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Canonical forms for positive discrete-time linear control systems[☆]

Rafael Bru*, Sergio Romero, Elena Sánchez

Dept. de Matemàtica Aplicada, ETSIA, Universitat Politècnica de València, Cami de Vera, s/n, 46022, València, Spain

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Abstract

In this paper, the properties of reachability, controllability and essential reachability of positive discrete-time linear control systems are studied. These properties are characterized in terms of the directed graph of the state matrix. From these characterizations canonical forms of those properties are deduced. © 2000 Published by Elsevier Science Inc. All rights reserved.

Keywords: Positive control systems; Canonical forms; Controllability; Reachability

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1. Introduction and preliminaries

In this paper, we will deal with positive discrete-time linear control systems in the state-space model, i.e., systems, whose states and inputs are nonnegative. The nonnegativity condition yields a different treatment of these control systems based upon the theory of nonnegative matrices. The nonnegative systems appear in many different real situations such as in economics, biological, enviromental and chemical processes, among others.

Many authors have studied different problems concerning positive systems. The invariant case has been studied by Ohta et al. [8], Coxson and Shapiro [5], van den Hof [10], Farina [7] and all references therein. Other researchers such as Bru and Hernández [2] and Bru et al. [3] deal with the positive periodic case.

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* Corresponding author. Tel.: +34 96 387 7660; fax: +34 96 387 7669.
E-mail address: rbru@mat.upv.es (R. Bru).

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Recently, Valcher [9] has provided reachability and controllability criteria and studied the canonical forms of reachable and controllable positive discrete-time systems. The approach used in [9] is combinatorial and the main tool is the notion of *deterministic path* in the directed graph of a state matrix. The aim of our work consists mainly of obtaining characterizations of the reachability and controllability properties of positive invariant systems by partitioning the set of vertices of the directed graph of the state matrix in different kinds of deterministic paths (Section 2). These characterizations provide canonical forms yielding a refinement of those given in [9]. Such a refinement allows us to determine positive and zero entries of those canonical forms and so we give more detailed blocks in the structure of the canonical forms. These forms have the advantage of being the same for all cases considered in [9] (with or without zero columns). Moreover, we construct a canonical form for the essential reachability property. The structure of these forms is block upper triangular (in terms of the state matrix of its associated canonical form). This is done for standard reachability in Section 3, standard controllability in Section 4 and essential reachability and controllability in Section 5, in all cases for the multi-input case. Analogous results for *periodic* positive discrete-time linear systems are the subject of the technical report [4].

Given a general matrix $A = [a_{ij}]$, we write $A \geqslant 0$ if $a_{ij} \geqslant 0\ \forall i, j;\ A > 0$ if at least some entry $a_{ij} > 0$ and $A \gg 0$ if $a_{ij} > 0\ \forall i, j$. Moreover, we must bear in mind that an *i-monomial vector* is a (nonzero) multiple of the unit vector e_i and that a *monomial matrix* has one and only one nonzero entry in each column and each row.

We consider a *positive invariant discrete-time linear control system* given by

$$x(k+1) = Fx(k) + Gu(k), \quad k \in \mathbb{N}, \tag{1}$$

where $F = [f_{ij}] \in \mathbb{R}_+^{n\times n}$, $G = [g_{ij}] \in \mathbb{R}_+^{n\times m}$, $x(t)$ is the nonnegative *state* vector and $u(t)$ is the nonnegative *control* or *input* vector. We denote that control system by $(F, G) \geqslant 0$. Note that if the initial state vector is nonnegative, that is $x_0 \geqslant 0$, and the input vector $u(k)$ is nonnegative for every $k \geqslant 0$, then the state vector $x(k)$ is also nonnegative in any other instant *k*.

We will study the structural properties of reachability and controllability for this kind of linear control systems. Usually (see [5]) a positive system (F, G) is said to be

(a) *reachable* (*from* 0) if for any nonnegative state x_f, there exist $k \in \mathbb{N}$ and a nonnegative input sequence $u(t) \geqslant 0,\ t = 0, \ldots, k-1$, transferring the state of the system from the origin at $t = 0$ to x_f at time $t = k$.

(b) (*completely*) *controllable* if for any pair of nonnegative states x_0 and x_f there exist $k \in \mathbb{N}$ and a nonnegative input sequence $u(t) \geqslant 0,\ t = 0, \ldots, k-1$, transferring the state of the system from x_0 at $t = 0$ to x_f at time $t = k$.

To study these structural properties, we introduce the reachability matrix in *k*-steps, defined as

$$\Re_k(F, G) = [G|FG|\cdots|F^{k-1}G],$$

and the reachability cone in *k*-steps defined as the polyhedral cone generated by the columns of the reachability matrix, which is denoted by $R_k(F, G)$. This cone is the set of all nonnegative states x_f which are reachable in *k*-steps, by means of a suitable sequence of nonnegative inputs $u(0), \dots, u(k-1)$. Then

$$R_\infty(F, G) = \bigcup_{n=1}^{\infty} R_k(F, G),$$

is the set of all the reachable states in finite time.

In [5] it was established that the pair $(F, G) \geqslant 0$ is reachable if and only if $R_\infty(F, G) = \mathbb{R}^n_+$, or equivalently, the reachability matrix $\Re_n(F, G)$ has a monomial submatrix of order *n*. Moreover, the pair $(F, G) \geqslant 0$ is controllable if and only if the pair $(F, G) \geqslant 0$ is reachable and *F* is nilpotent (see [5]).

It is known that (see [9]) if the pair $(F, G) \geqslant 0$ is reachable, then the matrix $[F|G]$ has a monomial submatrix of order *n*. Later, we will establish a converse result, i.e., we add a condition to the existence of the monomial submatrix in order to obtain a characterization of the property of reachability.

Given $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we denote by $\Gamma(A)$ the corresponding *direct graph*, consisting of a set of vertices $V = \{1, 2, \dots, n\}$ and a set of arcs. The arc $(i, j) \in \Gamma(A)$ if and only if $a_{ji} \neq 0$. A *path* from the vertex *i* to the vertex *j* of *V*, denoted by P_{ij}, is a set of arcs such that $(i, k_1), (k_1, k_2), \dots, (k_{r-1}, j)$. In this case, the *length* of the path, denoted by length(P_{ij}), is *r*. Each vertex *i* will be considered as an empty path of length 0. One kind of useful paths is the *deterministic* path (see [9]), that is, a path in which from each vertex there is at most one outgoing arc, except possibly for the last vertex. Define a *circuit* as a closed deterministic path, that is, is a deterministic path from *i* to *i*. Moreover, a communicating class *C* is a subset of vertices of *V* such that any two vertices of *C* are each accessible (i.e., there exists a path) from the other. The block of *A* such that the corresponding directed graph is the communicating class *C* is denoted by block$_C(A)$ and the spectral radius of that block by $\rho(C)$.

A matrix $A \geqslant 0$ is *irreducible* if and only if its direct graph is strongly connected, i.e., for any two vertices of *V* there exists a path connecting the former to the later (see [1]).

Finally, by col$_\alpha A$ we denote the αth column of the matrix *A*, *w* a vector in $\mathbb{R}^n$ with *j*th component w_j and by $\langle u_1, u_2, \dots, u_n\rangle$ the polyhedral cone generated by the vectors $u_1, u_2, \dots, u_n$.

2. Construction of a partition of the set *V*

Consider the pair $(F, G) \geqslant 0$. Since the pair (F, G) is reachable if and only if $\Re_n(F, G)$ has a monomial submatrix of order *n*, we focus our attention on deterministic paths starting from vertices associated with monomial vectors of *G*. Assume that $G \in \mathbb{R}^{n\times m}$ has *r*-monomial column vectors. Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of vertices associated with these α_j-monomial vectors $\forall\ j \in \{1, \dots, r\}$.

