2) with the iteration matrix

$$\widehat{T}_\theta = I - \theta \sum_{i=1}^p R_i^T \langle \tilde{A}_i \rangle^{-1} R_i \langle A \rangle$$

converges to the solution of $\langle A \rangle x = b$ for any initial vector x^0 , i.e., we have that $\rho(\widehat{T}_\theta) < 1$.

We are now ready to show the convergence of the damped additive Schwarz iteration matrix with inexact local solves for H -matrices.

Theorem 5. Let A be an H -matrix and let the inexact solves be such that the conditions (32)–(33) hold. Then, if $\theta \leq 1/q$, the damped additive Schwarz iteration with inexact local solves, defined by (2) with the iteration matrix (12) converges to the solution of (1) for any initial vector x^0 .

Proof. The proof is analogous to that of Theorem 2. Using the same techniques one shows that $|\tilde{T}_\theta| \leq \widehat{T}_\theta$, and using Theorem 4, we conclude that $\rho(\tilde{T}_\theta) \leq \rho(\widehat{T}_\theta) < 1$. \square

3. Multiplicative Schwarz iterations

We first observe that using (17), the iteration matrix for multiplicative Schwarz T_μ given in (11) can be written as

$$T_\mu = \prod_{i=p}^1 (I - R_i^T A_i^{-1} R_i A) = \prod_{i=p}^1 (I - E_i M_i^{-1} A). \quad (34)$$

As was the case for additive Schwarz, we consider the solution of the linear system (20) by multiplicative Schwarz iterations with the *same* blocks as for the system (1). Then, the new iteration matrix for the multiplicative Schwarz iteration is now

$$\widehat{T}_\mu = \prod_{i=p}^1 (I - R_i^T \langle A \rangle_i^{-1} R_i \langle A \rangle).$$

Using the results of [1] applied to the system (20) (whose coefficient matrix is an M -matrix), we have that

$$\widehat{T}_\mu = \prod_{i=p}^1 (I - E_i \langle M_i \rangle^{-1} \langle A \rangle) \geq O.$$

and that $\rho(\widehat{T}_\mu) < 1$.

Furthermore, using Theorem 1, we have that

$$\widehat{T}_\mu = \prod_{i=p}^1 (I - E_i + E_i \langle M_i \rangle^{-1} |N_i|). \quad (35)$$

We are ready to prove the convergence of multiplicative Schwarz iterations for H -matrices.

Theorem 6. *Let $A \in \mathbb{R}^{n \times n}$ be an H -matrix. Let the matrices R_i be of the form (4). Then, the multiplicative Schwarz iteration (2) with iteration matrix (11) converges to the solution of (1) for any initial vector x^0 .*

Proof. As in the proof of Theorem 2, to prove that $\rho(T_\mu) < 1$ we show that $|T_\mu| \leq \widehat{T}_\mu$. To that end, we bound

$$\begin{aligned} |I - E_i + E_i M_i^{-1} N_i| &\leq |I - E_i| + |E_i M_i^{-1} N_i| \\ &= |I - E_i| + |E_i M_i^{-1} N_i| \\ &\leq |I - E_i + E_i \langle M_i \rangle^{-1} |N_i||, \end{aligned} \quad (36)$$

where the last inequality follows from

$$\begin{aligned} |E_i M_i^{-1} N_i| &\leq |E_i| |M_i^{-1}| |N_i| \\ &\leq |E_i| \langle M_i \rangle^{-1} |N_i| = E_i \langle M_i \rangle^{-1} |N_i|. \end{aligned}$$

From (34), using that $A = M_i - N_i$, we have

$$\begin{aligned} |T_\mu| &= \left| \prod_{i=p}^1 (I - E_i M_i^{-1} A) \right| \leq \prod_{i=p}^1 |I - E_i M_i^{-1} A| \\ &= \prod_{i=p}^1 |I - E_i + E_i M_i^{-1} N_i|, \end{aligned}$$

and using 36 and (35) we conclude that

$$|T_\mu| \leq \prod_{i=p}^1 (I - E_i + E_i \langle M_i \rangle^{-1} |N_i|) = \widehat{T}_\mu. \quad (37)$$

The convergence of the multiplicative Schwarz iteration is proved. \square

3.1. Inexact local solves

In order to analyze the convergence of the multiplicative Schwarz iteration with inexact local solves, we first consider the solution of the auxiliary system (20), and apply [6, Theorem 4.5] to it. We thus obtain the following result.

Theorem 7. Let A be an H -matrix. Assume that $\langle \tilde{A}_i \rangle - \langle A_i \rangle \geq O$, $i = 1, \dots, p$, which hold if one imposes condition (33). Then the multiplicative Schwarz iteration matrix with inexact local solves, defined by (2) with the iteration matrix

$$\widehat{T}_\mu = \prod_{i=p}^1 (I - E_i \langle \tilde{M}_i \rangle^{-1} \langle A \rangle)$$

converges to the solution of $\langle A \rangle x = b$ for any choice of the initial vector x^0 , i.e., we have that $\rho(\widehat{T}_\mu) < 1$.

