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Modelling Nitrogen Dynamics in Citrus Trees

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Abstract—In this work, the dynamics of nitrogen absorption, distribution, and translocation in citrus trees is modelled by a positive periodic system, concretely by a compartmental system. Some properties of the model, such as reachability and stability, are studied. Furthermore, a simulation with the constructed model to predict some consequences is given.

Keywords—Compartmental systems, Nitrogen dynamics, Citrus trees, Modelling and simulation.

1. INTRODUCTION

Lately, too much nitrogen fertilizer is being applied in citrus trees because farmers generally observe that yield increases with higher doses. However, this response has a border from which the bigger outlay of fertilizer gives up compensating the yield improvement and, moreover, produces a fruit quality decrease. On the other hand, it leads to a serious environmental contamination, mainly in water bearings. So, studies on a rational application of fertilizer on citrus trees to improve its efficiency are necessary.

Mathematical modelling and simulation constitute a set of methods that are followed in research to solve or to understand a certain problem using mathematical knowledge. Concretely, they can help us to study the real behavior of nitrogen when it is applied to the soil with the irrigation water, and then to improve the criteria that are followed in the nitrogen fertilization of citrus at the moment.

As we will see, the dynamics of nitrogen absorption, distribution, and translocation in citrus trees can be modelled by a periodic compartmental system. *Compartmental systems* consist of a finite number of interconnected subsystems called *compartments*. The interactions between compartments and to/from the environment are transfers of material. Transfers that hold the *law of conservation of mass*. Compartmental systems are natural models for many areas of application that are subject to that law. For instance, they have been extensively used in chemistry,

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medicine, epidemiology, ecology, pharmacokinetics, and recently even in many social, economical, and marketing models (see [1–5]).

In Section 2 of this work, we recall the mathematical representation used for modelling the nitrogen flow in Section 3 where the compartmental system is displayed and the corresponding periodic model is constructed, ending the section studying some properties such as reachability and stability. Finally, in Section 4, a simulation is given with the constructed model in order to predict some consequences. Next, we give some notations that will be necessary. By $\mathbb{Z}$, we denote the set of integers, and by $\mathbb{R}_+^{m\times n}$, the set of $m \times n$ matrices where the entries are nonnegative real numbers. The matrix $A(k) = [a_{ij}(k)]$ is a matrix with a_{ij} entries at time k, $U = \mathrm{col}[u_1, \dots, u_n]$ is a column matrix with rows $u_1, \dots, u_n$ and $\phi_A(k, k_0)$ is the state transition matrix of a system defined by

$$\begin{aligned} \phi_A(k,k_0) &= A(k-1)A(k-2)\cdots A(k_0), \qquad k > k_0, \\ \phi_A(k_0,k_0) &= I. \end{aligned} \tag{1}$$

We denote by $\rho(A)$ the spectral radius of a square matrix A. Recall that it is the maximum of the module of the eigenvalues of A.

2. MATHEMATICAL MODEL

The state variables of compartmental systems represent the amount of material contained in each compartment, and hence, are bound to be nonnegative over time. Thus, compartmental systems belong to the broader class of *positive systems*, where the state and the output variables remain nonnegative for all positive input sequences [6].

A *positive discrete-time linear control system* is given by the equation

$$x(k+1) = A(k)x(k) + B(k)u(k), \qquad k \in \mathbb{Z}, \tag{2}$$

where $A(k) = [a_{ij}(k)] \in \mathbb{R}_+^{n\times n}$ is the *state matrix*, $B(k) = [b_{ij}(k)] \in \mathbb{R}_+^{n\times m}$ the *control matrix*, $x(k)$ the nonnegative *state vector*, and $u(k)$ the nonnegative *control* or *input vector* at time k. It is denoted by $(A(\cdot), B(\cdot))$.

The positive system (2) will be *N-periodic* with $N \in \mathbb{Z}$, if

$$\begin{aligned} A(k) &= A(k+N) \in \mathbb{R}_+^{n\times n}, \\ B(k) &= B(k+N) \in \mathbb{R}_+^{n\times m}, \end{aligned} \tag{3}$$

for all k. In the periodic case, it is enough to consider that k goes from 0 to $N-1$ since the periodicity. It is denoted by $(A(\cdot), B(\cdot))_N$. Particularly, when $N = 1$, the equations given by expressions (2) and (3) determine an *invariant* system, with $A(k) = A$ and $B(k) = B$, $k \in \mathbb{Z}$. This system is denoted by (A, B).

