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Gramian matrices and balanced model of generalized systems ☆

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Abstract

In this work, the generalization of reachability and observability Gramian matrices of control generalized systems have been studied. The properties of these matrices to be solutions of Lyapunov equations have been analyzed. In addition, an algorithm to obtain balanced generalized systems is given.

Keywords: Linear discrete-time generalized systems; Reachability; Observability; Lyapunov equation; Gramian matrix; Balanced systems

1. Introduction

The reachability and observability properties have special attention in control theory. Many classical problems of linear control systems have been studied using structural properties, for instance the balanced realization problem was introduced by Moore in [1]. Ober and McFarlane used Gramian matrices of standard systems by means of the corresponding reachability and observability matrices of the system in [2]. Gramian matrices have gained attention in the control theory because these matrices contain interesting information on the input–output behaviour of the system, but really there are no many papers dealing with this topic in the last decade. It is well known that

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Gramian matrices can be used for checking if a system is reachable and/or observable. In addition, these matrices play an important role in the study of the stability of a system using the respective Lyapunov equations.

Recently, different methods studying model reduction of standard system using Gramian matrices have appeared [3,4].

In [5], Bender gave a definition of reachability and observability Gramian matrices using the Laurent expansion of the matrix $(zE - A)^{-1}$. A sufficient condition on the orthogonality of some blocks of matrices B and C allows to state that Gramian matrices are the solution of the Lyapunov equations $EW_rE^{\mathrm{T}} - AW_rA^{\mathrm{T}} = BB^{\mathrm{T}}$ and $E^{\mathrm{T}}W_oE - A^{\mathrm{T}}W_oA = C^{\mathrm{T}}C$. In [6], Gramian matrices of a generalized system (E, A, B, C) have been introduced as the sum of the causal part

$$W_r^c = \sum_{k=0}^{\infty} \phi_k BB^{\mathrm{T}} \phi_k^{\mathrm{T}}, \qquad W_o^c = \sum_{k=0}^{\infty} \phi_k^{\mathrm{T}} C^{\mathrm{T}} C \phi_k$$

and the noncasual part

$$W_r^{\mathrm{nc}} = \sum_{k=-q}^{-1} \phi_k BB^{\mathrm{T}} \phi_k^{\mathrm{T}}, \qquad W_o^{\mathrm{nc}} = \sum_{k=-q}^{-1} \phi_k^{\mathrm{T}} C^{\mathrm{T}} C \phi_k,$$

where q is the nilpotence index and ϕ_k are the Laurent parameters of the serie expansion of the matrix $(zE - A)^{-1}$.

An extension of Gramian matrices to generalized linear discrete-time systems is discussed in this work. We use the reachability and observability Gramian matrices of generalized systems defined as a natural extension of the standard case [7]. We show that these matrices are solutions of the corresponding Lyapunov equations. Further, we use an algorithm to obtain balanced models of generalized systems.

First we provide some useful background which is used in the paper.

Let us consider a generalized system given by

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $k \in \mathbb{Z}$. If $\mathrm{rank}(E) < n$ the system (1) is called *singular system* and if $\mathrm{rank}(E) = n$, then we can consider the system

$$\begin{aligned} x(k+1) &= \widetilde{A}x(k) + \widetilde{B}u(k), \\ y(k) &= Cx(k), \end{aligned} \tag{2}$$

where $\widetilde{A} = E^{-1}A$ and $\widetilde{B} = E^{-1}B$. A system with the structure like that in the system (2) is called a *standard system*. We denote by (E, A, B, C) the singular system and by (A, B, C) the standard system. It is well known (see [8]), that if there exists an escalar $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0$, (i.e. the system (1) satisfies the regularity condition), then the system (1) is equivalent to the forward–backward canonical form given by

$$\begin{aligned} \overline{E}\bar{x}(k+1) &= \overline{A}\bar{x}(k) + \overline{B}u(k), \\ y(k) &= \overline{C}\bar{x}(k), \end{aligned} \tag{3}$$

with $n_1 + n_2 = n$, and

$$\overline{E} = \operatorname{diag}(I_{n_1}, N), \quad \overline{A} = \operatorname{diag}(A_1, I_{n_2}), \quad \overline{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \overline{C} = [\, C_1 \quad C_2 \,]$$

and N is nilpotent. We can split the system (3) into two subsystems, the forward subsystem given by