We proceed as in Theorem 6, using Theorem 3 one can prove that $|\tilde{T}_\mu| \leq \widehat{T}_\mu$ implying that $\rho(\tilde{T}_\mu) \leq \rho(\widehat{T}_\mu) < 1$. We have then the following convergence result.

Theorem 8. Let A be an H -matrix and let the inexact solves be such that condition (33) holds. Then, the multiplicative Schwarz iteration with inexact local solves, defined by (2) with the iteration matrix (13) converges to the solution of (1) for any initial vector x^0 .

4. Coarse grid corrections for H -matrices

In this section, we study the convergence of the additive and multiplicative Schwarz iterations when a coarse grid correction is applied. We follow the structure used in [1,6]. The coarse grid is represented algebraically by an additional subspace V_0 of dimension n_0 , with $p \leq n_0 < n$. For this subspace, we define a restriction operator R_0 as before with (4) ($i = 0$), A_0 as in (7), and E_0 as in (5), implying

$$O \leq E_0 \leq I. \quad (38)$$

We also define the corresponding matrix M_0 as in (15), and thus, by Theorem 1, the splitting $A = M_0 - N_0$ is also H -compatible. In other words, Theorem 1 holds for $i = 0, 1, \dots, p$, and this is how we use it throughout this section.

If the coarse grid equation $A_0 e_0 = r_0$, is solved approximately, we defined the matrix \tilde{A}_0 so that the approximation solves exactly $\tilde{A}_0 e_0 = r_0$. We assume that the conditions (32)–(33) hold (for $i = 0$). One can then define a matrix \tilde{M}_0 as in (31). The analysis of the two-level methods presented in the sequel can be applied to the case of inexact local solves, and/or inexact corrections using the same techniques. We omit the details.

4.1. Multiplicative Schwarz with multiplicative correction

Following [1], let the iteration matrix of the multiplicative Schwarz iteration with a multiplicative coarse grid correction be

$$H_\mu = (I - G_0A)T_\mu, \quad (39)$$

where $G_0 = R_0^T A_0^{-1} R_0$. It follows that (17) holds [1], i.e.,

$$E_0 M_0^{-1} A = G_0 A. \quad (40)$$

Let us consider the linear system (20). Applying the results of [1] to the M -matrix $\langle A \rangle$, we have that the iteration matrix

$$\widehat{H}_\mu = (I - \widehat{G}_0 \langle A \rangle) \widehat{T}_\mu \quad (41)$$

is nonnegative and has spectral radius less than one, where

$$\widehat{G}_0 = R_0^T \langle A \rangle_0^{-1} R_0 = R_0^T \langle A_0 \rangle^{-1} R_0 = E_0 \langle M_0 \rangle^{-1}.$$

By Theorem 1, we know that $\langle A \rangle = \langle M_0 \rangle - |N_0|$ and thus we can rewrite the matrix (41) as

$$\widehat{H}_\mu = (I - E_0 + E_0 \langle M_0 \rangle^{-1} |N_0|) \widehat{T}_\mu.$$

Theorem 9. *Let $A \in \mathbb{R}^{n \times n}$ be an H -matrix and consider the solution of (1) by multiplicative Schwarz iterations with multiplicative correction, i.e., the iteration (2) with $T = H_\mu$ as in (39). Then, the iterations converge to the solution, for any initial vector x^0 .*

Proof. We can rewrite the iteration matrix (39) using (40) and Theorem 1 as

$$H_\mu = (I - E_0 + E_0 M_0^{-1} N_0) T_\mu. \quad (42)$$

As in the proof of Theorem 6, we have

$$|I - E_0 + E_0 M_0^{-1} N_0| \leq I - E_0 + E_0 \langle M_0 \rangle^{-1} |N_0|. \quad (43)$$

From (42) and (43), we have that

$$\begin{aligned} |H_\mu| &= |(I - E_0 + E_0 M_0^{-1} N_0) T_\mu| \leq |I - E_0 + E_0 M_0^{-1} N_0| |T_\mu| \\ &\leq (I - E_0 + E_0 \langle M_0 \rangle^{-1} |N_0|) |T_\mu|. \end{aligned}$$

We use now (37), which says that $|T_\mu| \leq \widehat{T}_\mu$, then we have that

$$(I - E_0 + E_0 \langle M_0 \rangle^{-1} |N_0|) |T_\mu| \leq (I - E_0 + E_0 \langle M_0 \rangle^{-1} |N_0|) \widehat{T}_\mu = \widehat{H}_\mu.$$

Thus, $|H_\mu| \leq \widehat{H}_\mu$ and therefore $\rho(H_\mu) \leq \rho(\widehat{H}_\mu) < 1$. \square