Moreover, the positive periodic system (2),(3) will be *compartmental* if

$$\sum_{i=1}^{n} a_{ij}(k) \le 1, \qquad \text{for } j = 1, 2, \dots, n \text{ and for } k = 0, 1, \dots, N-1. \tag{4}$$

This condition comes from the balance of material which says that the output flow of material at time $k+1$ from a compartment cannot be larger than the whole mass of material that was present at time k, if the inflow from outside is assumed to be zero. A matrix $A(k) = [a_{ij}(k)] \in \mathbb{R}_+^{n\times n}$ satisfying the condition in equation (4) is called a *compartmental matrix*; where the entry $a_{ij}(k)$ $(i \ne j)$ is the rate inflow from the j^{th} to the i^{th} compartment between time k and time $k+1$, and the entry $a_{ii}(k)$ is the rate of material that was in compartment i at time k and is again in this compartment at time $k+1$.

To the linear compartmental N-periodic system (2)–(4), one can consider N discrete-time invariant linear systems associated with it (see [7]),

$$x_s(k+1) = A_s x(k) + B_s u(k), \tag{5}$$

where

$$\begin{aligned}
x_s &= x(kN+s),\\
u_s &= \mathrm{col}[u(kN+s+N-1), u(kN+s+N-2), \dots, u(kN+s)],\\
A_s &= \phi_A(s+N, s),\\
B_s &= [B(s+N-1), \phi_A(s+N, s+N-1)B(s+N-2), \dots, \phi_A(s+N, s+1)B(s)],
\end{aligned}$$

for every $s = 0, 1, \dots, N-1$, where $\phi_A(k, k_0)$ is the transition matrix of the periodic system given in (1). Therefore, A_s is a cyclic product of the matrices $A(k)$ and it is called the *monodromy matrix* at time s of the periodic system. These invariant systems are useful for studying the periodic system, as we will see later.

3. NITROGEN MODELIZATION

3.1. Preliminaries

Legaz (see [8]) studied different fundamental aspects to make a rational fertilizer practice in citrus trees, concretely in Valencia late orange trees: nitrogen absorption and distribution between the different organs of the plant through an annual growth cycle, and the mobilization and distribution of reserve nitrogen accumulated during the previous year. That work has given us the information we need to design a mathematical model which describes the dynamics of nitrogen in citrus trees. In that work [8], it was pointed out that the dynamics of nitrogen absorption, distribution, and translocation in citrus trees is not constant through an annual cycle, therefore, we will consider the following seasons of growth:

- *Rest*: from January 1–31 (we will denote it by R),
- *Flowering*: from February 1–April 30 (Fl),
- *Fruit Set*: from May 1–June 15 (S),
- *Second Growth Flush*: from June 16–August 15 ($2F$),
- *Third Growth Flush*: from August 16–October 31 ($3F$),
- *Winter Rest*: from November 1–December 31 ($2R$).

To design the model, we take the soil-plant system formed by several subsystems so that, the first subsystem is the soil, and the following subsystems are each organ of the plant, beginning with the fibrous roots and finishing with the third growth flush leaves. These subsystems are interconnected between them by flowing nitrogen. Nitrogen flows follow the law of conservation of mass. Thus, the dynamics of nitrogen absorption, distribution and translocation in citrus trees can be modelled by a *compartmental system* formed by the compartments showed in Figure 1.

In Figure 1, we have drawn the compartments in levels, labelled from the top to the bottom. We denote by

- $u(k)$: amount of nitrogen applied to the soil at time k,
- $a_{ij}(k)$: nitrogen rate flowing from compartment j to compartment i at time k due to nitrogen absorption by the plant,
- $t_{ij}(k)$: nitrogen rate flowing from compartment j to compartment i at time k due to nitrogen traslocation (mobilization of reserve nitrogen).

The discontinuous line-dot arrows indicate nitrogen losses by organ death, and the discontinuous line arrow indicates that the flow direction changes depending on the season of growth we