Now, we consider the deterministic paths of $\Gamma(F)$ generated by $\{\alpha_1, \dots, \alpha_r\}$,

$$\begin{array}{ccccccc} \alpha_1=\alpha_1^1 & \to & \alpha_1^2 & \to & \dots & \to & \alpha_1^{k_1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_r=\alpha_r^1 & \to & \alpha_r^2 & \to & \dots & \to & \alpha_r^{k_r}. \end{array} \tag{2}$$

In what follows, we can assume that there are no repeated deterministic paths, if not, we eliminate them. *The set of all vertices in these different deterministic paths given in* (2) *is denoted by* $A=\{\alpha_1^1,\ \alpha_1^2,\dots,\alpha_1^{k_1},\dots,\alpha_r^1,\ \alpha_r^2,\dots,\ \alpha_r^{k_r}\}$. Note that the vertices of A determine the corresponding monomial vectors in $\Re_n(F,G)$. Note that the deterministic paths in (2) may or may not be connected. This fact allows us to make a finer partition of the set A as follows. The vertices of those deterministic paths that have no access to the other deterministic paths are grouped in the following set:

$$A_1=\{\alpha_j^m\in A|\operatorname{col}_{\alpha_j^{k_j}}F=0\},$$

or equivalently

$$A_1=\{\alpha_j^m\in A|(\alpha_j^{k_j},i)\notin\Gamma(F)\ \forall i\in V\}.$$

The vertices of A which belong to deterministic paths that have access to some vertex of A_1 are grouped in the set

$$A_2=\left\{\alpha_j^m\in A|\exists\alpha\in A_1:\ (\alpha_j^{k_j},\alpha)\in\Gamma(F),\ \text{and}\ (\alpha_j^{k_j},i)\notin\Gamma(F)\ \forall i\notin A_1\right\}.$$

By proceeding in this way, we introduce the sets

$$\begin{aligned} A_h=\Big\{&\alpha_j^m\in A|\exists\alpha\in A_{h-1}:\ (\alpha_j^{k_j},\alpha)\in\Gamma(F),\ \text{and}\\ &(\alpha_j^{k_j},i)\notin\Gamma(F)\ \forall i\notin A_1\cup\dots\cup A_{h-1}\Big\}, \end{aligned}$$

for $h=3,\dots,n-1$.

Moreover, in order to achieve a partition of A, we define $A_n=A-\{A_1\cup\dots\cup A_{n-1}\}$. Note that the vertices of A_n are in deterministic paths such that all of them are connected to some vertex included in $V-\{A_1\cup\dots\cup A_{n-1}\}$.

Next, we consider a second kind of deterministic paths starting from vertices β which are not associated with monomial vectors of G, and such that there exists a vertex k of A_n having access to β and access to some vertex of the set $A_1\cup\dots\cup A_{n-1}$. That is, $\beta\in V-A$ is such that

$$\begin{aligned} &\exists k\in A_n,\ \ \exists\alpha\in A_1\cup\dots\cup A_{n-1}:\ (k,\beta)\in\Gamma(F),\ (k,\alpha)\in\Gamma(F)\\ &\quad\text{and}\ \forall i\notin A_1\cup\dots\cup A_{n-1},\ i\neq\beta,\ (k,i)\notin\Gamma(F). \end{aligned} \tag{3}$$

The set of all vertices in this second kind of deterministic paths

$$\begin{array}{ccccccc} \beta_1=\beta_1^1 & \to & \beta_1^2 & \to & \cdots & \to & \beta_1^{p_1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \beta_s=\beta_s^1 & \to & \beta_s^2 & \to & \cdots & \to & \beta_s^{p_s}. \end{array} \tag{4}$$

is denoted by $B = \{\beta_1^1,\ \beta_1^2, \ldots, \beta_1^{p_1}, \ldots, \beta_s^1,\ \beta_s^2, \ldots, \beta_s^{p_s}\}$.

Additionally, we consider those deterministic paths starting from vertices $\gamma \in V - \{A \cup B\}$ such that there exists a column of G which can be written as

$$\begin{aligned} &\operatorname{col} G = \tau e_\gamma + w,\ \text{for some real number } \tau > 0,\ \text{and for some} \\ &\quad \text{real vector } w > 0, \quad \text{with } j \in A_1 \cup \cdots \cup A_{n-1} \ \text{if } w_j \neq 0. \end{aligned} \tag{5}$$

The set of all vertices in this third kind of deterministic paths

$$\begin{array}{ccccccc} \gamma_1 = \gamma_1^1 & \to & \gamma_1^2 & \to & \cdots & \to & \gamma_1^{q_1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \gamma_t = \gamma_t^1 & \to & \gamma_t^2 & \to & \cdots & \to & \gamma_t^{q_t}. \end{array} \tag{6}$$

is denoted by $C = \{\gamma_1^1,\ \gamma_1^2, \ldots, \gamma_1^{q_1}, \ldots, \gamma_t^1,\ \gamma_t^2, \ldots, \gamma_t^{q_t}\}$.

We will see in Proposition 2 that the deterministic paths given in (4) and (6) are closed deterministic paths, that is circuits, so the corresponding vertices correspond to monomial vectors in the reachability matrix.

When studying essential reachability, we will consider circuits, whose communicating classes (blocks) have a spectral radius with bordering properties. If τ belongs to a circuit, by C_τ we denote the communicating class associated with this vertex (and hence with this circuit).

We consider then the circuits whose vertices are not in $A \cup B \cup C$, and the set of their vertices

$$\tilde{H} = \{\tau \in V - \{A \cup B \cup C\} | \exists\, C_\tau : j \in V - \{A \cup B \cup C\}\ \forall j \in C_\tau\}.$$

Now, we shall only focus on those circuits having a unique vertex accesible from any vertex of A_n, that is, we shall consider vertices $\tau \in \tilde{H}$ such that there exists $k \in A_n$ with

$$\begin{aligned} &\operatorname{col}_k F = \alpha e_\tau + w, \text{for some real number } \alpha > 0, \\ &\text{and some vector } w \in \mathbb{R}_+^n,\ \text{where } w_j = 0\ \forall j \in C_\tau, \end{aligned} \tag{7}$$

or equivalently

$$(k, \tau) \in \Gamma(F) \quad \text{and} \quad (k, j) \notin \Gamma(F)\ \ \forall j \in C_\tau,$$

and with the three following conditions for any vertex $k \in A_n$ connected to τ:

(i) k does not have access to any vertex $\eta \in V - C_\tau$ such that from η there exist two outgoing arcs which reach two different vertices of C_τ.

(ii) If $\eta \in V - C_\tau$ is accessible from k, and there exists only one arc from η to a vertex $\bar{\tau}$ of C_τ, then

$$\text{length}(P_{k\eta}) = \text{length}(P_{\tau\bar{\tau}}) + q \cdot \text{length}(C_\tau), \quad q \in \mathbb{Z}_+,$$

for all paths $P_{k\eta}$.

(iii) For every communicating class $\bar{C}$ accessible from k, $\rho(\bar{C}) \leqslant \rho(C_\tau)$. Moreover, if $\rho(\bar{C}) = \rho(C_\tau)$, then C_τ is accessible from $\bar{C}$.

The set of all vertices in these circuits

$$\begin{tikzcd} \tau_1^1 \arrow[r] & \tau_1^2 \arrow[r] & \cdots \arrow[r] & \tau_1^{\epsilon_1} \arrow[lll, bend left] \\ & & \vdots & \\ \tau_l^1 \arrow[r] & \tau_l^2 \arrow[r] & \cdots \arrow[r] & \tau_l^{\epsilon_l}. \arrow[lll, bend left] \end{tikzcd} \tag{8}$$

is denoted by $D_1 = \{\tau_1^1,\ \tau_1^2, \ldots, \tau_1^{\epsilon_1}, \ldots, \tau_l^1,\ \tau_l^2, \ldots, \tau_l^{\epsilon_l}\}$.

We will construct the last set D_2. Previously, we take the deterministic paths starting from vertices δ of $\tilde{H} - D_1$ such that there exists a column of G written as

$$\begin{aligned} &\operatorname{col} G = \alpha e_\delta + w,\ \text{ for some real number } \alpha > 0, \\ &\text{and some } w \in \mathbb{R}_+^n \text{ with } w_j = 0 \quad \forall j \in C_\delta, \end{aligned} \tag{9}$$

and with the additional condition

(iv) If $\operatorname{col} G = \alpha e_\delta + w,\ \alpha > 0,\ w \in \mathbb{R}_+^n,$ with $w_j = 0\ \forall j \in C_\delta$, then any communicating class $\bar{C}$ accessible from any vertex j such that $w_j \neq 0$ satisfies $\rho(\bar{C}) < \rho(C_\delta)$.