4.2. Additive Schwarz with multiplicative correction

The multiplicative correction $(I - G_0A)$ applied to the additive Schwarz iterations gives rise to the following iteration matrix [1]

$$H_\theta = (I - G_0A)T_\theta. \quad (44)$$

As it can be appreciated, the structure of this iteration matrix is the same as that of (39). Using the same logic as in the previous subsection, and using the fact that $|T_\theta| \leq \widehat{T}_\theta$, one can compare (44) with $\widehat{H}_\theta = (I - \widehat{G}_0(A))\widehat{T}_\theta$, the iteration matrix of additive Schwarz iterations for the system (20), for which $\rho(\widehat{H}_\theta) < 1$ if $\theta \leq 1/q$; see [1]. Thus, we have the counterpart to Theorem 9.

Theorem 10. *Let $A \in \mathbb{R}^{n \times n}$ be an H -matrix and consider the solution of (1) by additive Schwarz iterations with multiplicative correction, i.e., the iteration (2) with $T = H_\theta$ as in (44). Then, if $\theta \leq 1/q$, the iterations converge to the solution for any initial vector x^0 .*

We remark that unlike the situation in the M -matrix case [1], the coarse grid correction not always leads to a decrease of the spectral radius of the iteration matrix. The following simple example illustrates this situation. Consider the following H -matrix

$$A = \begin{bmatrix} 14/15 & 1/15 & 7/90 & 1/20 \\ 1/5 & 4/5 & 7/45 & 3/20 \\ 1/30 & 4/15 & 29/30 & 3/10 \\ 1/10 & 1/45 & 1/10 & 89/90 \end{bmatrix},$$

and let

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the coarse grid correction given by

$$R_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The spectral radii of the iteration operators are

$$\begin{aligned} \rho(T_\mu) &\approx 0.0004, & \rho(H_\mu) &\approx 0.0028, \\ \rho(T_\theta) &\approx 0.6667, & \rho(H_\theta) &\approx 0.6674 \end{aligned}$$

for the value $\theta = 1/3$.

On the other hand, for the case of A symmetric positive definite, one can say the following. It holds that $I - G_0A$ is an orthogonal projection onto V_0 using the A -inner product, and thus $\|I - G_0A\|_A = 1$. Therefore $\|H_\mu\|_A = \|(I - G_0A)T_\mu\|_A \leq \|T_\mu\|_A$ and $\|H_\theta\|_A = \|(I - G_0A)T_\theta\|_A \leq \|T_\theta\|_A$.

4.3. Additive Schwarz with additive coarse grid correction

Having a coarse grid correction additively consists of having an extra term in (10) corresponding to the correction in the subspace V_0 . The corresponding iteration matrix is thus

$$\bar{T}_\theta = I - \theta \sum_{i=0}^p R_i^T A_i^{-1} R_i A. \quad (45)$$

In [6], this iteration was shown to be convergent for systems where the coefficient is an M -matrix, and $\theta \leq 1/(q+1)$. Thus, if we consider the solution of system (20) by the additive Schwarz with additive coarse grid correction, we have that the matrix

$$\widehat{T}_\theta = I - \theta \sum_{i=0}^p R_i^T \langle A \rangle_i^{-1} R_i \langle A \rangle$$

is nonnegative and has spectral radius less than one, if $\theta \leq 1/(q+1)$. Using Theorem 1 (for $i = 0, 1, \dots, p$) we can rewrite it as

$$\widehat{T}_\theta = I - \theta \sum_{i=0}^p (E_i - E_i \langle M_i \rangle^{-1} |N_i|) \geq O.$$

Theorem 11. *Let $A \in \mathbb{R}^{n \times n}$ be an H -matrix. Then, if $\theta \leq 1/(q+1)$, the damped additive Schwarz iteration with additive correction defined by the iteration matrix \bar{T}_θ of (45) converges to the solution of (1) for any initial vector x^0 .*

Proof. The proof is analogous to that of Theorem 2. On one side we have that

$$\widehat{T}_\theta - \bar{T}_\theta = \theta \sum_{i=0}^p (\langle M_i \rangle^{-1} |N_i| - M_i^{-1} N_i) \geq O.$$

On the other, we write

$$\widehat{T}_\theta + \bar{T}_\theta = 2 \left[I - \theta \sum_{i=0}^p E_i \right] + \theta \sum_{i=0}^p E_i \left[\langle M_i \rangle^{-1} |N_i| + M_i^{-1} N_i \right],$$

which is nonnegative since from (6) and (38), for $\theta \leq 1/(q+1)$,

$$I \geq \theta(q+1)I \geq \theta \sum_{i=0}^p E_i.$$

Then, we conclude that $|\bar{T}_\theta| \leq \widehat{T}_\theta$ and thus $\rho(\bar{T}_\theta) \leq \rho(\widehat{T}_\theta) < 1$. \square

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