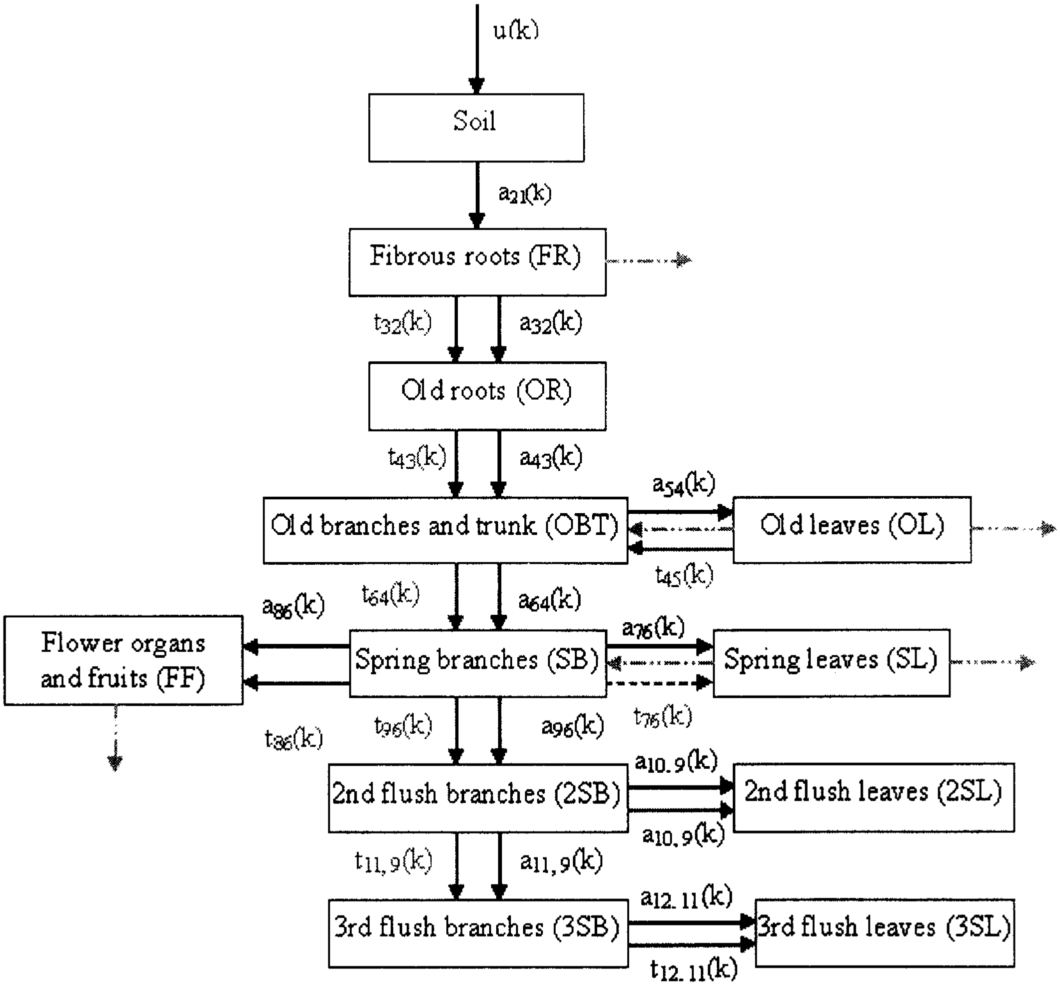

Figure 1. Soil-plant system (compartmental system).

stay. Moreover, we must comment that the general diagram shown in Figure 1 also changes through the annual growth cycle. At the beginning of the year, during the rest, the system consists on the first five compartments (Levels 1–4). During the following season, in the flowering, spring growth flush and flowering happen, appearing the next three compartments (Level 5). In the fruit set season, we continue with the same compartments. After, in summer, the second growth flush takes place (compartments of Level 6) and finally in autumn, the third growth flush (compartments of Level 7), completing the diagram of the figure. So, the system is formed by 12 compartments. Next year, the diagram will be the same, since organs from the previous flushes are now considered as old organs. Again, we have a system composed by the first five compartments, and like last year, the following compartments will appear with the next flushes of the annual cycle, completing the diagram of the second year. It will happen equally the third year, and so on, year after year. Then, it is a periodic process with one year periodicity.

3.2. Nitrogen Flow Model

We have just seen that the dynamics of nitrogen in citrus trees changes through the annual cycle periodically. Therefore, it can be represented by a periodic compartmental system of period $N = 365$, as

$$x(k+1) = A(k)x(k) + B(k)u(k), \tag{6}$$

where $A(k) = A(k+365) \in \mathbb{R}_+^{12\times 12}$, with $\sum_{i=1}^{12} a_{ij}(k) \le 1$, $j = 1, 2, \ldots, 12$, and $B(k) = B(k+$

$365) \in \mathbb{R}_+^{12\times 1}$, for all $k \in \mathbb{Z}$. We denote by

- $x(k)$: amount of nitrogen present in each compartment at time k, $x(k) \in \mathbb{R}_+^{12}$ because the system has 12 compartments;
- $x(k+1)$: amount of nitrogen present in each compartment at time $k+1$;
- $u(k)$: the control, amount of nitrogen applied to the soil by fertigation at time k. It is a positive real number because only one fertilizer is applied to the system.

According to the experiences of [8], the periodic matrices $A(k)$ can be considered invariant during each season of growth. As we have six seasons of growth in an annual cycle, we have six different matrices $A(k)$ that we indicate now:

- $A(0) = \cdots = A(30) = AR$,
- $A(31) = \cdots = A(119) = A\mathrm{Fl}$,
- $A(120) = \cdots = A(165) = AS$,
- $A(166) = \cdots = A(226) = A2F$,
- $A(227) = \cdots = A(303) = A3F$,
- $A(304) = \cdots = A(364) = A2R$,
- again $A(365) = A(0)$, and so on.