Then, *the set of all vertices in these circuits*

$$\begin{tikzcd} \delta_1^1 \arrow[r] & \delta_1^2 \arrow[r] & \cdots \arrow[r] & \delta_1^{\nu_1} \arrow[lll, bend left] \\ & & \vdots & \\ \delta_h^1 \arrow[r] & \delta_h^2 \arrow[r] & \cdots \arrow[r] & \delta_h^{\nu_h}. \arrow[lll, bend left] \end{tikzcd} \tag{10}$$

is denoted by $D_2 = \{\delta_1^1,\ \delta_1^2, \ldots, \delta_1^{\nu_1}, \ldots, \delta_h^1,\ \delta_h^2, \ldots, \delta_h^{\nu_h}\}$.

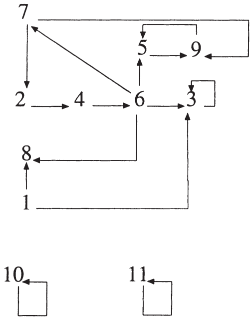
We illustrate the definitions given above by means of the following example.

Example 1. Consider the pair $(F, G) \geqslant 0$ of order $n = 11$, where

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{4} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G=\begin{bmatrix}0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&1&0&0&0&1\\0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}.$$

The directed graph associated with the matrix F is



Note that the monomial vectors of the matrix G correspond to the set of vertices $\{1, 2, 7, 8\}$. Moreover, G has the columns $g_1 = e_{10} + e_8$ and $g_2 = e_{11} + e_8 + e_7 + e_5 + e_3$.

We construct the deterministic paths generated by the vertices $\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 7,\ \alpha_4 = 8$. We have

$$\alpha_1 = \alpha_1^1 = 1$$

$$\alpha_2 = \alpha_2^1 = 2 \quad \to \quad \alpha_2^2 = 4 \quad \to \quad \alpha_2^3 = 6$$

$$\alpha_3 = \alpha_3^1 = 7$$

$$\alpha_4 = \alpha_4^1 = 8.$$

Then, $A = \{1, 2, 4, 6, 7, 8\}$. Now, we will construct the sets $A_j,\ j = 0, 1, \ldots, 11$. The set of vertices of deterministic paths such that the corresponding last vertex has no outgoing arc is $A_1 = \{8\}$. The set of vertices of deterministic paths such that from the last vertex all outgoing arcs reach vertices of the set A_1 is $A_2 = \emptyset$. Moreover, $A_3 = A_4 = \cdots = A_{10} = \emptyset$. Then, $A_{11} = \{1, 2, 4, 6, 7\}$.

Notice that for $\beta_1 = 3$ there exists a vertex $k = 1 \in A_{11}$ such that $(1, 3) \in \Gamma(F)$ and $(1, 8) \in \Gamma(F)$ with $8 \in A_1 \cup \cdots \cup A_{10}$. So, we consider the deterministic path associated with the vertex $\beta_1 = 3$

$$\beta_1 = \beta_1^1 = 3.$$

Then, $B = \{3\}$.

It is clear that for $\gamma_1 = 10$, there exists a column in G, $g_1 = e_{10} + e_8$ with $8 \in A_1 \cup \cdots \cup A_{10}$. The deterministic path generated by $\gamma_1 = 10$ is

$$\gamma_1 = \gamma_1^1 = 10.$$

Therefore, $C = \{10\}$.

Note that $V - \{A \cup B \cup C\} = \{5, 9, 11\}$, so $\tilde{H} = \{5, 9, 11\}$ since $C_5 = C_9 = \{5, 9\}$ and $C_{11} = \{11\}$. Observe that for $\tau_1 = 5$, there exists an arc $(6, 5) \in \Gamma(F)$ with $k = 6 \in A_{11}$ and $(6, 9) \notin \Gamma(F)$, $(6, 11) \notin \Gamma(F)$. Note that the set of vertices accessible from $k = 6$ is $\{5, 9, 3, 8, 7, 2, 4, 6\}$. None of these vertices has two or more outgoing arcs to different vertices of C_5 and only the vertex 7 has an outgoing arc $(7, 9)$ to C_9 ($\bar{\tau} = 9$). Between 6 and 7, we have two possible paths:

$$P_{67}^1 : 6 \to 7 \quad \text{or} \quad P_{67}^2 : 6 \to 7 \to 2 \to 4 \to 6 \to 7$$

with $\text{length}(P_{67}^1) = 1 = \text{length}(P_{59})$ and $\text{length}(P_{67}^2) = 5 = \text{length}(P_{59}) + 2\,\text{length}(C_5)$.

Moreover, the accessible communicating classes from 6 are:

$$\bar{C}_1 = \{2, 4, 6, 7\}, \quad \rho(\bar{C}_1) = 0.5,$$

$$\bar{C}_2 = \{3\}, \quad \rho(\bar{C}_1) = 1/4,$$

$$\bar{C}_3 = \{8\}, \quad \rho(\bar{C}_1) = 0$$

while $\rho(C_5) = 0.5$. With these properties, we deduce that the conditions (i)–(iii) hold for $\tau_1 = 5$. Therefore, we construct the circuit generated by $\tau_1 = 5$

$$\tau_1 = \tau_1^1 = 5 \to \tau_1^2 = 9.$$

Then, $D_1 = \{5, 9\}$.

Finally, we construct the set D_2. Note that for $\delta_1 = 11$, there exists a column of G, $g_2 = e_{11} + w$ with $w = e_8 + e_5 + e_7 + e_3$. Moreover, the accessible communicating classes from 8, 5, 7, 3 are $\bar{C}_1, \bar{C}_2, \bar{C}_3, C_5$ having each one of them spectral radius less than $\rho(C_{11}) = 1$. Therefore, the corresponding circuit is

$$\delta_1 = \delta_1^1 = 11.$$

Then, $D_2 = \{11\}$.

Proposition 1. *Let $(F, G) \geqslant 0$ be a positive system of order n such that $[F|G]$ has a monomial submatrix of order n. For every $i_1 \in V - \{A \cup B \cup C\}$, there exists a unique deterministic path containing i_1. Moreover, this deterministic path is a circuit and all its vertices are in the set $V - \{A \cup B \cup C\}$.*

Proof. Since $i_1 \notin A$, from the definition of the set A, the i_1-monomial vector is not in the matrix G, and since $[F|G]$ has a monomial submatrix of order n then the i_1-monomial vector is in the matrix F, i.e., there exists a unique vertex i_2 such that the arc

$$i_2 \to i_1,$$

is in $\Gamma(F)$. If $i_2 = i_1$, then the proof is completed. Otherwise, $i_2 \notin A \cup B \cup C$ since $i_1 \notin A \cup B \cup C$ and both vertices are in the same deterministic path. By reasoning in this way, we obtain a deterministic path

$$i_l \to \cdots \to i_j \to \cdots \to i_2 \to i_1.$$

with $i_j \notin A \cup B \cup C \ \forall j \in \{1, \ldots, l\}$. Moreover, this deterministic path is closed since A is of finite order and the first vertex i_l always has a unique ingoing arc from any other vertex of $\Gamma(F)$. Finally, note that the constructed deterministic path is unique since the vertices $i_2, \ldots, \ i_l$ correspond to monomial vectors in the matrix F. □

From the construction of the sets $A, \ B$ and C and the previous result, if a vertex of a deterministic path is in $A, \ B, \ C$ or $V - \{A \cup B \cup C\}$, then all vertices of such deterministic path are in A, B, C or $V - \{A \cup B \cup C\}$, respectively. Then we can establish the following results.