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + B_1 u(k), \\ y_1(k) &= C_1 x_1(k) \end{aligned} \tag{4}$$

and the backward subsystem given by

$$\begin{aligned} N x_2(k+1) &= x_2(k) + B_2 u(k), \\ y_2(k) &= C_2 x_2(k). \end{aligned} \tag{5}$$

In the following we suppose that singular systems satisfy the regularity condition. Note that, if $\operatorname{rank}(E) = n$, then $\det(\lambda I - E^{-1}A) \neq 0$ and so $\det(\lambda E - A) \neq 0$, for some $\lambda \in \mathbb{C}$. That is, a standard system can be interpreted as a generalized system with only the forward part. Then, we focus our attention on singular systems.

Definition 1. Consider the singular system (E, A, B, C).

(i) A state $x \in \mathbb{R}^n$ is reachable if there exists a time $k \in \mathbb{Z}$ and a control sequence $u(j)$, $j = 0, 1, \ldots, k-1$ transferring the state of the system from the origin at time 0 to x at time k. The system (E, A, B, C) is reachable if every $x \in \mathbb{R}^n$ is reachable.
(ii) The system (E, A, B, C) is observable if the initial condition $x(0)$ may be uniquely determined by control and output sequences, $u(j)$ and $y(j)$, $j = 0, 1, \ldots, k-1$, $k \in \mathbb{Z}$.

The reachability matrices of subsystems (4) and (5) are given by $\Omega_r(A_1, B_1) = [B_1\ A_1B_1\ A_1^2B_1\ \cdots]$ and by $\Omega_r(N, B_2) = [\cdots\ N^2B_2\ NB_2\ B_2]$, respectively. The observability matrices of subsystem (4) and (5) are given by

$$\Omega_o(A_1, C_1) = \begin{bmatrix} C_1 \\ C_1A_1 \\ C_1A_1^2 \\ \vdots \end{bmatrix} \quad \text{and} \quad \Omega_o(N, C_2) = \begin{bmatrix} \vdots \\ C_2N^2 \\ C_2N \\ C_2 \end{bmatrix}, \text{ respectively.}$$

It is known that the system (3) is reachable if and only if $\mathrm{rank}[\Omega_r(A_1,B_1)] = n_1$ and $\mathrm{rank}[\Omega_r(N,B_2)] = n_2$ and the system (3) is observable if and only if $\mathrm{rank}[\Omega_o(A_1,C_1)] = n_1$ and $\mathrm{rank}[\Omega_o(N,C_2)] = n_2$. A reachable and observable system is minimal.

In the following the spectral radius of a square matrix M will be denoted by $\rho(M)$. The set $\mathscr{S}_n^{m,p}$ denote the systems (E,A,B,C) satisfying the following properties: (i) $E,A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, (ii) (E,A,B,C) is a minimal system and (iii) (E,A,B,C) is asymptotically stable.

Now we define the Gramian matrices associated to generalized systems.

Definition 2. Let $(E,A,B,C) \in \mathscr{S}_n^{m,p}$ and consider its associated forward–backward system $(\overline{E},\overline{A},\overline{B},\overline{C})$. The reachability Gramian matrix of the system (E,A,B,C) is given by

$$W_r(\overline{E},\overline{A},\overline{B}) = \mathrm{diag}(W_r(A_1,B_1), W_r(N,B_2)),$$

where $W_r(A_1,B_1) = \Omega_r(A_1,B_1)\Omega_r^{\mathrm{T}}(A_1,B_1)$ and $W_r(N,B_2) = -\Omega_r(N,B_2)\Omega_r^{\mathrm{T}}(N,B_2)$. The observability Gramian matrix of the system (E,A,B,C) is given by

$$W_o(\overline{E},\overline{A},\overline{C}) = \mathrm{diag}(W_o(A_1,C_1), W_o(N,C_2)),$$

where $W_o(A_1,C_1) = \Omega_o^{\mathrm{T}}(A_1,C_1)\Omega_o(A_1,C_1)$ and $W_o(N,C_2) = -\Omega_o^{\mathrm{T}}(N,C_2)\Omega_o(N,C_2)$.