As we said in Section 2, the different entries of matrices $A(k)$ are all nitrogen rates. To calculate these rates, we need to know the nitrogen flows between the different organs of the tree due to nitrogen absorption and nitrogen traslocation at the different seasons of growth. Taking into account these data and the diagram of Figure 1, the nitrogen rates are calculated as follows:

$$f_{ij}(k) = \frac{F_{ij}(k)}{XM_j(k)}, \tag{7}$$

with $i \neq j$, $i = 0, 2, 3, \dots, 12$ (the index $i = 0$ corresponds to the environment) and $j = 1, 2, \dots, 6, 9, 11$. Where $f_{ij}(k)$ is the nitrogen rate inflow from the j^{th} to the i^{th} compartment between time k and time $k+1$; $F_{ij}(k)$ is the amount of nitrogen that flows from the compartment j to i at this time, it corresponds to the sum of nitrogen mobilized by absorption and by traslocation; and $XM_j(k)$ is the average amount of nitrogen present at the compartment j at time k, i.e.,

$$XM_j(k) = \frac{(XF_j(k) + X\exp_j(k))}{2}. \tag{8}$$

Parameters $XF_j(k)$ and $X\exp_j(k)$ are the amount of nitrogen in the compartment j at the end of the previous season of growth and approximately at the end of the present season, respectively. The first one was calculated with the mathematical model and the second one was obtained experimentally with five year old trees. The process was the following. We began calculating the nitrogen rates from the rest season, considering that $XM_j(k)$, $k = 0, 1, \dots, 30$, was equal to the amount of nitrogen in each compartment j obtained experimentally. Then, the entries of the matrices $A(k)$ was obtained by

$$a_{ij}(k) = f_{ij}(k), \qquad i, j = 1, 2, \dots, 12\ (i \neq j),$$
$$a_{ii}(k) = 1 - \sum_{j=0}^{12} f_{ji}(k).$$

Recall that $A(0) = \cdots = A(30) = AR$. The next step was to program the compartmental model (6) in MATLAB to obtain the vector $x(k)$ from $k = 0$ to 30. For that, we considered a typical fertilizer program (see [9,10]) where fertilizer is not applied until March, i.e.,

$$x(k+1) = AR \cdot x(k), \qquad k = 0, 1, \dots, 29.$$

The vector $x(30)$ is the state vector at the end of January 31, that is, at the end of the rest season. So, it is the value of the parameter $XF_j(k)$, we needed to calculate the nitrogen rates from the following season. Afterwards, we obtained the matrices $A(31) = \cdots = A(119) = A\mathrm{Fl}$, and again, the state vector $x(k)$ at the end of the season. The control $u(k)$ was always applied according to a typical fertilizer program and the nutrient needs of a five year old trees.

In that way, all the matrices $A(k)$ for the different seasons were successively obtained taking into account that the periodic matrices $B(k) \in \mathbb{R}_+^{12}$ were all the same because we always applied fertilizer (control) to the soil only, that is, to the first compartment. Hence,

- $B(0) = \cdots = B(364) = B = [\,1\;0\;\cdots\;0\,]^\top$,
- again $B(365) = B(0)$, and so on.

Here we display the matrices $A(k)$ obtained for each season of growth.

REST.

$$AR = \begin{bmatrix} 0.9995 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0005 & 0.9839 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0143 & 0.9988 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0012 & 0.9989 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0011 & 0.9999 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

FLOWERING.

$$A\mathrm{Fl} = \begin{bmatrix} 0.9961 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0029 & 0.9188 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0781 & 0.9854 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0146 & 0.9730 & 0.0047 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0032 & 0.9947 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0023 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0116 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0064 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

FRUIT SET.

$$AS = \begin{bmatrix} 0.9912 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0058 & 0.8588 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1346 & 0.9806 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0194 & 0.9783 & 0.0031 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0055 & 0.9945 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0162 & 0 & 0.9117 & 0.0002 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0393 & 0.9991 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0490 & 0 & 0.9670 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

SECOND GROWTH FLUSH.

$$A2F = \begin{bmatrix}
0.9933 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0041 & 0.8863 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.1079 & 0.9836 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0164 & 0.9793 & 0.0072 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0015 & 0.9902 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0192 & 0 & 0.8825 & 0.0002 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0222 & 0.9991 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0356 & 0 & 0.9926 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0045 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0551 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

THIRD GROWTH FLUSH.

$$A3F = \begin{bmatrix}
0.9958 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0032 & 0.9142 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0845 & 0.9879 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0121 & 0.9832 & 0.0048 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0014 & 0.9941 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0154 & 0 & 0.8611 & 0.0042 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0081 & 0.9951 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0508 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0800 & 0 & 0 & 0.8198 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0593 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0137 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1073 & 0 & 0 & 1
\end{bmatrix}.$$