Proposition 2. *Let $(F, G) \geqslant 0$ be a positive system of order n such that $[F|G]$ has a monomial submatrix of order n. For every $i_1 \in B \ (i_1 \in C)$ there exists a unique deterministic path containing i_1. Moreover, this deterministic path is a circuit and all its vertices are in $B \ (C)$.*

Proof. If $i_1 \in B$ $(i_1 \in C)$, then $i_1 \notin A$, and hence we can apply a similar reasoning as in the proof of Proposition 1. □

Proposition 3. *Let $(F, G) \geqslant 0$ be a positive system of order n such that $[F|G]$ has a monomial submatrix of order n. For every $i_1 \in V$ such that $(i_1, i) \notin \Gamma(F) \ \forall i \in V$, there exists a deterministic path containing i_1 with its vertices in A_1.*

Proof. If $(i_1, i) \notin \Gamma(F) \ \forall i \in V$, then the deterministic path ending in i_1 is not a circuit. Thus, all vertices of that deterministic path must be in A. More precisely, by the construction of sets $A_1, \ldots, A_n$, all these vertices must be in A_1. □

3. Reachability property

Consider a positive linear discrete-time system (F, G) given in (1). Note that the unit vector e_i is reachable in a finite number of steps if and only if there exists an *i*-monomial vector in the reachability matrix. In [9] the set of vertices *i* such that there is an *i*-monomial vector in the reachability matrix is denoted by $I(F,G)$. Thus, the positive system (F,G) is reachable if and only if $I(F,G)=\{1,\dots,n\}$.

From the definitions of the sets A, B and C, and the propositions given in Section 2, we establish the following results in order to obtain a reachability canonical form, which has the structure of that given in [9], but with more detail.

First, we relate the set $I(F,G)$ to the sets A, B and C.

Proposition 4. *Assume that the matrix* $[F|G]$ *has a monomial submatrix of order n. Then,* $I(F,G)=A\cup B\cup C$.

Proof. We will show that if the vertex $i_1\in V$ is such that $i_1\notin A\cup B\cup C$, the i_1-monomial vector is not in the reachability matrix $\Re_n(F,G)=[G|FG|F^2G|\cdots|F^{n-1}G]$, that is, $i_1\notin I(F,G)$.

To reach this objective, it is sufficient to prove that the i_1-monomial column is not in F^kG $\forall k=1,\dots,n-1$. Since $i_1\notin C$, then any column *g* of *G* can be written either as $g=w$, where $w_j\neq 0$ if $j\in A_1\cup\cdots\cup A_{n-1}$, or as $g=\tau e_{i_1}+w$, where $\tau\geqslant 0$, $w\in\mathbb{R}^n_+$, $w_{i_1}=0$, and $\exists j\notin A_1\cup\cdots\cup A_{n-1}$ such that $w_j\neq 0$. Consider these two cases:

Case 1: Suppose that $g=w$, where $w_j\neq 0$ if $j\in A_1\cup\cdots\cup A_{n-1}$. If $w=0$, then $F^kg=0$ and therefore we do not obtain the corresponding i_1-monomial vector, that is, $i_1\notin I(F,G)$. So consider $w\neq 0$. In this case, it is sufficient to prove that F^ke_j is not an i_1-monomial vector $\forall k\in\{1,\dots,n-1\}$ and $\forall j\in A_1\cup\cdots\cup A_{n-1}$.

If $j\in A_1$, then *j* is in one of the deterministic paths in (2)

$$\alpha^1_{j'}\to\alpha^2_{j'}\to\alpha^{m'}_{j'}=j\to\cdots\to\alpha^{k_{j'}}_{j'}$$

with all vertices in A_1, and so they are different from i_1. Then, $\operatorname{col}_{\alpha^{k_{j'}}_{j'}}F=0$.

Moreover, we observe that

$$\begin{aligned}
Fe_j = \ Fe_{\alpha^{m'}_{j'}} &\in \left\langle e_{\alpha^{m'+1}_{j'}}\right\rangle\\
F^2e_j &\in \left\langle Fe_{\alpha^{m'+1}_{j'}}\right\rangle = \left\langle e_{\alpha^{m'+2}_{j'}}\right\rangle\\
&\ \vdots\\
F^{k_{j'}-m'+1}e_j &\in \left\langle Fe_{\alpha^{k_{j'}}_{j'}}\right\rangle = 0,
\end{aligned}$$

and thus, F^ke_j is not an i_1-monomial vector $\forall k\in\{1,\dots,n-1\}$ and $\forall j\in A_1$.

Now let us study the case in which $j\in A_2$. In this case *j* also belongs to a deterministic path in (2), such that the last vertex α is connected only to vertices of A_1, i.e., $\operatorname{col}_\alpha F=v$, such that $j\in A_1$ if $v_j\neq 0$. Thus, we find ourselves in the same situation as before, and we can conclude that F^ke_j is not an i_1-monomial vector $\forall k\in\{1,\dots,n-1\}$ and $\forall j\in A_2$.

With the analogous line of reasoning, we deduce the same result for the other situations when the vertex j is in the sets A_h, $h \in \{3, \dots, n-1\}$.

Case 2: Let $g = \tau e_{i_1} + w$, where $\tau \geqslant 0$, $w \in \mathbb{R}^n_+$, $w_{i_1} = 0$ and $\exists j \notin A_1 \cup \cdots \cup A_{n-1}$ such that $w_j \neq 0$. We only need to prove that $F^k(\tau e_{i_1} + e_j)$ $\forall k \in \{1, \dots, n-1\}$ is not an i_1-monomial vector $\forall j \notin A_1 \cup \cdots \cup A_{n-1}$. We will study the following different possibilities: $j \in B$ or $j \in C$ or $j \in A_n$ or $j \notin A \cup B \cup C$. First, note from Proposition 1 that since $i_1 \notin A \cup B \cup C$, the vertex i_1 is in a circuit

$$\begin{tikzcd} i_l \arrow[r] & \cdots \arrow[r] & i_2 \arrow[r] & i_1 \arrow[lll, bend left] \end{tikzcd} \tag{11}$$

having no vertices in $A \cup B \cup C$.

If $j \in B$ $(j \in C)$, from Proposition 2 we know that j is in a circuit

$$\begin{tikzcd} j_{l'} \arrow[r] & \cdots \arrow[r] & j_2 \arrow[r] & j_1 = j \arrow[lll, bend left] \end{tikzcd}$$

with all vertices in B (C) and hence different from i_1. Therefore, we obtain

$$F^k(\tau e_{i_1} + e_j) \in \langle F^k e_{i_1}, F^k e_j \rangle = \left\langle e_{i_{l+1-k}}, e_{j_{l'-k+1}} \right\rangle \quad \forall k \in \{1, \dots, l\}. \tag{12}$$

Thus, $F^k(\tau e_{i_1} + e_j)$ is not an i_1-monomial vector $\forall k \in \{1, \dots, n-1\}$.

From Proposition 1, if $j \notin A \cup B \cup C$, by the same line of reasoning and supposing that j is not in the same circuit as i_1 given in (11), then $F^k(\tau e_{i_1} + e_j)$ is not an i_1-monomial vector $\forall k \in \{1, \dots, n-1\}$. Otherwise, if j and i_1 are in the same circuit, there is an index $d \in \{1, \dots, l\}$ such that $j = i_d \neq i_1$ and so

$$F^k(\tau e_{i_1} + e_{i_d}) \in \langle F^k e_{i_2}, F^k e_{i_{d+1}} \rangle = \langle e_{i_{l-k+1}}, e_{i_{d-k}} \rangle \quad \forall k \in \{1, \dots, n-1\},$$

which is not an i_1-monomial vector, since τ is positive because $j \notin A$.

Finally, we consider the case $j \in A_n$. In this case the vertex j is in a deterministic path in (2), whose last vertex α is connected to some vertex included in $V - \{A_1 \cup \cdots \cup A_{n-1}\}$.

First, we suppose that from the last vertex α of the deterministic path there exists a unique outgoing arc, that is, there exists a unique vertex ϕ such that $\mathrm{col}_\alpha F = v$ with $v_\phi \neq 0$. Then, ϕ is in the same path of j. In this case we have a circuit, and thus, the vector $F^k(\tau e_{i_1} + e_j)$ is not an i_1-monomial vector $\forall k \in \{1, \dots, n-1\}$.