Note that, the uncoupled structure of Gramian matrices in the above definition permits to construct parallel algorithms for computing them, taking advantage respect to the other definitions.

2. Reachability and observability Gramian matrices

In this section we study the reachability Gramian matrix for singular systems. We start with the following lemma.

Lemma 1. *Let (E,A,B,C) be a singular system and consider its associated forward–backward system $(\overline{E},\overline{A},\overline{B},\overline{C})$. If $\rho(A_1) < 1$ then*

(a) *$W_r(A_1,B_1)$ is the only nonnegative definite solution of the Lyapunov equation*

$$P - A_1PA_1^{\mathrm{T}} = B_1B_1^{\mathrm{T}}.$$

(b) *$-W_r(N,B_2)$ is the only nonnegative definite solution of the Lyapunov equation*

$$Q - NQN^{\mathrm{T}} = B_2B_2^{\mathrm{T}}.$$

Proof. Using the stability characterization of standard systems, it is easy to prove part (a). Now, we show condition (b). Since N is a nilpotent matrix, then

$\rho(N) < 1$. So the solution of the Lyapunov equation $Q - NQN^{\mathrm{T}} = B_2B_2^{\mathrm{T}}$ is $\Omega_r(N, B_2)\Omega_r^{\mathrm{T}}(N, B_2)$. Hence $-W_r(N, B_2)$ is the required nonnegative definite solution. □

Using the above lemma, the matrix $W_r(N, B_2)$ satisfies

$$NW_r(N, B_2)N^{\mathrm{T}} - W_r(N, B_2) = B_2B_2^{\mathrm{T}}.$$

To obtain the solution of the Lyapunov equation

$$\overline{E}P\overline{E}^{\mathrm{T}} - \overline{A}P\overline{A}^{\mathrm{T}} = \overline{B}\overline{B}^{\mathrm{T}},$$

where $\overline{E}$, $\overline{A}$ and $\overline{B}$ are given in (4) and (5) and $P = [P_{ij}]_{i,j=1,2}$, we must solve the following equations

$$P_{11} - A_1P_{11}^{\mathrm{T}}A_1^{\mathrm{T}} = B_1B_1^{\mathrm{T}}, \tag{6}$$

$$P_{12}N^{\mathrm{T}} - A_1P_{12} = B_1B_2^{\mathrm{T}}, \tag{7}$$

$$NP_{21} - P_{21}A_1^{\mathrm{T}} = B_2B_1^{\mathrm{T}}, \tag{8}$$

$$NP_{22}N^{\mathrm{T}} - P_{22} = B_2B_2^{\mathrm{T}}. \tag{9}$$

When $\rho(A_1) < 1$, the Eq. (6) has only one solution but Eqs. (7) and (8) can even not have solution. A sufficient condition to guarantee the solution of Eq. (7) is that $B_1B_2^{\mathrm{T}} = O$. We use this condition in the following result.

Theorem 2. *Let (E, A, B, C) be a singular system and consider its associated forward–backward system $(\overline{E}, \overline{A}, \overline{B}, \overline{C})$. If $\rho(A_1) < 1$ and $B_1B_2^{\mathrm{T}} = O$, then the reachability Gramian matrix $W_r(\overline{E}, \overline{A}, \overline{B})$ satisfies the Lyapunov equation*

$$\overline{E}P\overline{E}^{\mathrm{T}} - \overline{A}P\overline{A}^{\mathrm{T}} = \overline{B}\overline{B}^{\mathrm{T}}.$$