WINTER REST.

$$A2R = \begin{bmatrix}
0.9978 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0022 & 0.9534 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0466 & 0.9930 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0070 & 0.9906 & 0.0007 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0003 & 0.9943 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0091 & 0 & 0.9253 & 0.0004 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0049 & 0.9989 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0634 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0064 & 0 & 0 & 0.9896 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0096 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0008 & 0 & 0.9996 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0004 & 1
\end{bmatrix}.$$

Note that the main diagonal entries $a_{ii}(k)$ represent the nitrogen rate which was in the compartment i at time k and remains in this compartment at time $k+1$, so null diagonal values mean that we do not have the corresponding organs at this season yet. On the other hand, null values of entries $a_{ij}(k)$, $i \neq j$, will mean that the corresponding compartments are not connected.

To fit the model as well as possible, the obtained results were compared with the experimental data. Tables 1–3 show the differences between the last results achieved by the model and experimental data through an annual growth cycle, starting with the experimental data from the rest season. We denote by X_{exp} and X_{comp}, the experimental and computed data of the amount of nitrogen inside the different organs approximately at the end of each season of growth, respectively.

Table 1. Data corresponding to the flowering and fruit set seasons.

Organs	XF_{exp}	XF_{comp}	Relative Error	XS_{exp}	XS_{comp}	Relative Error
3SL	-	-	-	-	-	-
3SB	-	-	-	-	-	-
2SL	-	-	-	-	-	-
2SB	-	-	-	-	-	-
FF	9121	9110	0.12	5793	5880	1.50
SL	15779	15540	1.51	20767	20740	0.13
SB	3158	3140	0.57	3568	3670	2.86
OL	21495	21150	1.61	20444	20570	0.62
OBT	16916	16090	4.88	22214	20990	5.51
OR	20906	19690	5.82	26018	25920	0.38
FR	3653	3550	2.82	4525	4700	3.87

Table 2. Data corresponding to the second and third growth flushes.

Organs	$X2_{exp}$	$X2_{comp}$	Relative Error	$X3_{exp}$	$X3_{comp}$	Relative Error
3SL	-	-	-	7412	7710	4.02
3SB	-	-	-	963	980	1.77
2SL	10152	9490	6.52	17421	17420	0.01
2SB	798	800	0.25	1601	1570	1.94
FF	9477	9390	0.92	20665	19080	7.67
SL	23943	23540	1.68	20873	20860	0.06
SB	3786	3950	4.33	3533	3540	0.20
OL	13493	14490	7.39	11611	11320	2.51
OBT	25227	25050	0.70	26713	26200	1.92
OR	30775	28720	6.68	34718	33710	2.90
FR	4805	5020	4.47	5342	5740	7.45

Table 3. Data corresponding to the winter rest season.

Organs	$X2R_{exp}$	$X2R_{comp}$	Relative Error
3SL	12098	12980	7.29
3SB	1578	1690	7.10
2SL	20085	21120	5.15
2SB	1700	1790	5.29
FF	36453	35570	2.42
SL	19167	18560	3.17
SB	3280	3420	4.27
OL	7904	8230	4.12
OBT	27456	27330	0.46
OR	38044	37390	1.72
FR	6455	6340	1.78

Relative error is given in percentage, and the rows of each table represent the different organs of the tree with the same notation as Figure 1. Table 1 corresponds to the flowering and fruit set seasons, Table 2 corresponds to the second and third growth flushes, and Table 3 correspond to the winter rest season.

As we can see, model fit to experimental data. Computed data differ from experimental data in at most 7.67%.

3.3. Properties of the Model

In this section, we study the structural properties and the stability of the nitrogen flow model.

3.3.1. Structural properties

First, we recall the reachability and controllability concepts for positive periodic systems (see [11]). It is known that a positive periodic system $(A(\cdot), B(\cdot))_N$ is *completely controllable at time* s, if any final state $x_f \in \mathbb{R}^n_+$ can be reached in finite time, from any initial state $x_0 \in \mathbb{R}^n_+$ at time s, with a nonnegative control sequence. It is *completely controllable* if it is completely controllable at time s for all $s \in \mathbb{Z}$. On the other hand, it is *reachable* if $x_0 = 0$ and it is *null-controllable* if $x_f = 0$.

In [7], the structural properties of positive N-periodic systems by means of the associated N invariant systems defined by (5) are given. They are summarized in the following results.

PROPOSITION 1. *(See [7].) A positive periodic system $(A(\cdot), B(\cdot))_N$ is completely controllable at time s if and only if the positive invariant system (A_s, B_s) corresponding to index s is completely controllable, for $s = 0, 1, \dots, N-1$.*

REMARK 2. This proposition holds for every final state $x_f \geq 0$ and for every initial state $x_0 \geq 0$. In particular, if $x_f = 0$, we will have the equivalence for null-controllable, and if $x_0 = 0$ for reachable.