Now, we suppose that from the last vertex α of the deterministic path there are two outgoing arcs. When one of the arcs is leading to a vertex of $A_1 \cup \cdots \cup A_{n-1}$, we can prove that the other vertex is in a circuit not including the vertex i_1 since $i_1 \notin B$. Thus, $F^k(\tau e_{i_1} + e_j)$ is not an i_1-monomial vector $\forall k \in \{1, \dots, n-1\}$. Then, we consider that the outgoing arcs from the last vertex α are leading to vertices not included in $A_1 \cup \cdots \cup A_{n-1}$. This study is reduced to the two following cases:

(a) $\mathrm{col}_\alpha F = v$ with only two entries nonzero, $v_\phi \neq 0$ and $v_{i_d} \neq 0$, where $\phi \notin A_1 \cup \cdots \cup A_{n-1}$, i_d is in the same circuit of i_1 with $\phi \neq i_d$, and

(b) $\mathrm{col}_\alpha F = v$ with only two entries nonzero, $v_{\phi_1} \neq 0$ and $v_{\phi_2} \neq 0$, where $\phi_1, \phi_2 \notin A_1 \cup \cdots \cup A_{n-1}$, $\phi_1 \neq \phi_2$ and they are not in the same circuit of i_1.

Then, if ϕ, ϕ_1 and $\phi_2 \notin A_n$, reasoning as before and using Propositions 1 and 2, we obtain the desired result. Finally, if ϕ, ϕ_1 and $\phi_2 \in A_n$, then we will be reiterating the process. Then, we prove that $F^k(\tau e_{i_1} + e_j)$ is not an i_1-monomial vector $\forall k \in \{1, \ldots, n-1\}$ if $j \in A_n$.

Conversely, suppose that the vertex i_1 is in the set $A \cup B \cup C$. We only outline the case when $i_1 \in C$ (the other cases are similar). By definition of the set *C*, there exists a column *g* in *G* such that $g = \rho e_j + w$, with $\alpha \in A_1 \cup \cdots \cup A_{n-1}$ if $w_\alpha \neq 0$, and vertices i_1 and *j* are in the same circuit. Then, for some *k*, $F^k g$ is i_1-monomial since the vector sequence $F^k w,\ k \in \mathbb{Z}$, eventually becomes zero. Hence, $i_1 \in I(F, G)$. □

With this result, and Lemma 2 of [9], we can establish the following characterization of the reachability property of the positive system (F, G).

Theorem 1. *Assume $(F, G) \geqslant 0$. Then, (F, G) is reachable if and only if the matrix $[F|G]$ has a monomial submatrix of order n and $A \cup B \cup C = \{1, \ldots, n\}$.*

Now, we are going to construct a canonical form of reachability as follows.

Suppose that $(F, G) \geqslant 0$ is reachable, then by Theorem 1, all vertices are in $A \cup B \cup C$. Next, we are going to construct some permutation matrices *P* and *Q* from the sets $A = A_1 \cup \cdots \cup A_n$, *B* and *C* in order to get the canonical form of the reachability property.

We introduce $A' = A_1 \cup \cdots \cup A_{n-1}$, so that $A = A' \cup A_n$, and we take a partition of $A_n = A_{n_B} \cup A_{n_R}$, where A_{n_B} is the set of vertices of A_n belonging to deterministic paths in (2) connected to vertices of set *B*, according to the definition of set *B*.

We denote the number of deterministic paths in *A*, A', $A_i,\ i = 1, \ldots, n,\ A_{n_B}$, A_{n_R}, *B* and *C* by *r*, r', $r_i,\ i = 1, \ldots, n,\ r_{n_B}, r_{n_R}$, *s*, and *t*, respectively. Thus,

$$r = r' + r_n,\ r' = r_1 + \cdots + r_{n-1},\ r_n = r_{n_B} + r_{n_R}.$$

Then, we relabel the vertices in decreasing order, from *n* to 1, considering the following order of the sets: $A_{n_R}, A_{n_B}, A_{n-1}, \ldots, A_1$, *B* and *C*, and in each set starting from the longest deterministic path and finishing with the shortest deterministic path.

Define *P* as the permutation matrix associated with this relabelling, and let *Q* be the matrix which places the *r*-monomial vectors of *G* as the first *r* columns in the order that we will show within this proof. Then, the pair $[P^{\mathrm{T}} F P | P^{\mathrm{T}} G Q]$ has the following structure:

$$\left[\begin{array}{c|c|c|c|c|c}
\mathscr{C} & O & O & O & \Delta & \\ \cline{1-5}
O & \mathscr{B} & O & \Sigma & \Delta & \\ \cline{1-5}
O & O & \mathscr{A}' & \Delta & \Delta & \mathscr{G} \\ \cline{1-5}
O & O & O & \mathscr{A}_{n_B} & \Delta & \\ \cline{1-5}
O & O & O & O & \mathscr{A}_{n_R} &
\end{array}\right], \tag{13}$$

where $\mathscr{C}$, $\mathscr{B}$, $\mathscr{A}'$, $\mathscr{A}_{n_B}$ and $\mathscr{A}_{n_R}$ are the submatrices associated with the deterministic paths of the sets C, B, A', A_{n_B} and A_{n_R}, respectively. By construction of these sets we have

$$\mathscr{C} = \operatorname{diag}\left[\mathscr{C}(t), \ldots, \mathscr{C}(1)\right] \quad \text{and} \quad \mathscr{B} = \operatorname{diag}\left[\mathscr{B}(s), \ldots, \mathscr{B}(1)\right], \tag{14}$$

where $\mathscr{C}(i)$ and $\mathscr{B}(j)$, $\forall i = 1, \ldots, t$ and $\forall j = 1, \ldots, s$, are cyclic irreducible blocks, according to Proposition 2.

Moreover,

$$\mathscr{A}' = \left[\begin{array}{c|c|c|c|c} \mathscr{A}_1 & \varDelta & \cdots & \varDelta & \varDelta \\ \hline O & \mathscr{A}_2 & \cdots & \varDelta & \varDelta \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline O & O & \cdots & \mathscr{A}_{n-2} & \varDelta \\ \hline O & O & \cdots & O & \mathscr{A}_{n-1} \end{array}\right], \tag{15}$$

where

$$\varDelta = [\varPsi, \ldots, \varPsi], \text{ with } \varPsi = \begin{bmatrix} * & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \ldots & 0 \end{bmatrix}, \tag{16}$$

and $*$ denotes a nonnegative entry. By construction of the sets A_h, $h = 2, \ldots, n-1$, note that each block $\varPsi$ of the blocks $\varDelta$ in the superdiagonal of $\mathscr{A}'$ in expression (15) has at least one positive entry. Moreover,

$$\mathscr{A}_i = \operatorname{diag}\left[\mathscr{A}_i(r_i), \mathscr{A}_i(r_i - 1), \ldots, \mathscr{A}_i(1)\right], \quad i = 1, \ldots, n-1,$$

where

$$\mathscr{A}_i(j) = \begin{bmatrix} 0 & + & 0 & \ldots & 0 \\ 0 & 0 & + & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & + \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad j = 1, \ldots, r_i, \quad i = 1, \ldots, n-1, \tag{17}$$

and $+$ denotes a positive entry.

In addition,

$$\mathscr{A}_{n_B} = \operatorname{diag}\left[\mathscr{A}_{n_B}(s), \mathscr{A}_{n_B}(s-1), \ldots, \mathscr{A}_{n_B}(1)\right], \tag{18}$$

where $\mathscr{A}_{n_B}(j)$, $\forall j = 1, \ldots, s$, has the same structure as the matrix given in (17). The relationship between set A_{n_B} and set B is expressed in the structure of the matrix $\varSigma$, with s diagonal blocks,

$$\varSigma = \operatorname{diag}\left[\varPhi, \ldots, \varPhi\right], \quad \text{where} \quad \varPhi = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \\ + & 0 & \ldots & 0 \end{bmatrix}, \tag{19}$$

with $+$ denoting a positive entry. Moreover, the remaining relationships between the sets are given by the matrix Δ in (16). Note that these matrices are of appropriate dimensions in each case.

The last diagonal block of $P^{\mathrm{T}}FP$, $\mathscr{A}_{n_R}$, is a block matrix, where all off-diagonal blocks are Ψ and the diagonal blocks are given by

$$\mathscr{A}_{n_R}(j)=\begin{bmatrix} * & + & 0 & 0 & \ldots & 0 \\ * & 0 & + & 0 & \ldots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ * & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & + \\ * & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \quad \forall j=1,\ldots,r_R, \tag{20}$$

with $+$ denoting a positive entry and $*$ denoting a nonnegative entry.