Proof. We must prove that the matrix $W_r(\overline{E}, \overline{A}, \overline{B})$ satisfies the Lyapunov equation. In fact,

$$\begin{aligned}
&\overline{E}W_r(\overline{E}, \overline{A}, \overline{B})\overline{E}^{\mathrm{T}} - \overline{A}W_r(\overline{E}, \overline{A}, \overline{B})\overline{A}^{\mathrm{T}} \\
&\quad = \begin{bmatrix} I_{n_1} & O \\ O & N \end{bmatrix} \begin{bmatrix} W_r(A_1, B_1) & O \\ O & W_r(N, B_2) \end{bmatrix} \begin{bmatrix} I_{n_1} & O \\ O & N^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} A_1 & O \\ O & I_{n_2} \end{bmatrix} \\
&\qquad \times \begin{bmatrix} W_r(A_1, B_1) & O \\ O & W_r(N, B_2) \end{bmatrix} \begin{bmatrix} A_1^{\mathrm{T}} & O \\ O & I_{n_2} \end{bmatrix} \\
&\quad = \begin{bmatrix} W_r(A_1, B_1) - A_1W_r(A_1, B_1)A_1^{\mathrm{T}} & O \\ O & NW_r(N, B_2)N^{\mathrm{T}} - W_r(N, B_2) \end{bmatrix}.
\end{aligned}$$

Using part (a) of the above lemma, $W_r(A_1, B_1) - A_1W_r(A_1, B_1)A_1^{\mathrm{T}} = B_1B_1^{\mathrm{T}}$ and also $NW_r(N, B_2)N^{\mathrm{T}} - W_r(N, B_2) = B_2B_2^{\mathrm{T}}$. Further,

$$\overline{BB}^{\mathrm{T}} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1^{\mathrm{T}} & B_2^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} B_1B_1^{\mathrm{T}} & B_1B_2^{\mathrm{T}} \\ B_2B_1^{\mathrm{T}} & B_2B_2^{\mathrm{T}} \end{bmatrix}$$

and since $B_1B_2^{\mathrm{T}} = O$ we obtain the result. □

Now, we give the corresponding results of the observability Gramian matrix for singular systems, for which proofs are similar to the reachability case.

Lemma 3. *Let* (E, A, B, C) *and consider its associated forward–backward system* $(\overline{E}, \overline{A}, \overline{B}, \overline{C})$. *If* $\rho(A_1) < 1$ *then*:

(a) $W_o(A_1, C_1)$ *is the only nonnegative solution of the Lyapunov equation*

$$P - A_1^{\mathrm{T}}PA_1 = C_1^{\mathrm{T}}C_1,$$

(b) $-W_o(N, C_2)$ *is the only nonnegative solution of the Lyapunov equation*

$$Q - N^{\mathrm{T}}QN = C_2^{\mathrm{T}}C_2.$$

Theorem 4. *Let* (E, A, B, C) *be a singular system and consider its associated forward–backward system* $(\overline{E}, \overline{A}, \overline{B}, \overline{C})$. *If* $\rho(A_1) < 1$ *and* $C_1^{\mathrm{T}}C_2 = O$, *then observability Gramian* $W_o(\overline{E}, \overline{A}, \overline{C})$ *satisfies the Lyapunov equation*

$$\overline{E}^{\mathrm{T}}Q\overline{E} - \overline{A}^{\mathrm{T}}Q\overline{A} = \overline{C}^{\mathrm{T}}\overline{C}.$$

3. Balanced model

The need of balancing the gains between inputs and states just like that between states and outputs permits to develop algorithms for obtaining balanced standard system, that is, linear standard systems whose reachability and observability Gramian matrices are diagonal and coincide. This kind of system is used to obtain models of smaller size (and then cheaper) than the original system by different techniques. For instance, in [9] singular perturbation approximation and direct truncation are used to obtain reduced model of balanced standard system.

Different algorithms have been developed to obtain balanced standard realizations, for instance using Markov parameters in [10] and using singular value decomposition of the Hankel matrix in [11]. The purpose of this section is to construct a balanced singular system associated to a generalized system. First, we give the following definition.

Definition 3. Let $(E, A, B, C) \in \mathscr{S}_n^{m,p}$. The system (E, A, B, C) is called balanced if

$$W_r(\overline{E},\overline{A},\overline{C}) = W_o(\overline{E},\overline{A},\overline{C}) = \mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_n),$$

where $\sigma_1,\sigma_2,\ldots,\sigma_n$ are called the Hankel singular values of the system.

A first step to obtain a balanced system associated to the system $(E,A,B,C)\in\mathscr{S}_n^{m,p}$ is to compute the reachability and observability Gramian matrices of this system.