Our model is a compartmental periodic system of period $N = 365$ so we can represent it by 365 different invariant systems, that is, $s = 0, 1, \dots, 364$.

For invariant systems, positive reachability, null-controllability, and controllability have being studied by different authors. We will use some of those results, where the notion of monomial vectors (matrices) is of fundamental importance. Recall that a vector with exactly one nonzero entry is called *monomial*, so the unit basic vectors e_i are clearly monomial. Moreover, an $n \times m$ real matrix is said to be monomial if it consists of linearly independent monomial columns.

PROPOSITION 3. *(See [12].) A positive linear invariant system (A, B) is reachable if and only if the reachability matrix $R_n(A, B) = [B \;\; AB \;\; \cdots \;\; A^{n-1}B]$ has an $n \times n$ monomial submatrix.*

Applying Remark 2 and Proposition 3, we obtain the following result for the periodic system (6), which represents the dynamics of nitrogen in citrus trees described in Section 3.2.

COROLLARY 4. *The periodic system (6) is not reachable.*

PROOF. The periodic system (6) is reachable if all the associated invariant systems are reachable, i.e., their reachability matrices,

$$R_{12}^{(s)} = [B_s \quad A_s B_s \quad \cdots \quad A_s^{11} B_s],$$

$s = 0, 1, \dots, 364$, have a 12×12 monomial submatrix. It is enough to check that one invariant system is not reachable. Let us work with $s = 1$. Using (5), we have that the matrix A_1 is the monodromy matrix at time 1 of the periodic system,

$$\begin{aligned} A_1 &= \phi_A(366, 1) = A(365) \cdot A(364) \cdots A(1) \\ &= AR \cdot A2R^{61} \cdot A3F^{77} \cdot A2F^{61} \cdot AS^{46} \cdot AFl^{89} \cdot AR^{30}, \end{aligned}$$

where AR is the block matrix

$$AR = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix}, \tag{9}$$

where $\hat{A}$ is the 5×5 nonnegative leading principal submatrix. As A_1 is the product of AR with other matrices, we obtain

$$A_1 = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ 0 & 0 \end{bmatrix}, \tag{10}$$

where C_{11} is a 5×5 nonnegative leading principal submatrix, and C_{12} is a 5×2 nonnegative submatrix. Following the same reasoning, the structure of the matrices A_1B_1, $A_1^2B_1, \dots, A_1^{11}B_1$, for any matrix B_1, coincides with (10). Therefore, we will not be able to obtain from the products among matrices A_1 and B_1, at least, the monomial vector $e_{12} = [0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1]^\top$.

On the other hand, B_1 is (see Section 2)

$$B_1 = [\, B(365) \quad \phi_A(366,365)B(364) \quad \cdots \quad \phi_A(366,2)B(1)\,],$$

where the matrices $B(\cdot)$ are all $B = [1 \;\; 0 \;\; \cdots \;\; 0]^\top$ (see Section 3.2). Furthermore, we have just seen that the matrices $\phi_A(366, k_0)$, $k_0 = 2, 3, \dots, 365$, are the result of the product of the matrix AR with other matrices. Thus, reasoning as before, we conclude that the matrix B_1 has the same structure than the matrix A_1.

In consequence, we will never have a 12×12 monomial submatrix in the reachability matrix $R_{12}^{(1)} = [B_1 \;\; A_1B_1 \;\; A_1^2B_1 \;\; \cdots \;\; A_1^{11}B_1]$, hence, this invariant system will not be reachable and so, the periodic system (6). ∎

We note that the same proof can be made for both matrices A_s and B_s of the invariant systems, $s = 2, 3, \dots, 227$, because the monodromy matrix of these systems are products of matrices where the first one has the same partition into blocks as AR (see Section 3.2, matrices AFl, AS, and $A2F$), but consistently changing the size of $\hat{A}$.

Let us see now the null-controllability property of our system. We can find the following result.

PROPOSITION 5. *(See [13].) A positive invariant linear system (A, B) is null-controllable if and only if A is a nilpotent matrix, i.e., $A^k = 0$ for some integer $k \le n$.*

Application of Remark 2 and Proposition 5 to the system of the nitrogen flow model yields the next result.

COROLLARY 6. *The periodic system (6) is not null-controllable.*

PROOF. The *periodic system* (6) will be null-controllable if and only if all the associated invariant systems are null-controllable. It is known that a square matrix A is nilpotent if and only if its eigenvalues are all 0. Note that none of the matrices A_s holds this condition, because they are the product of nonnegative matrices, which have at least the first five entries of the main diagonal positive. So, none of the invariant systems are null-controllable, and then, the periodic system either. ∎

We finish these properties studying the complete controllability of our positive periodic system.