Finally, associated with each block of the submatrices $\mathscr{A}'$, $\mathscr{A}_{n_B}$ and $\mathscr{A}_{n_R}$, we have a monomial vector in the matrix $\mathscr{G}$. That is, r columns of $\mathscr{G}$ are multiples of distinct unit vectors. From the order established in the renumbering of the vertices, we can assume that the first column of $\mathscr{G}$ is a multiple of e_n, the second is a multiple of e_{n-l}, where l is the number of vertices of the longest deterministic path of $\mathscr{A}_{n_R}$, and, in general, the ith-column, $i=1,\ldots,r$, is a multiple of e_{n-l_i}, where l_i is the sum of the number of vertices of all previous deterministic paths, according to the above order. Moreover, the following t columns of $\mathscr{G}$ have the structure considered in (5). This ordering of the columns of $\mathscr{G}$ defines the permutation matrix Q.

Then, the pair $(P^{\mathrm{T}}FP, P^{\mathrm{T}}GQ)$ is similar to the pair (F, G) and we have constructed a reachable canonical form.

Moreover, it is easy to check that the reachability matrix of the similar pair $(P^{\mathrm{T}}FP, P^{\mathrm{T}}GQ)$ given in (13) has a monomial submatrix of order n, and hence the pair is reachable.

We summarize all the last results in the following theorem.

Theorem 2. *Assume* $(F, G) \geqslant 0$. *Then,* (F, G) *is reachable if and only if there exist permutation matrices P and Q such that the matrix* $[P^{\mathrm{T}}FP|P^{\mathrm{T}}GQ]$ *has the structure given in* (13), *where the blocks are given in* (14)–(20).

The characterization of the reachability of the pair $(F, G) \geqslant 0$ in terms of directed graph $\Gamma(F)$ is given in the next result.

Theorem 3 (Reachability in terms of $\Gamma(F)$). *Let* (F, G) *be a positive discrete-time control system of order n. Consider the sets* $\{\beta_1,\ldots,\beta_s\}$ *and* $\{\gamma_1,\ldots,\gamma_t\}$ *given in* (3) *and* (5), *respectively, and the set of vertices* $\{\alpha_1,\ldots,\alpha_r\}$ *such that for each* $i\in\{1,\ldots,r\}$, *there exists a column of G which is an* α_i*-monomial vector. Then,* (F, G) *is reachable if and only if there exist in* $\Gamma(F)$, *r-deterministic paths generated from the vertices* $\{\alpha_1,\ldots,\alpha_r\}$, *and* $s+t$ *closed deterministic paths* (*circuits*) *generated from the vertices* $\{\beta_1, \ldots, \beta_s\}$ *and* $\{\gamma_1, \ldots, \gamma_t\}$, *such that all these paths cover all the n vertices of* $\Gamma(F)$.

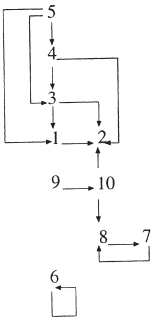
Example 2. Consider the control system $(F, G) \geqslant 0$, of order $n = 10$, given by

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The following graph is associated with the matrix F.



We construct the deterministic paths generated by $\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3, \alpha_4 = 4,\ \alpha_5 = 5,\ \alpha_6 = 9$, which are

$1 \to 2$
2
3
4
5
$9 \to 10.$

The unrepeated deterministic paths are

$$\alpha_1^1 = 1 \to \alpha_1^2 = 2$$

$$\alpha_2^1 = 3$$

$$\alpha_3^1 = 4$$

$$\alpha_4^1 = 5$$

$$\alpha_5^1 = 9 \to \alpha_5^2 = 10.$$

Note that $A = \{1, 2, 3, 4, 5, 9, 10\}$. Following the steps of Example 1, we construct the deterministic paths in B

$$\beta_1^1 = 8 \to \beta_1^2 = 7.$$

Then, $s = 1,\ p_1 = 2$ and $B = \{8, 7\}$. And finally, we have the deterministic path in C given by

$$\gamma_1^1 = 6.$$

Therefore, $t = 1$, $q_1 = 1$ and $C = \{6\}$.

Since $A \cup B \cup C = \{1, \dots, 10\}$ and the pair (F, G) has a monomial submatrix of order 10, then the pair (F, G) is reachable and its canonical form $[P^{\mathrm{T}}FP|P^{\mathrm{T}}GQ]$ is given by

$$\begin{array}{c|cc|cc|c|c|c|cc|c|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 1 \\ \cline{1-3}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 \\ \cline{2-5}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & & 0 & 0 & 0 & 0 & 1 & 1 \\ \cline{4-6}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 & 0 & 1 & 0 & 0 \\ \cline{6-7}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & & 0 & 0 & 1 & 0 & 0 & 1 \\ \cline{7-8}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 & 0 & 1 \\ \cline{8-10}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 & 0 & 0 \\
\\
\mathscr{C} & \mathscr{B} & & \mathscr{A}_1 & & \mathscr{A}_2 & \mathscr{A}_3 & \mathscr{A}_4 & \mathscr{A}_{10_B} & & & & & \mathscr{G} & & &
\end{array}$$

4. Complete controllability

Now we shall study the complete controllability property of a positive discrete-time control system. By the relationship between this property and the reachability property (see [5]), we can establish the following characterization in terms of the sets $A = A_1 \cup \cdots \cup A_n$, B, and C.

Theorem 4. *Let (F, G) be a positive discrete-time control system. Then, (F, G) is completely controllable if and only if $A_1 \cup A_2 \cup \cdots \cup A_{n-1} = \{1, \ldots, n\}$.*

Proof. ($\Longrightarrow$): Suppose the pair (F, G) is completely controllable, then we know (see [5]) that the pair (F, G) is reachable and the matrix F is nilpotent. Then, by Theorem 1, the matrix $[F|G]$ has a monomial submatrix of order n and $A \cup B \cup C = \{1, \ldots, n\}$. To prove that $A_1 \cup A_2 \cup \cdots \cup A_{n-1} = \{1, \ldots, n\}$, we suppose, by contradiction, that there exists a vertex $j \in B$ or $j \in C$ or $j \in A_n$. In all these cases, by Proposition 2 and the construction of the set A_n, one can find a circuit in $\Gamma(F)$, and thus, F is not nilpotent.

($\Longleftarrow$): We have to show that the pair (F, G) is reachable and the matrix F is nilpotent. To see reachability, from Theorem 1, we have to prove that the pair (F, G) has a monomial submatrix of order n. That is true by the construction of the sets $\{A_h,\ h = 1, \ldots, n-1\}$ and the condition $A_1 \cup A_2 \cup \cdots \cup A_{n-1} = \{1, \ldots, n\}$. In this case, the obtained canonical form is $[P^{\mathrm{T}}FP|P^{\mathrm{T}}GQ] = [\mathscr{A}'|\mathscr{G}]$ where the matrix $\mathscr{A}'$, given in (15), is strictly upper triangular. Thus, the matrix F is nilpotent. □

From the proof of Theorem 4, we can deduce the complete controllability canonical form.

Theorem 5. *Assume $(F, G) \geqslant 0$. Then, (F, G) is completely controllable if and only if there exists permutation matrices P and Q such that the matrix $[P^{\mathrm{T}}FP|P^{\mathrm{T}} GQ] = [\mathscr{A}', \mathscr{G}]$, where the matrix $\mathscr{A}'$ is given in* (15).

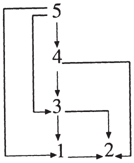
In the following result, we characterize the complete controllability of the pair $(F, G) \geqslant 0$ in terms of the directed graph $\Gamma(F)$. We recall that a matrix A is nilpotent if and only if there exists a permutation matrix P such that $P^{\mathrm{T}}AP$ is strictly upper triangular, equivalently the vertices of $\Gamma(A)$ can be relabelled $1, 2, \ldots, n$, in such a way that each arc (i, j) satisfies $j < i$.