Thus, consider a minimal and stable system $(\overline{E},\overline{A},\overline{B},\overline{C})$ given by (4) and (5), with $B_1B_2^{\mathrm{T}} = O$ and $C_1^{\mathrm{T}}C_2 = O$. An algorithm to compute the reachability (observability) Gramian matrix is as follows:

To take $X_0 = O$ and $Y_0 = O$. The matrix $W_r(A_1,B_1)$ is obtained using the iterative squeme $X_{i+1} = A_1X_iA_1^{\mathrm{T}} + B_1B_1^{\mathrm{T}}$, and the matrix $W_r(N,B_2)$ is obtained using the iterative squeme $Y_{i+1} = NY_iN^{\mathrm{T}} + B_2B_2^{\mathrm{T}}$, and making iterations until convergence.

Assign $W_r(A_1,B_1) = X_i$, $W_r(N,B_2) = -Y_i$ and construct the matrix

$$W_r(\overline{E},\overline{A},\overline{B}) = \begin{bmatrix} W_r(A_1,B_1) & O \\ O & W_r(N,B_2) \end{bmatrix}.$$

By a similar way the corresponding observability Gramian matrices can be obtained, only changing the matrices involving in the iterative squemes. Remember that

$$W_o(\overline{E},\overline{A},\overline{C}) = \begin{bmatrix} W_o(A_1,C_1) & O \\ O & W_o(N,C_2) \end{bmatrix}.$$

Now, we give the algorithm to obtain the balanced realization.

Step 1: To obtain the singular value decomposition of $W_r(\overline{E},\overline{B})$ using the block structure of this matrix. Then there exists an orthogonal matrix $U = \mathrm{diag}(U_1,U_2)$ such that $W_r(\overline{E},\overline{B}) = \mathrm{diag}(U_1,U_2)\Sigma_r^2\mathrm{diag}(U_1^{\mathrm{T}},U_2^{\mathrm{T}})$.

Step 2: To construct matrix $M = (U\Sigma_r)^{\mathrm{T}}W_o(\overline{E},\overline{A},\overline{C})U\Sigma_r$. Then

$$M = \mathrm{diag}(M_1,M_2).$$

Step 3: To obtain the singular value decomposition of M, that is

$$M = \mathrm{diag}(V_1,V_2)\Sigma_o^2\mathrm{diag}(V_1^{\mathrm{T}},V_2^{\mathrm{T}}).$$

Step 4: To construct matrix $T = \mathrm{diag}(T_1,T_2)$

$$T = \Sigma_o^{\frac{1}{2}}\mathrm{diag}(V_1^{\mathrm{T}},V_2^{\mathrm{T}})\Sigma_r^{-1}\mathrm{diag}(U_1^{\mathrm{T}},U_2^{\mathrm{T}}).$$

Step 5: To construct matrices

$$\overline{E}_b = T\overline{E}T^{-1}, \quad \overline{A}_b = T\overline{A}T^{-1}, \quad \overline{B}_b = T\overline{B} \quad \overline{C}_b = \overline{C}T^{-1}.$$

Note that, all the above matrices are diagonal matrices and the system satisfies that $B_{b1}B_{b2}^{\mathrm{T}} = O$ and $C_{b1}^{\mathrm{T}}C_{b2} = O$. Then, the obtained balanced model keeps the uncoupled structure between the forward and backward subsystems and Gramian matrices of the balanced model satisfy the Lyapunov equations

$$\overline{E}_b W_r(\overline{E}_b, \overline{A}_b, \overline{B}_b)\overline{E}_b^{\mathrm{T}} - \overline{A}_b W_r(\overline{E}_b, \overline{A}_b, \overline{B}_b)\overline{A}_b^{\mathrm{T}} = \overline{B}_b\overline{B}_b^{\mathrm{T}},$$

$$\overline{E}_b^{\mathrm{T}} W_o(\overline{E}_b, \overline{A}_b, \overline{C}_b)\overline{E}_b - \overline{A}_b^{\mathrm{T}} W_o(\overline{E}_b, \overline{A}_b, \overline{C}_b)\overline{A}_b = \overline{C}_b^{\mathrm{T}}\overline{C}_b.$$

Note that the Gramian matrices satisfy the following relationships

$$W_r(\overline{E}_b, \overline{A}_b, \overline{B}_b) = TW_r(\overline{E}, \overline{A}, \overline{B})T^{\mathrm{T}} \text{ and } W_o(\overline{E}_b, \overline{A}_b, \overline{C}_b) = T^{-T}W_o(\overline{E}, \overline{A}, \overline{C})T^{-1}.$$

Next an example illustrates the above algorithm.