PROPOSITION 7. *(See [13].) A positive linear system (A, B) is completely controllable if and only if it is reachable and null-controllable.*

Combining Propositions 1 and 7 with Corollaries 4 and 6, we obtain the following result.

COROLLARY 8. *The periodic system (6) is not completely controllable.*

PROOF. We have just seen in Corollaries 4 and 6 that not all the associated invariant systems are neither reachable nor null-controllable, then by Proposition 7, they are not completely controllable. By Proposition 1, the result follows. ∎

That our system is not completely controllable means that not every state (amount of nitrogen) can be achieved with any control (fertilizer program).

3.3.2. Stability of positive periodic systems

In the same way that happens with the structural properties, an N-periodic system $(A(\cdot),B(\cdot))_N$ will be stable if and only if the associated invariant systems (A_s, B_s) are all stable.

First, we recall (see [3]) that a system is *asymptotically stable* if and only if for any input u, there exists a single equilibrium state x_e and $x(k)$ satisfies the condition

$$\lim_{k\to\infty} x(k) = x_e,$$

for every $x(0)$ when $u(k) = u$.

We recall that a discrete time linear system (A, B) is *asymptotically stable* if and only if all eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the matrix $A(k) \in \mathbb{R}_+^{n\times n}$ have module less than 1. This is equivalent to say that the spectral radius $\rho(A) < 1$.

COROLLARY 9. *The periodic system (6) is asymptotically stable.*

PROOF. It is known that the spectral radius satisfies that for any two matrices M and N, $\rho(MN) = \rho(NM)$ (see [14]). Hence, the spectral radius of a product of matrices is invariant under cyclic permutations of the product factors. In other words,

$$\rho(A_s) = \rho(A_{s+1}), \qquad s = 0, 1, \dots, 363.$$

Therefore, it will be sufficient to study the stability of one of the invariant systems. We study the case for $s = 1$. Recall that the state matrix of this system is the block matrix A_1 given in (10). As

$$A_1 = AR \cdot A2R^{61} \cdot A3F^{77} \cdot A2F^{61} \cdot AS^{46} \cdot AF1^{89} \cdot AR^{30},$$

note that the last matrix is again AR given in (9), so the block C_{12} in the matrix A_1 is a 5×2 zero submatrix, i.e.,

$$A_1 = \begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$C_{11} = \begin{bmatrix} 0.1943 & 0 & 0 & 0 & 0 \\ 0.0095 & 0.0000 & 0 & 0 & 0 \\ 0.0788 & 0.0112 & 0.0103 & 0 & 0 \\ 0.0739 & 0.0355 & 0.0342 & 0.0110 & 0.0498 \\ 0.0149 & 0.0302 & 0.0313 & 0.0264 & 0.1355 \end{bmatrix}.$$

It is easy to check that $\rho(A_1) = 0.1943 < 1$. Then, the invariant systems is asymptotically stable, and, in the same way, the rest of the invariant systems. Consequently, the periodic system is asymptotically stable. ∎

This result is natural because it means that without any application of nitrogen, at the infinity, any compartment (organs of the tree) will not contain nitrogen.

4. SIMULATION

As we comment in the introduction, the mathematical model of the nitrogen dynamics can help us to improve the criteria that are followed in the nitrogen fertilization of citrus at present. The model permits study of the real behavior of nitrogen when it is applied to the soil with the irrigation water. For instance, we may predict what would happen if we applied another nitrogen different from the optimum. We assume that this is the fertilizer program that has been applied with the original model.

Let us see an example. Consider that we applied less nitrogen, i.e., 75% of the optimum rate in each time. If we run the model with the new control sequence, we find that the amount of nitrogen mainly decreases in old organs (see Figure 2) but it does not practically change in young

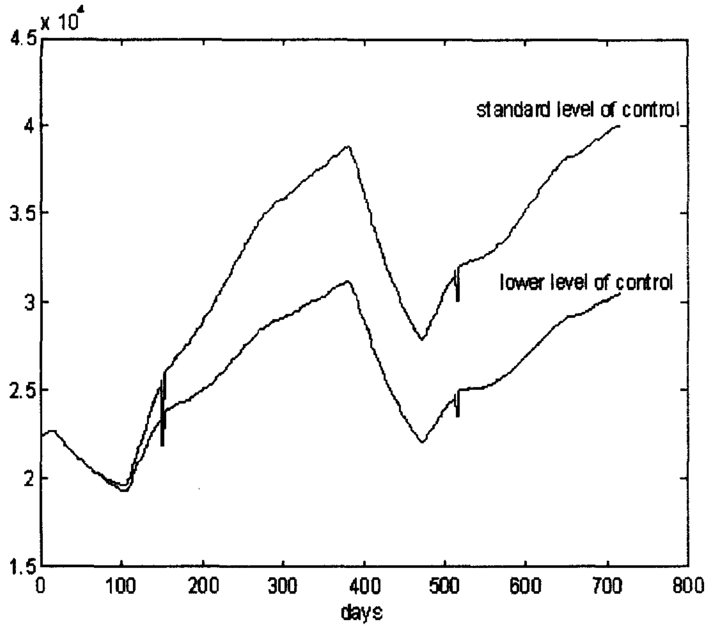

Figure 2. Amount of nitrogen in old roots.