Theorem 6. *Let (F, G) be a positive discrete-time control system and let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of vertices such that, for each α_i with $i \in \{1, \ldots, r\}$, there exists a column of G which is an α_i-monomial. Then (F, G) is completely controllable if and only if there exists in $\Gamma(F)$ r-deterministic paths generated from the vertices $\{\alpha_1, \ldots, \alpha_r\}$, such that these paths cover all vertices in $\Gamma(F)$ and the vertices of $\Gamma(F)$ can be relabelled in such a way that each arc (i, j) satisfies $j < i$.*

Example 3. Consider the control system $(F, G) \geqslant 0$, with $n = 5$, given by

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The graph associated with the matrix F is



From the above graph, we have $A_1 = \{1, 2\},\ A_2 = \{3\},\ A_3 = \{4\},\ A_4 = \{5\}$. Since $A_1 \cup \cdots \cup A_4 = \{1, 2, 3, 4, 5\}$, then the pair (F, G) is completely controllable and its canonical form is given by

$$[P^{\mathrm{T}} F P | P^{\mathrm{T}} G Q] = \left[\begin{array}{cc|c|c|c||cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{1} & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & \underline{0} & \underline{1} & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \underline{0} & \underline{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{0} & 1 & 0 & 0 & 0 \end{array}\right].$$

Theorem 7. *Let (F, G) be a positive discrete-time control system. Then (F, G) is completely controllable if and only if (F, G) is reachable and for each column of G, there exists a vertex $k \in \{1, \ldots, n\}$ such that $F^k g = 0$.*

5. Essential reachability

When not all nonnegative states can be reached in finite time, one considers the essential reachability property. A positive system (F, G) is said to be *essentially reachable* (see [5]) if all positive states are asymptotically reachable. From the construction of the reachability cones, one can say that (F, G) is essentially reachable if and only if $\overline{R_\infty(F, G)} = \mathbb{R}^n_+$. That is, the states not reachable in a finite number of steps are limits of states which are reachable in a finite number of steps.

In [9] it is established that a positive system (F, G) is essentially reachable if and only if for every vertex $i \notin I(F, G)$ the following facts hold:

(a) i belongs to some communicating class C_i of $\Gamma(F)$ consisting either of a single vertex i or of a circuit,

(b) there exists some column vector g_{τ_i} in G such that for every positive integer $t \geqslant n$ the block of the components of $F^t g_{\tau_i}$ corresponding to C_i, denoted by

$\text{block}_{C_i}(F^t g_{\tau_i})$ (as we said in Section 1), constitutes a monomial vector. Moreover, for each communicating class $\bar{C} \neq C_i$ such that $\text{block}_{\bar{C}}(F^t g_{\tau_i}) > 0$ for some $t \in \mathbb{Z}^+$ we have $\rho(\bar{C}) \leqslant \rho(C_i)$. In addition, if $\rho(\bar{C}) = \rho(C_i)$, then $\bar{C}$ has access to C_i.

We will establish conditions equivalent to those conditions (a) and (b) of the above result, and we will construct an essential reachability canonical form. First, we shall show the following result.

Proposition 5. *Assume* $(F, G) \geqslant 0$*. Then, for each vertex* $i \notin I(F, G)$ *the condition* (a) *holds if and only if the matrix* $[F|G]$ *has a monomial submatrix of order n.*

Proof. ($\Longrightarrow$): We will show that the matrix $[F|G]$ includes, among its columns, all i-mononial vectors, $i = 1, \ldots, n$. If $i \in I(F, G)$, then we know that *i*-monomial vector is in $[F|G]$ (see [9]). And if $i \notin I(F, G)$ then, by hypothesis, we know that *i* is in some communicating class which consists of either the single vertex *i* or of h_i vertices connected by a single circuit, i.e.

$$\begin{tikzcd} i_{h_i} \arrow[r] & \cdots \arrow[r] & i_2 \arrow[r] & i_1 = i \arrow[lll, bend left] & \text{or} & i_1 = i \arrow[loop below] \end{tikzcd}$$

Note that in both cases the *i*-monomial vector appears in *F*.

($\Longleftarrow$): Since the pair $[F|G]$ has a monomial submatrix of order *n*, from Proposition 4 we have $I(F, G) = A \cup B \cup C$. Then, by Proposition 1, for all $i \notin I(F, G) = A \cup B \cup C$, there exists a closed deterministic path containing the vertex *i* which is the circuit that we are looking for. $\square$

Moreover, we have the following characterization of the condition (b) in terms of the subsets of $V = \{1, \ldots, n\}$.

Proposition 6. *Assume that* $(F, G) \geqslant 0$ *is such that* $[F|G]$ *has a monomial submatrix. Then, the condition* (b) *holds if and only if* $A \cup B \cup C \cup D_1 \cup D_2 = \{1, \ldots, n\}$.

Proof. Since $[F|G]$ has a monomial submatrix of order *n*, by Proposition 4 we have $I(F, G) = A \cup B \cup C$, and moreover, by Proposition 5, condition (a) holds for all $i \notin I(F, G)$.

($\Longrightarrow$): We have to prove that $D_1 \cup D_2 = \{1, \ldots, n\} - I(F, G)$. Suppose $i \notin I(F, G)$ and $i \notin D_1 \cup D_2$. Then, by Proposition 1, there exists a circuit,

$$\begin{tikzcd} i_p \arrow[r] & \cdots \arrow[r] & i_2 \arrow[r] & i_1 = i \arrow[lll, bend left] \end{tikzcd}$$

with $i_d \notin I(F, G)\ \forall\, d = 1, \ldots, p$. Moreover, by definition of sets D_1 and D_2, $i_d \notin D_1 \cup D_2\ \forall d = 1, \ldots, p$.

Denoting by C_i the communicating class associated with the above circuit, and since $i_d \notin D_2$, then none of the columns of G have the special form indicated in the definition of set D_2. We can distinguish three cases:

Case 1: If $g = \operatorname{col} G$ with at least two nonzero entries in the $\operatorname{block}_{C_i}(g)$, then the $\operatorname{block}_{C_i}(F^t g)$ is not monomial for $t \geqslant n$, in contradiction with condition (b).

Case 2: Suppose $g = \operatorname{col} G$ with only one nonzero entry in the $\operatorname{block}_{C_i}(g)$ and with some nonzero entry outside this block such that the corresponding vertex has access to a class $\bar{C}$ satisfying $\rho(\bar{C}) \geqslant \rho(C_i)$. Then, by condition (b), $\rho(\bar{C}) = \rho(C_i)$ and so, the class $\bar{C}$ has access to the class C_i.

By the construction of the sets A, B and C, and since the matrix $[F|G]$ has a monomial submatrix of order n, we see that all vertices of the class $\bar{C}$ are in A_n, since the vertices in B, C, $A_1, \ldots, A_{n-1}$ and $V - (A \cup B \cup C)$ are only connected to vertices in such sets. Thus, let k be the last vertex of a deterministic path of the set A_n which is in the class C and such that from k there exists one outgoing arc reaching some vertex i_j, for some $j \in \{1, \ldots, p\}$ (note that from k there cannot be two or more outgoing arcs leading to the class C_i since the $\operatorname{block}_{C_i}(F^t g)$ is a monomial vector for some $t \in \mathbb{Z}^+$ by condition (b)).

Earlier, we noted that if $i_d \notin D_1 \cup D_2$ then one at least, among conditions (i)–(iii) of the construction of the set D_1 does not hold. In this case, condition (iii) holds, thus we can suppose that either condition (i) or condition (ii) is not satified. If (i) does not hold, then there exists an index $t \in \{1, \ldots, n\}$ such that $\operatorname{block}_{C_i}(F^t g)$ is not a monomial vector, which is not possible. If condition (ii) is not fulfilled, then there exists a vertex $\eta \in V - C_i$, accessible from k, such that from η there exists a unique outgoing arc to a vertex $i_{j'}$ of C_i, and

$$\operatorname{length}(P_{d\eta}) \neq \operatorname{length}(P_{i_j i_{j'}}) + q \cdot \operatorname{length}(C_i), \; q \in \mathbb{Z}_+,$$

for some path $P_{d\eta}$. The difference between the lengths allows us to assert that $\operatorname{block}_{C_i}(F^t g)$ is not a monomial vector for some $t \in \mathbb{Z}^+$, in contradiction with the hypothesis.