Example 1. Consider the system (E, A, B, C) given by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

Using the algorithm, the matrix T is given by

$$T = \begin{bmatrix} 0.3915 & 0.4612 & 0.3696 & 0 & 0 & 0 \\ -0.5478 & 0.7040 & -0.7831 & 0 & 0 & 0 \\ -0.2613 & 0.6634 & -1.0880 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5177 & 0.3379 & -0.9730 \\ 0 & 0 & 0 & -0.3051 & -0.8785 & -0.4674 \\ 0 & 0 & 0 & 0.4183 & -0.2226 & 0.1453 \end{bmatrix}$$

and the balanced realization is given by

$$E_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1158 & -0.5656 & -0.2693 \\ 0 & 0 & 0 & 0.5656 & 0.6265 & -0.2431 \\ 0 & 0 & 0 & 0.2693 & -0.2431 & -0.5107 \end{bmatrix},$$

$$A_b = \begin{bmatrix} 0.0065 & -0.3412 & -0.0240 & 0 & 0 & 0 \\ 0.3412 & 0.6303 & 0.2381 & 0 & 0 & 0 \\ -0.0240 & -0.2381 & 0.9632 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_b = \begin{bmatrix} 1.6834 & 1.6834 \\ 0.0771 & 0.0771 \\ -0.0226 & -0.0226 \\ -0.9730 & 0.9739 \\ -0.4674 & 0.4674 \\ 0.1453 & -0.1453 \end{bmatrix},$$

$$C_b = \begin{bmatrix} 2.3807 & -0.1090 & -0.0319 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.3760 & -0.6611 & 0.2054 \end{bmatrix}.$$

Finally, the reachability and observability Gramian matrices are

$$\begin{aligned} W_r(E_b, A_b, B_b) &= W_o(E_b, A_b, C_b) \\ &= \mathrm{diag}(5.8129, 1.2397, 1.0336, -2.6579, -2.1667, -0.4912). \end{aligned}$$

References

[1] B.C. Moore, Principal component analysis in linear systems: controllability, observability, and model reduction, IEEE Trans. Autom. Control AC-26 (1) (1981) 17–32.
[2] R. Ober, D. McFarlane, Balanced canonical forms minimal systems a normalized coprime factor approach, Linear Algebra Appl. 122/123/124 (1989) 23–64.
[3] E.J. Ang, V. Sreeram, W.Q. Liu, Identification/reduction to balanced realization via the extended impulse response Gramian, IEEE Trans. Autom. Control 40 (12) (1995) 2153–2158.
[4] M. Diab, V. Sreeram, W.Q. Liu, Frequency weighted identification and model reduction via extended impulse response Gramian, Int. J. Control 70 (1) (1998) 103–122.

[5] D. Bender, Lyapunov-like equations and reachability/observability Gramians for descriptor systems, IEEE Trans. Autom. Control AC-32 (4) (1987) 343–348.
[6] L. Zhang, J. Lam, Q. Zhang, Lyapunov and Riccati equations of discrete-time descriptor systems, IEEE Trans. Autom. Control. 44 (11) (1999) 2134–2139.
[7] D.F. Delchamps, State-Space and Input–Output Linear Systems, Springer-Verlag, New York, 1988.
[8] L. Dai, Singular Control Systems, Springer-Verlag, NewYork, 1989.
[9] Y. Liu, B.D.O. Anderson, Singular perturbation approximation of balanced systems, Int. J. Control 50 (4) (1989) 1379–1405.
[10] J. Sveinsson, F. Fairman, Minimal balanced realization of transfer function matrices using Markov parameters, IEEE Trans. Autom. Control AC-30 (10) (1985) 1014–1016.
[11] C. Yang, F. Yeh, A simple algorithm on minimal balanced realization for transfer function matrices, IEEE Trans. Autom. Control 34 (8) (1989) 879–882.