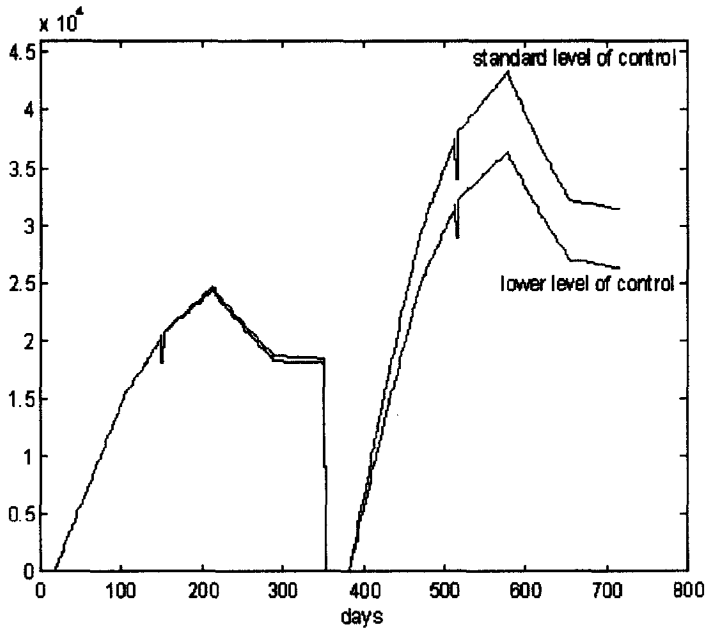

Figure 3. Amount of nitrogen in spring leaves.

organs (Figure 3). This event seems natural because most of the absorbed nitrogen normally goes to developing organs, and the remainder goes to old organs. This last amount will constitute the nitrogen reserve that will be translocated to developing organs next flushes. Then, we will observe the consequences in the following years, given some insights. For that we only run it again. But as we saw in Section 3.1, taking into account that organs from the last flushes are now considered as old organs, i.e., at the beginning of the second cycle, the amounts of nitrogen in each compartment are

$$\begin{aligned} X_{\mathrm{FR}} &= X2R_{\mathrm{FR}}, \\ X_{\mathrm{OR}} &= X2R_{\mathrm{OR}}, \\ X_{\mathrm{OBT}} &= X2R_{\mathrm{OBT}} + X2R_{\mathrm{SB}} + X2R_{\mathrm{2SB}} + X2R_{\mathrm{3SB}}, \\ X_{\mathrm{OL}} &= X2R_{\mathrm{OL}} + X2R_{\mathrm{SL}} + X2R_{\mathrm{2SL}} + X2R_{\mathrm{3SL}}, \end{aligned}$$

$$X_{\mathrm{SB}} = 0, \qquad X_{\mathrm{SL}} = 0, \qquad X_{\mathrm{FF}} = 0, \qquad X_{2\mathrm{SB}} = 0,$$
$$X_{2\mathrm{SL}} = 0, \qquad X_{3\mathrm{SB}} = 0, \qquad X_{3\mathrm{SL}} = 0,$$

where $X2R$ are the amount of nitrogen in each organ at the end of the winter rest.

To simplify, we only show the results of an old organ (reserve organ) and a young organ (flush). Figure 2 shows the results of old roots and Figure 3 shows the results of spring leaves.

Note that differences between both treatments increase in the second cycle. As we expect, there is an important decrease in the amount of nitrogen in young organs. Differences are very significative in all organs now.

5. CONCLUSIONS

A compartmental periodic system has been constructed to model the dynamics of nitrogen in citrus trees. The model has been validated using previous studies in Valencia late orange trees, obtaining relative errors smaller than 8%. The properties of the model have been stated seeing that it is not reachable nor null-controllable but it is asymptotically stable, which means that not every wanted amount of nitrogen can be achieved with any fertilizer program, and without any fertilization, at the end, any organ of the tree will not contain nitrogen. A simulation with an other level of nitrogen has been achieved showing that the model can be used to predict the consequences of applying other fertilizer programs, and hence, to fit it. We saw that it is not convenient to apply so little nitrogen to the trees because eventually yield lowers.

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