Case 3: Finally, assume that $g = \operatorname{col} G$ has all zero entries in the $\operatorname{block}_{C_i}(g)$. If a vertex $h \in A_n$ such that $g_h \neq 0$ and h has access to a unique i_d does not exist, then there does not exist any integer t such that the $\operatorname{block}_{C_i}(F^t g)$ is a monomial vector, in contradiction with the hypothesis. If such h exists, then by the same reasoning, we find that $i_d \in D_1$, which is in contradiction with our assumption.

($\Longleftarrow$): We have to prove that condition (b) holds for the vertices in D_1 and D_2. If $i \in D_2$ then i is in a circuit (communicating class C_i) of the type given in (10),

$$\begin{tikzcd} \delta^1 \arrow[r] & \delta^2 \arrow[r] & \cdots \arrow[r] & \delta^\nu \arrow[lll, bend left] \end{tikzcd}$$

By construction of the set D_2, this path is associated with a column of G, $g = \alpha e_{\delta^1} + w$ with $\alpha > 0$ and $w \in \mathbb{R}^n_+$ such that $w_j = 0 \; \forall j \in C_i$ and satisfying condition (iv). Then such a column g satisfies the condition (b).

If $i \in D_1$, then i is in a circuit (communicating class C_i) of the type given in (8). Moreover, by condition (7) there exists a vertex in A_n having access to this path. Then, the monomial column in the matrix G associated with this vertex satisfies the condition (b). $\square$

From the above propositions, we can establish the following result.

Theorem 8. *Assume* $(F, G) \geqslant 0$. *Then,* (F, G) *is essentially reachable if and only if* $[F|G]$ *has a monomial submatrix of order* n *and* $A \cup B \cup C \cup D_1 \cup D_2 = \{1, \ldots, n\}$.

Next, we give the essential reachability canonical form.

For that, we construct some permutation matrices P and Q from the sets A, B, C, D_1 and D_2, which constitute a partition of the set $V = \{1, \ldots, n\}$, following the steps of the proof of Theorem 2 but adding the sets D_1 and D_2.

We keep the notation corresponding to the sets $A = A' \cup A_n$, B and C, but we take a finer partition of $A_n = A_{n_B} \cup A_{n_{D_1}} \cup A_{n_R}$, where A_{n_B} is defined as in the proof of Theorem 2 and $A_{n_{D_1}}$ is set of the vertices of A_n in deterministic paths associated with vertices of set D_1, according to the definition of condition (7). Then, we denote the number of deterministic paths in $A_{n_{D_1}}$, D_1 and D_2 by $r_{n_{D_1}}$, l and h, respectively. Thus, $r_n = r_{n_B} + r_{n_{D_1}} + r_{n_R}$.

Then, we relabel in decreasing order the vertices, from n to 1, considering the sets ordered in the following way: A_{n_R}, $A_{n_{D_1}}$, A_{n_B}, $A_{n-1}, \ldots, A_1$, D_1, B, C and D_2, and in each set starting from the longest deterministic path and finishing with the shortest deterministic path.

Define P as the permutation matrix associated with this relabelling, and let Q be the matrix reordering the columns of $\mathscr{G}$ in the order used in Theorem 2, that is, the first columns of $\mathscr{G}$ are monomial vectors, the following t columns of $\mathscr{G}$ have the structure considered in (3) and in addition, the following h columns have the form given in (9). Then, the pair $[P^{\mathrm{T}}FP|P^{\mathrm{T}}GQ]$ has the following structure

$$\left[\begin{array}{c|c|c|c|c|c|c|c||c}
\mathscr{D}_2 & O & O & O & O & O & \mathit{\Delta} & \mathit{\Delta} & \\ \cline{1-8}
O & \mathscr{C} & O & O & O & O & \mathit{\Delta} & \mathit{\Delta} & \\ \cline{1-8}
O & O & \mathscr{B} & O & O & \Sigma & \mathit{\Delta} & \mathit{\Delta} & \\ \cline{1-8}
O & O & O & \mathscr{D}_1 & O & O & \Sigma & \mathit{\Delta} & \\ \cline{1-8}
O & O & O & O & \mathscr{A}' & \mathit{\Delta} & \mathit{\Delta} & \mathit{\Delta} & \mathscr{G} \\ \cline{1-8}
O & O & O & O & O & \mathscr{A}_{n_B} & \mathit{\Delta} & \mathit{\Delta} & \\ \cline{1-8}
O & O & O & O & O & O & \mathscr{A}_{n_{D_1}} & \mathit{\Delta} & \\ \cline{1-8}
O & O & O & O & O & O & \mathit{\Delta} & \mathscr{A}_{n_R} &
\end{array}\right], \tag{21}$$

where $\mathscr{C}$, $\mathscr{B}$, $\mathscr{A}'$, $\mathit{\Delta}$, $\mathscr{A}_{n_B}$, and Σ are the matrices given in (14)–(16), (18) and (19), respectively. Moreover, $\mathscr{A}_{n_R}$, is a block matrix where all off-diagonal blocks are Ψ and the blocks in the diagonal have the structure given in (20). In addition,

$$\mathscr{D}_1 = \mathrm{diag}[\mathscr{D}_1(l), \ldots, \mathscr{D}_1(1)] \quad \text{and} \quad \mathscr{D}_2 = \mathrm{diag}\,[\mathscr{D}_2(h), \ldots, \mathscr{D}_2(1)], \tag{22}$$

where $\mathscr{D}_1(i)$ and $\mathscr{D}_2(j)$, $\forall i = 1, \ldots, l$ and $\forall j = 1, \ldots, h$, are cyclic irreducible blocks. Further

$$\mathscr{A}_{n_{D_1}} = \operatorname{diag}\left[\mathscr{A}_{n_{D_1}}(l), \ldots, \mathscr{A}_{n_{D_1}}(1)\right], \tag{23}$$

where $\mathscr{A}_{n_{D_1}}(j)$, $j = 1, \ldots, l$, has the same structure as the matrices given in (17).

Note that if there exists some nonzero entry in the matrix Δ, connecting $\mathscr{A}_{n_{D_1}}$ with $\mathscr{A}_{n_R}$, then there exists restrictions given by the definition of set D_1 on the nonzero entries in the column block Δ associated with $\mathscr{A}_{n_R}$.

Moreover, it is possible to check that the pair $(P^{\mathrm{T}}FP, P^{\mathrm{T}}GQ)$ given in (21) is essentially reachable.

Therefore, we summarize the above discusion in the following theorem.

Theorem 9. *Assume* $(F, G) \geqslant 0$. *Then*, (F, G) *is essentially reachable if and only if there exists permutation matrices* P *and* Q *such that* $[P^{\mathrm{T}}FP \mid P^{\mathrm{T}}GQ]$ *has the structure given in* (21), *where the blocks are given in* (14)–(23).

Example 4. Consider the control system $(F, G) \geqslant 0$ given in Example 1.

We note that $A \cup B \cup C \cup D_1 \cup D_2 = \{1, \ldots, 11\}$. Then, the pair (F, G) has a monomial submatrix of order 11. Therefore, the pair (F, G) is essentially reachable and its canonical form is given by

$$\left[P^{\mathrm{T}}FP|P^{\mathrm{T}}GQ\right]$$

$$= \left[\begin{array}{c|c|c|cc|c|c|ccc|c||cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \cline{1-2}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \cline{2-3}
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \cline{3-5}
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \cline{4-6}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \cline{6-7}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \cline{7-10}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \cline{8-11}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\mathscr{D}_2 & \mathscr{C} & \mathscr{B} & \multicolumn{2}{c}{\mathscr{D}_1} & \mathscr{A}' & \mathscr{A}_{n_{\mathscr{B}}} & \multicolumn{3}{c}{\mathscr{A}_{n_{\mathscr{D}_1}}} & \mathscr{A}_{n_R} & \multicolumn{6}{c}{\mathscr{G}}
\end{array}\right]$$

Finally, since the essential reachability property together with the stability property is equivalent to the essential controllability property, the canonical form of this last property coincides with that of the former property adding that the spectral radius of F is less than 1.

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